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THE APPROXIMATION OF THE EQUATION'S SOLUTION IN LINEAR NORMED SPACES USING APPROXIMANT SEQUENCES (II)

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. Considering a function between two linear normed spaces and a arbitrary approximant sequence, we will study the conditions for the convergence of this sequence towards one solution of the equation generated by this function. The speed of convergence should be of a big enough order, characterized by a number $p \in \mathbb{N}$.

1. Introduction

One of the most often used methods for the approximation of an equation's solutions is that of constructing a sequence that is convergent to that solution. In order to do that it is necessary to know this solution and maybe also its quality of being the only one existing near a determined point.

A sequence having the quality described above will be called an approximant sequence.

From the practical point of view, in order to make an approximation of the solution with an error that doesn't exceed the maximum admissible value, it is important not to use too many terms of the approximant sequence, that is to obtain a good speed of approximation.

In order to make the concepts above clear, let us consider X and Y two normed linear spaces, their norm $\|\cdot\|_X$ and respectively $\|\cdot\|_Y$ a set $D \subseteq X$, a function $f : D \longrightarrow Y$, θ_Y, the null element of the space Y and, using these elements, the equation:

$$f(x) = \theta_Y \tag{1}$$

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To clarify these notions, we will have:

Definition 1.1. *In addition to the data above, let us also consider* $p \in \mathbb{N}$, *and* $(x_n)_{n\in\mathbb{N}} \subseteq D$. *We say that the sequence is an approximant sequence of the order* p *of a solution of the equation* (1), *if there exist* $\alpha, \beta \geq 0$ *so that for any* $n \in \mathbb{N}$ *we have:*

$$\begin{aligned} \|f(x_{n+1})\|_Y &\leq \alpha \|f(x_n)\|_Y^p ; \\ \|x_{n+1} - x_n\|_X &\leq \beta \|f(x_n)\|_Y . \end{aligned} \tag{2}$$

As we showed in papers [3] and [4], if $(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the order p, $p \geq 2$; X is a Banach space; $f : D \to Y$ is continuous, and the constants α and β that verify **Definition 1.1** are chosen so that:

$$\begin{aligned} \rho_0 &= \alpha^{\frac{1}{p-1}} \parallel f(x_0) \parallel_Y , \\ S(x_0, \delta) &= \{x \in X / \parallel x - x_0 \parallel_X \leq \delta\} \subseteq D, \end{aligned} \tag{3}$$

with:

$$\delta = \frac{\beta\alpha^{\frac{1}{p-1}}}{1 - \rho_0^{p-1}},$$

then the approximant sequence is convergent towards the element x^* which, together with all the terms of the sequence $(x_n)_{n\in\mathbb{N}}$ is placed in the ball $S(x_0, \delta)$ and x^* is a solution of the equation (1). For any $n \in \mathbb{N}$ the following inequalities take place:

$$\begin{aligned} \parallel x_{n+1} - x_n \parallel_X &\leq \beta\alpha^{\frac{1}{p-1}} \rho_0^{p^n} \\ \parallel x^* - x_n \parallel &\leq \frac{\beta\alpha^{\frac{1}{p-1}} \rho_0^{p^n}}{1 - \rho_0^{p^n\,(p-1)}} . \end{aligned} \tag{4}$$

These inequalities justify the fact of calling it an approximant sequence of the order p ; the last inequality will also give an evaluation of a superior margin of the error through which x_n approximates x^*.

Above x_0 is the initial element of the sequence, the starting element of the approximation proceeding.

The convergence or the non-convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ as well as the convergence speed, materialized through the number p, depend on the fact of correctly choosing x_0.

In order to verify the inequalities (4) as well as the affirmations preceding them we have to make the inequalities (2) true. But this often proves to be difficult, and this is the reason for which we will try to replace them with more practical conditions. Nevertheless we will consider that the function $f : D \to Y$ admits Fréchet derivatives up to the order p included.

As a series of iterative methods known in practice use the inverse of the Fréchet derivative of the first order of the mapping $[f\ '(x_n)]^{-1}$, an unpractical condition, as the existence of this mapping implies solving the linear equation $f'(x_n)h = q$; $h \in X$, $q \in Y$, we will try to eliminate the conditions about the inverse of the Fréchet derivative from the hypothesis, but we will try to demonstrate this existence.

