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SOME APPLICATIONS OF WEAKLY PICARD OPERATORS

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} anniversary

Abstract. In this paper we give some applications of weakly Picard operators theory to linear positive approximation operators, to difference equations with deviating argument and to functional-integral equations.

1. Introduction

Let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$$F_A := \{x \in X |\ A(x) = x\};$$

$$I(A) := \{Y \subset X |\ A(Y) \subset Y,\ Y \neq \emptyset\};$$

$$A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A \circ A^n, \quad n \in N.$$

By definition an operator A is weakly Picard operator (WPO) if the sequence of successive approximations, $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit is a fixed point of A. If the operator A is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator (PO). For an WPO A we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

We have the following characterization of the WPOs.

Theorem 1.1 (I. A. Rus [6], [7], [12]). *An operator A is WPO if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A)$, $\forall\ \lambda \in \Lambda$;

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(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is PO,* $\forall\ \lambda \in \Lambda$.

The aim of this paper is to give some applications of this theorem.

2. Iterates of two variables Bernstein operator

Let $\overline{D} = \{(x,y) \in R^2 |\ x, y \in R_+,\ x+y \le 1\}$ and $e_{ij} : \overline{D} \to R_+$ be defined by $e_{ij} := x^i y^j$, $i, j \in N$.

Let we denote by $\|\cdot\|_C$ the Chebyshev norm on $C(\overline{D})$.

In what follow we consider the two variables Bernstein operator (see D. D. Stancu [13])

$$B_n : C(\overline{D}) \to C(\overline{D}), \quad n \in N^*$$

defined by

$$B_n(f)(x,y) := \sum_{0 \le i+j \le n} \frac{n!}{i!j!(n-i-j)!} x^i y^j (1-x-y)^{n-i-j} f\left(\frac{i}{n}, \frac{j}{n}\right). \qquad (2.1)$$

It is well known that ([13]):

$$e_{00}, e_{01}, e_{10} \in F_{B_n}, \quad n \in N^*.$$

We have

Theorem 2.1. *The operator* B_n *is WPO and*

$$B_n^\infty(f)(x,y) = f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y, \quad x, y \in \overline{D};\ f \in C(\overline{D}).$$

Proof. Let

$$X_{\alpha,\beta,\gamma} := \{f \in C(\overline{D}) |\ f(0,0) = \alpha,\ f(1,0) = \beta,\ f(0,1) = \gamma\},$$

$$f_{\alpha,\beta,\gamma}(x,y) := \alpha + (\beta - \alpha)x + (\gamma - \alpha)y, \quad x, y \in \overline{D},$$

for all $\alpha, \beta, \gamma \in R$.

We remark that

(i) $X_{\alpha,\beta,\gamma}$ is a closed subset of $C(\overline{D})$;

(ii) $X_{\alpha,\beta,\gamma}$ is an invariant subset of B_n, for all $\alpha, \beta, \gamma \in R$ and $n \in N^*$;

(iii) $C(\overline{D}) = \bigcup_{\alpha,\beta,\gamma \in R} X_{\alpha,\beta,\gamma}$ is a partition of $C(\overline{D})$;

(iv) $f_{\alpha,\beta,\gamma} \in X_{\alpha,\beta,\gamma} \cap F_{B_n}$.

Now we prove that

$$B_n|_{X_{\alpha,\beta,\gamma}} : X_{\alpha,\beta,\gamma} \to X_{\alpha,\beta,\gamma}$$

is a contraction for all $\alpha, \beta, \gamma \in R$ and $n \in N^*$.

Let $f, g \in X_{\alpha,\beta,\gamma}$. From (2.1) we have

$$|B_n(f)(x,y) - B_n(g)(x,y)| = |B_n(f-g)(x,y)| \leq$$

$$\leq |1-(1-x-y)^n - x^n - y^n| \cdot \|f-g\|_C \leq$$
$$\leq \left(1 - \frac{1}{2^{n-1}}\right) \|f-g\|_C, \ \forall \ x, y \in \overline{D}.$$

So,

$$\|B_n(f) - B_n(g)\|_C \leq \left(1 - \frac{1}{2^{n-1}}\right) \|f-g\|_C, \ \forall \ f, g \in X_{\alpha,\beta,\gamma};$$

i.e., $B_n|_{X_{\alpha,\beta,\gamma}}$ is a contraction for all $\alpha, \beta, \gamma \in R$.

From the contraction principle $f_{\alpha,\beta,\gamma}$ is the unique fixed point of B_n in $X_{\alpha,\beta,\gamma}$ and that $B_n|_{X_{\alpha,\beta,\gamma}}$ is a PO.

From the Theorem 1.1 the proof follows.

Remark 2.1. For the one dimensional case see I. A. Rus [10], [11], [12] and O. Agratini and I. A. Rus [1]. See also R.P. Kelisky and T.J. Rivlin [4].

Remark 2.2. The case $\overline{D} = [0,1] \times [0,1]$ (see P. L. Butzer [3]) will be presented elsewhere.

Remark 2.3. A similar result for Bernstein operators on a simplex we have.