From the results that have inspired this work of research we will mention primarily the well-known theorem of **L. V. Kantorovich** for the case when the approximant sequence $(x_n)_{n\in N}$ is generated by the **Newton - Kantorovich** method [5], [6]. In this case the existence of the mapping $[f'(x)]^{-1} \in (Y, X)^*$ is supposed only for $x = x_0$, as this is the initial point of the iterative method . In what the convergence of the same method is concerned, we also mention the result obtained by **Misovski, I. P.** , [7], where from a certain point of view the conditions of the convergence are simpler, but the existence of the mapping $[f\ '(x)]^{-1}$ and of a constant $M > 0$ satisfying the inequality $\left\|[f'(x)]^{-1}\right\| \leq M$ for any x - an element of a certain ball centered in the initial element x_0 - is imposed. Then **Păvăloiu, I.**, in [8] , [9], generalizes these results for the convergence of a sequence generated by the relation of recurrence:

$$x_{n+1} = Q\left(x_n\right) \tag{5}$$

where $Q : X \to X$ verifies certain conditions. In the result obtained by **Păvăloiu, I.**,. **Misovski's** condition mentioned above does not appear explicitly, but the use of the result in concrete cases makes it necessary. Thus this general result can be applied in the case of the **Newton-Kantorovich** method to obtain **Misovski's** result and in the case of **Chebischev's** method, obtaining a corresponding result.

By changing one of the conditions our result is more easily applicable than that of **Păvăloiu, I.** for concrete methods. We also succeed to show that for any

$n \in \mathbb{N}$, $[f'(x_n)]^{-1}$ exists and these mappings taken for any $n \in \mathbb{N}$ form an equally margined set.

2. **Main results**

We will proceed in the same way as in our papers [1] , [2].

Let us now note by $(X^p, Y)^*$ the set of p -linear and continuous mappings defined on

$$X^p = \underbrace{X \times \cdots \times X}_{p\ times}$$

(the p times Cartesian product), taking values in Y.

The fact that the mapping $f^{\,(p)} : D \to (X^p, Y)^*$ verifies **Lipschitz's** condition is resumed to the existence of the constant $L > 0$, so that for any $x, y \in D$ we can have:

$$\left\| f^{(p)}(x) - f^{(p)}(y) \right\| \le L \left\| x - y \right\|_X \tag{6}$$

so that L will be called **Lipschitz's** constant.

From the verification of such a condition with the constant $L > 0$ we can easily deduce that for any $x, y \in D$ the following inequality takes place:

$$\left\| f(x) - f(y) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(y)(x-y)^i \right\|_Y \le \frac{L}{(p+1)!} \parallel x - y \parallel_X^{p+1} . \tag{7}$$

Then if we take $x_0 \in D$ and $\delta > 0$ so that:

$$S(x_0, \delta) = \{ x \in X / \left\| x - x_0 \right\| \le \delta \} \subseteq D$$

and we define the numbers $L_0, ..., L_p > 0$ through:

$$L_k = \left\| f^{(k)}(x_0) \right\| + L_{k+1}\delta; \quad k = 0, 1, .., p \tag{8}$$

with $L_{p+1} = L$, then for any $x \in S(x_0, \delta)$ we have:

$$\left\| f^{(k)}(x) \right\| \le L_{k+1}\delta \tag{9}$$

for any $k \in \{0, 1, ..., p\}$ and for any $x, y \in S(x_0, \delta)$ we have:

$$\left\| f^{(k-1)}(x) - f^{(k-1)}(y) \right\| \le L_k \left\| x - y \right\|_X ,$$

for any $k \in \{1, 2, ...p + 1\}$.

Under the conditions mentioned above, the following takes place:

Theorem 2.1. *In addition to the data above we consider* $p \in N$, $\delta > 0$, $(x_n)_{n\in N} \subseteq D$.

If:

i) X *is a Banach space and* $S(x_0, \delta) \subseteq D$, $S(x_0, \delta)$ *representing the ball with the center* x_0 *and radius* δ*;*

ii) *the function* $f : D \to Y$ *admits Fréchet derivatives up to the order p including it , and, for* $f^{(p)} : D \to (X^p, Y)^*$ *the number* $L > 0$ *exists so that for any* $x, y \in D$ *the following inequality* (6) *is verifies:*

iii) $a, b \geq 0$ *exist so that for any* $n \in \mathbb{N}$ *we have the inequalities:*

$$\left\| f(x_n) + \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_n)(x_{n+1} - x_n)^i \right\|_Y \leq a \left\| f\left(x_n\right) \right\|_Y^{p+1} \tag{10}$$

and:

$$\left\| f'\left(x_n\right)\left(x_{n+1} - x_n\right) \right\|_Y \leq b \left\| f\left(x_n\right) \right\|_Y ; \tag{11}$$

iv) *the mapping* $f'(x_0) \in (X, Y)^*$ *is invertible;*

v) *if we note:*

$$\begin{gathered} \rho_0 = \left\| f\left(x_0\right) \right\|_Y, \quad B_0 = \left\| \left[f'\left(x_0\right)\right]^{-1} \right\|, \quad h_0 = bL_2 B_0^2 \rho_0 \\ M = \left\| \left[f'\left(x_0\right)\right]^{-1} \right\| e^{1+2^{-2p=3}}, \quad \alpha = a + L\frac{(bM)^{p+1}}{(p+1)!} \end{gathered} \tag{12}$$

the following inequalities are verified:

$$h_0 \leq \frac{1}{2}, \quad \alpha^{\frac{1}{p}} \rho_0 < \frac{1}{4}, \quad \delta \geq \frac{bM\rho_o}{1 - \alpha\rho_0^p} \tag{13}$$

then:

j) $x_n \in S(x_0, \delta)$, $[f'(x_n)]^{-1}$ *exists and* $\left\|[f'(x_n)]^{-1}\right\| \geq M$ *for any* $n \in \mathbb{N}$*;*

jj) *the equation* (1) *admits a solution* $x^* \in S(x_0, \delta)$*;*

jjj) *the sequence* $(x_n)_{n\in N}$ *is an approximant sequence of the order* $p+1$ *of this solution of the equation* (1)*;*

jv) *the following estimates hold:*

$$\max\left\{ \left\| f(x_n) \right\|_Y, \frac{1}{Mb} \left\| x_{n+1} - x_n \right\|_X \right\} \leq \alpha^{\frac{(p+1)^n - 1}{p}} \left\| f(x_0) \right\|_Y^{(p+1)^n} \tag{14}$$

and:

$$\|x^* - x_n\|_X \le \frac{bM\alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{1-(\alpha\rho_0^p)^{(p+1)^n}} \tag{15}$$

for any $n \in \mathbb{N}$.

Proof. From the invertibility of the mapping $f'(x_0) \in (X,Y)^*$ we clearly deduce that:

$$\|f'(x_0)\|, \ \left\|[f'(x_0)]^{-1}\right\| > 0.$$

Let the sequences $(\rho_n)_{n\in N}$, $(B_n)_{n\in N}$ and $(h_n)_{n\in N}$ be so that:

$$\rho_0 = \|f(x_0)\|_Y, \ B_0 = \left\|[f'(x_0)]^{-1}\right\|$$

and for any $n \in \mathbb{N}$, we have:

$$h_n = bL_2B_n^2\rho_n, \ \rho_{n+1} = \alpha\rho_n^{p+1}, \ B_{n+1} = \frac{B_n}{1-h_n}.$$

We will show that for any $n \in \mathbb{N}$ the following statements are true:

$$\begin{aligned}
&\mathbf{a)} \quad x \in S(x_0,\delta),\\
&\mathbf{b)} \quad [f'(x_n)]^{-1} \in (Y,X)^* \text{ exists, and } \left\|[f'(x_n)]^{-1}\right\| \le B_n,\\
&\mathbf{c)} \quad \|f(x_n)\|_Y \le \rho_n = \alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n},\\
&\mathbf{d)} \quad h_n \le \min\left\{\frac{1}{2}, \ \beta^{-\frac{1}{p}}(\beta h_0)^{(p+1)^n}\right\}, \text{ where } \beta = \frac{4}{(4h_0)^p},\\
&\mathbf{e)} \quad B_0 \le B_n \le M.
\end{aligned} \tag{16}$$

Using mathematical induction we notice that for $n = 0$ the statements **a)**−**e)** are evidently true from the hypotheses of the theorem with the notations we have introduced.