3. Difference equations in $C([0,1], X)$

Let X be a Banach space. We denote by $\|\cdot\|_C$ the Chebyshev norm on $C([0,1], X)$. Let $h \in C([0,1] \times X \times X, X)$ and $g \in C([0,1] \times X, X)$ be two operators. In what follow we consider the following difference equation with deviating argument, in $C([0,1], X)$,

$$x_{n+1}(t) = h(t, x_n(t), x_n(0)) + g(t, x_n(t)), \quad t \in [0,1), \ n \in N^* \tag{3.1}$$

For to study this equation we consider the operator

$$A : C([0,1], X) \to C([0,1], X)$$

$$A(x)(t) := h(t, x(t), x(0)) + g(t, x(t)).$$

We have

Theorem 3.1. *We suppose that*

(i) $h(0,\lambda,\lambda)=\lambda,\ \forall\ \lambda\in X$*;*

(ii) $g(0,\lambda)=0,\ \forall\ \lambda\in X$*;*

(iii) $g(t,\cdot)$ *is an* α*-contraction for all* $t\in[0,1]$*;*

(iv) $h(t,\cdot,\lambda)$ *is a* β*-contraction for all* $t\in[0,1]$, $\lambda\in X$*;*

(v) $\alpha+\beta<1$.

Then the operator A *is WPO.*

Proof. Let

$$X_\lambda := \{x\in C([0,1],X) |\ x(0)=\lambda\},\quad \lambda\in X.$$

Then

(a) X_λ is a closed subset of $C([0,1],X)$;

(b) $X_\lambda\in I(A)$, for all $\lambda\in X$;

(c) $C([0,1],X)=\bigcup\limits_{\lambda\in\Lambda} X_\lambda$ is a partition of $C([0,1],X)$.

From (i)-(v) we have that the restriction of A to X_λ is an $(\alpha+\beta)$-contraction.

By the Theorem 1.1 we have that the operator A is WPO.

Let x_λ^* be the unique fixed point of the operator A in X_λ. It is clear that $cardF_A = cardX$, and that F_A is the equilibrium solution set of the equation (3.1).

From the Theorem 3.1 we have

Theorem 3.2. *In the conditions of the Theorem 3.1, let* $(x_n)_{n\in N}$ *be a solution of the equation (3.1). If* $x_0\in X_\lambda$*, then* $x_n\in X_\lambda$*, for all* $n\in N$*. Moreover*

$$x_n\to x_\lambda^* \text{ as } n\to\infty.$$

Remark 3.1. In the conditions of Theorem 3.1 the equilibrium solution x_λ^* is globally asymptotically stable relative to X_λ.

Remark 3.2. For the fixed point technique in the theory of difference equations see M. A. Şerban [14].

Remark 3.3. The following example is in the conditions of the Theorem 3.1:

$$x_{n+1}(t)=\frac{1}{2}t\sin x_n(t)+x_n(0),\quad n\in N$$

$$x_0 \in C[0,1]$$

4. **Functional-integral equations**

Let X be a Banach space $f \in C([a,b] \times X, X)$ and $K \in C([a,b] \times [a,b] \times X, X)$. Consider the following functional-integral equation

$$x(t) = x(a) + \int_a^t f(s, x(s))ds + \int_\alpha^t \int_a^s K(s, u, x(u))duds \tag{4.2}$$

$$t \in [a,b]; \quad x \in C([a,b], X)$$

Let $X_\lambda := \{x \in C([a,b], X) | \ x(a) = \lambda\}$, $\lambda \in X$ and $A : C([a,b], X) \to C([a,b], X)$ defined by $A(x)(t) :=$ second part of (4.1).

If we denote by S the solution set of the eq. (4.1) then $S = F_A$.

We remark that

(a) X_λ is a closed subset of $C([0,1], X)$ for all $\lambda \in X$;

(b) $X_\lambda \in I(A)$;

(c) $C([0,1], X) = \bigcup\limits_{\lambda \in X} X_\lambda$ is partition of $C([0,1], X)$;

(d) if $f(s, \cdot)$ is L_f-Lipschitz and $K(s, u, \cdot)$ is L_K-Lipschitz for all $s, u \in [a,b]$ then the restriction of A to X_λ is a contraction with respect to a suitable Bielecki's norm. More exactly if we denote

$$\|x\|_B = \max_{a \leq t \leq b} (\|x(t)\| e^{-\tau(t-a)})$$

then we have

$$\|A(x) - A(y)\|_B \leq \left(\frac{L_f}{\tau} + \frac{L_K}{\tau^2} \right) \|x - y\|_B, \ \forall \ x, y \in X_\lambda; \ \lambda \in X.$$

Let x^*_λ be the unique fixed point of A in X_λ. From the Theorem 1.1 it follows that the operator A is WPO and $cardF_A = cardX$.

So, we have

Theorem 4.1. *In the above conditions*

(1) $cardS = cardX$

(2) the solution x^*_λ *is globally asimptotically stable with respect to* X_λ.

Remark 4.1. For other types of functional integral equations see R. Precup [5], I. A. Rus [8] and [9].

Remark 4.2. For other applications of the WPO see A. Buică [2], I. A. Rus [6], [7].

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