Let us suppose that for any $n \le k$ the assertions **a)** − **e)** are true, and let us demonstrate them for $n = k+1$.

a)We notice that for any $n \in N, \ n \le k$ we have:

$$\|x_{n+1} - x_n\|_X = \left\|[f'(x_n)]^{-1}f'(x_n)(x_{n+1}-x_n)\right\|_X \le$$

$$\le \left\|[f'(x_n)]^{-1}\right\| . \|f'(x_n)(x_{n+1}-x_n)\|_Y \le Mb\|f(x_n)\|_Y \le Mb\rho_n.$$

So:

$$\|x_{n+1} - x_n\|_X \leq Mb\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}} \rho_0\right)^{(p+1)^n}, \tag{17}$$

from where:

$$\|x_{n+1} - x_0\|_X \leq \sum_{n=0}^{k} \|x_{n+1} - x_n\|_X \leq Mb\alpha^{-\frac{1}{p}} \sum_{n=0}^{k} \left(\alpha^{\frac{1}{p}} \rho_0\right)^{(p+1)^n - 1}$$

From $p \geq 1$ we deduce that $(p+1)^n - 1 > np$ for any $n \in \mathbb{N},\ n > 0$ and as $\rho_0 < 1$ we deduce that:

$$\sum_{n=0}^{k} \left(\alpha^{\frac{1}{p}} \rho_0\right)^{(p+1)^n - 1} < \sum_{n=0}^{k} (\alpha\rho_0^p)^n = \frac{1 - (\alpha\rho_0^p)^{k+1}}{1 - \alpha\rho_0^p} < \frac{1}{1 - \alpha\rho_0^p}.$$

So:

$$\|x_{n+1} - x_0\|_X \leq Mb\frac{\rho_0}{1 - \alpha\rho_0^p} \leq \delta$$

from where it results immediately that $x_{n+1} \in S(x_0, \delta)$.

b)Let:

$$H_k = [f'(x_k)]^{-1} (f'(x_k) - f'(x_{k+1})) \in (X, X)^*,$$

its existence and its belonging to $(X, X)^*$ are guaranteed by the hypothesis of the induction. It is obvious that:

$$\|H_k\| \leq \left\|[f'(x_k)]^{-1}\right\| \cdot \|f'(x_k) - f'(x_{k+1})\| \leq B_k L_2 \|x_{k+1} - x_k\|_X$$

But:

$$\|x_{k+1} - x_k\|_X \leq \left\|[f'(x_k)]^{-1}\right\| \cdot \|f'(x_k)(x_{k+1} - x_k)\|_Y \leq bB_k \|f(x_k)\|_Y \leq$$

$$\leq bB_k\rho_k,$$

from where:

$$\|H_k\| \leq bL_2 B_k^2 \rho_k = h_k \leq \frac{1}{2} < 1$$

and according to the well known **Banach's theorem** we deduce that:

$$(I_k - H_k)^{-1} \in (X, X)^*$$

and:

$$\left\|(I_k - H_k)^{-1}\right\| \leq \frac{1}{1 - \|H_k\|} \leq \frac{1}{1 - h_k}$$

(here $I_X : X \to X$ represents the identical mapping of the space X).

Obviously:

$$I_X - H_k = [f'(x_k)]^{-1} f'(x_{k+1}),$$

from where:

$$f'(x_{k+1}) = f'(x_k)(I_k - H_k).$$

The hypothesis of the induction guarantees the existence of the mapping $[f'(x_k)]^{-1} \in (Y,X)^*$, so, from the above, the mapping $(I_k - H_k)^{-1}$ will exist, so the mapping $[f'(x_{k+1})]^{-1} = (I_k - H_k)^{-1}\ [f'(x_k)]^{-1}$ will exist as well, and:

$$\left\|[f'(x_{k+1})]^{-1}\right\| \leq \left\|[f'(x_k)]^{-1}\right\| \cdot \left\|(I_X - H_k)^{-1}\right\| \leq \frac{B_k}{1-h_k} = B_{k+1}.$$

c) Clearly:

$$\|f(x_{k+1})\|_Y \leq \left\| f(x_{k+1}) - f(x_k) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_k)(x_{k+1}-x_k)^i \right\|_Y + \\ + \left\| f(x_k) + \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_k)(x_{k+1}-x_k)^i \right\|_Y.$$

Because of the fact that $x_k, x_{k+1} \in S(x_0,\delta) \subseteq D$, of the hypothesis **ii**) and using the remark that precedes the text of the theorem we deduce that:

$$\left\| f(x_{k+1}) - f(x_k) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_k)(x_{k+1}-x_k)^i \right\|_Y \leq \frac{L}{(p+1)!} \|x_{k+1}-x_k\|_X^{p+1},$$

also using the first inequality from the hypothesis **iii**) we deduce that:

$$\|f(x_{k+1})\|_Y \leq \frac{L}{(p+1)!} \|x_{k+1}-x_k\|_X^{p+1} + a\,\|f(x_k)\|_Y^{p+1} \leq \\ \leq \left[a + \frac{L(Mb)^{p+1}}{(p+1)!}\right] \|f(x_k)\|_X^{p+1} \leq \alpha\rho_k^{p+1} = \rho_{k+1}.$$

As $\rho_{k+1} = \alpha\rho_k^{p+1}$ and $\rho_k = \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^k}$ we deduce that:

$$\alpha^{\frac{1}{p}}\rho_{k+1} = \left(\alpha^{\frac{1}{p}}\rho_k\right)^{p+1} = \left(\alpha^{\frac{1}{p}}\rho_k\right)^{(p+1)^{k+1}},$$

so:

$$\rho_{k+1} = \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_k\right)^{(p+1)^{k+1}}.$$

d) We have the equalities:

$$h_{k+1} = L_2 b B_{k+1}^2 \rho_{k+1} = L_2 b \alpha \rho_k^{p+1} \left(\frac{B_k}{1-h_k}\right)^2 = \alpha h_k \frac{\rho_k^p}{(1-h_k)^2}.$$

From $h_k \leq \dfrac{1}{2}$, we deduce that:

$$\frac{h_k}{(1-h_k)^2} \leq 2$$

so:

$$h_{k+1} \leq 2\alpha\rho_k^p.$$

We have:

$$\alpha^{\frac{1}{p}}\rho_0 < 1 \Rightarrow \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^k} < \alpha^{\frac{1}{p}}\rho_0 \Rightarrow \rho_k < \rho_0 \Rightarrow h_{k+1} \leq 2\alpha\rho_0^p \Rightarrow$$

$$\Rightarrow h_{k+1} \leq 2\left(\alpha^{\frac{1}{p}}\rho_0\right)^p < \frac{1}{2^{2p-1}} \leq \frac{1}{2}.$$

Meanwhile:

$$h_{k+1} = \frac{\alpha h_k}{(1-h_k)^2} \cdot \frac{h_k^p}{(bL_2B_k^2)^p} = \frac{\alpha}{(bL_2)^p} \cdot \frac{1}{B_k^{2p}} \cdot \frac{h_k^{p+1}}{(1-h_k)^2}$$

From $B_k \geq B_0$ and:

$$\frac{1}{(1-h_k)^2} \leq 4$$

we deduce that:

$$h_{k+1} \leq \frac{4\alpha h_k^{p+1}}{(bL_2)^p\ B_0^{2p}} < \frac{4h_k^{p+1}}{(bL_2B_0^2)^p 4^p \rho_0^p} = \beta h_k^{p+1}$$

and then, it the same way as in the proof of **c**) we deduce that:

$$h_{k+1} = \beta^{-\frac{1}{p}}\left(\beta^{\frac{1}{p}}h_0\right)^{(p+1)^{k+1}}$$

e) Because $B_{k+1} = \dfrac{B_k}{1-h_k}$ and $h_k \in]0, \dfrac{1}{2}]$ we have $B_{k+1} \geq B_k$, so $B_{k+1} \geq B_0$.

The same initial relation implies:

$$B_{k+1} = \frac{B_k}{(1-h_0)(1-h_1)\ldots(1-h_k)}.$$

Using the inequality between the geometric mean and the arithmetic mean we deduce:

$$\frac{1}{(1-h_0)(1-h_1)\ldots(1-h_k\ )} \leq \left[\frac{1}{k+1}\sum_{i=0}^{k}\frac{1}{1\ -\ h_i}\right]^{k+1} =$$

$$= \left[1 + \frac{1}{k+1}\sum_{i=0}^{k}\frac{h_i}{1 \; - \; h_i}\right]^{k+1}.$$

As $\beta^{\frac{1}{p}} h_0 = \frac{4^{\frac{1}{p}}}{4} \leq 1$ we deduce that:

$$\max\left\{\beta^{-\frac{1}{p}}\left(\beta^{\frac{1}{p}} h_0\right)^{(p+1)^n} \Big/ n \in \mathbb{N}\right\} = \beta^{-\frac{1}{p}}\left(\beta^{\frac{1}{p}} h_0\right) = h_0$$

and:

$$\sum_{i=0}^{k}\frac{h_i}{1-h_i} \leq \sum_{i=0}^{k}\frac{h_i}{1-\beta^{-\frac{1}{p}}(\beta^{\frac{1}{p}} h_0)^{(p+1)^i}} \leq \frac{1}{1-h_0}\sum_{i=0}^{k} h_i.$$

But for $k \in \mathbb{N}$ we have:

$$h_{k+1} = \frac{\alpha h_k \rho_k^p}{(1-h_k)^2} \leq 2\alpha\alpha^{-1}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{p(p+1)^k} = 2\left(\alpha\rho_0^p\right)^{(p+1)^k},$$

and so:

$$\sum_{i=0}^{k} h_i = h_0 + 2\sum_{i=1}^{k}\left(\alpha\rho_0^p\right)^{(p+1)^{i-1}} = h_0 + 2\alpha\rho_0^p\sum_{i=1}^{k}\left(\alpha\rho_0^p\right)^{(p+1)^{i-1}-1}.$$

For $i \geq 2$ we have:

$$(p+1)^{i-1} - 1 = p\left[1 + (p+1) + ... + (p+1)^{i-2}\right] \geq p\,(i-1),$$

so:

$$\sum_{i=0}^{k} h_i \leq h_0 + 2\alpha\rho_0^p\left[1 + \sum_{i=2}^{k}\left(\alpha^p\rho_0^{p^2}\right)^{i-1}\right] < h_0 + \frac{2\alpha\rho_0^p}{1-\alpha^p\rho_0^{p^2}} < \frac{1}{2} + \frac{2^{2p^2-2p+1}}{2^{2p^2}-1}$$

But, as $p \geq 1$ we have :

$$2^{2p^2} - 1 = 1 + 2 + 2^2 + ... + 2^{2p^2-1} \geq 2^{2p^2-1}$$

so, evidently:

$$\sum_{i=0}^{k} h_i < \frac{1}{2} + \frac{2^{2p^2-2p+1}}{2^{2p^2-1}} = \frac{1}{2} + 2^{-2p+2}$$

and:

$$\sum_{i=0}^{k}\frac{h_i}{1-h_i} \leq \frac{1}{1-h_0}\left(\frac{1}{2} + 2^{-2p+2}\right) \leq 1 + 2^{-2p+3},$$

from where:

$$\left(1 + \frac{1}{k+1}\sum_{i=0}^{k}\frac{h_i}{1-h_i}\right)^{k+1} \leq \left(1 + \frac{1+2^{-2p+3}}{k+1}\right)^{k+1} \leq \exp\left(1 + 2^{-2p+3}\right)$$

and:

$$B_{k+1} \leq B_0 \exp\left(1 + 2^{-2p+3}\right).$$

From the above we deduce that the statements **a)-e)** from (16) are true for $n = k+1$. According to the principle of mathematical induction these statements are true for any $n \in \mathbb{N}$.

Now we will deduce that, that sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, because:

$$\|x_{n+m} - x_n\|_X < \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\|_X \leq \sum_{i=n}^{n+m-1} Mb\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^i} =$$

$$= bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n} \sum_{j=0}^{m-1} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^{n+j}-(p+1)^n}.$$

But, for any $j \in \{0, 1, ..., m-1\}$ we have:

$$(p+1)^{n+j} - (p+1)^n = (p+1)^n \left[(p+1)^j - 1\right] =$$

$$= p\,(p+1)^n \left[1 + (p+1) + ... + (p+1)^{j-1}\right] \geq jp\,(p+1)^n,$$

so:

$$\|x_{n+m} - x_n\|_X < bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n} \sum_{j=0}^{m-1} \left[(\alpha\rho_0^p)^{(p+1)^n}\right]^j$$

and so:

$$\|x_{n+m} - x_n\|_X < \frac{bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{1 - (\alpha\rho_0^p)^{(p+1)^n}} \tag{18}$$

The last inequality and the condition:

$$\alpha^{\frac{1}{p}}\rho_0 < \frac{1}{4} < 1$$

determine the fact that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space X, so $(x_n)_{n\in\mathbb{N}}$ is convergent. If we note:

$$x^* = \lim_{n\to\infty} x_n \in X$$

and if we make so that $m \to \infty$ in the inequality (18) we deduce that:

$$\|x^* - x_n\|_X \leq \frac{bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{1 - (\alpha\rho_0^p)^{(p+1)^n}},$$

(this is the inequality (15)), from where for $n = 0$ we can deduce:

$$\|x^* - x_0\|_X \leq \frac{bM\rho_0}{1-\alpha\rho_0^p} \leq \delta,$$

so $x^* \in S(x_0, \delta)$.

From:

$$\|f(x_n)\|_Y \leq \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}} \rho_0\right)^{(p+1)^n}$$

and the condition $\alpha^{\frac{1}{p}} \rho_0 < 1$ we deduce that:

$$\lim_{n\to\infty} \|f(x_n)\|_Y = 0,$$

from where:

$$f(x^*) = \theta_Y,$$

so x^* is a solution of the equation (1).

The inequalities:

$$\|x_{n+m} - x_n\|_X \leq Mb\,\|f(x_n)\|_Y\,, \quad \|f(x_{n+1})\|_Y \leq \alpha\,\|f(x_n)\|_Y^{p+1}\,,$$

show that the sequence $(x_n)_{n\in N}$ is a approximant sequence of the order $p+1$ for the solution x^*.

Form the inequality **c**) from (16) together with (17) we deduce the inequality (14).

In this way the theorem is proven.

3. **Special cases**

Now we will see how **Theorem 2.1** is applied in the case of particular proceedings of approximation.

Let us first suppose that the function $f : D \to Y$ admits for any $x \in D$ a Fréchet derivative of the first order, an $L > 0$ exists so that:

$$\|f'(x) - f'(y)\| \leq L\,\|x-y\|_X$$

for any $x, y \in D$, and the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ verifies for any $n \in \mathbb{N}$ the equality:

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y. \tag{19}$$

Obviously, if for any $n \in \mathbb{N}$, $[f'(x_n)]^{-1}$ exists, the relation (19) is equivalent to:

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n), \tag{20}$$

form under which the **Newton-Kantorovich method** is well known. But the form (20)of the relation (19) will be one of the conclusions of the statement that will be established.

Because:

$$\|f(x_n) + f'(x_n)(x_{n+1} - x_n)\|_Y = 0 \leq 0 \cdot \|f(x_n)\|_Y^2$$

and:

$$\|f'(x_n)(x_{n+1} - x_n)\|_Y = 1 \cdot \|f(x_n)\|_Y ,$$

we deduce that the inequalities (10) and (11) of the hypothesis **iii)** of **Theorem 2.1** are verified for $a = 0$ and $b = 1$. In this case:

$$p = 1,\ L_2 = L,\ h_0 = 2LB_0^2\rho_0,\ \alpha = \frac{LM^2}{2},\ M = \left\|[f'(x_0)]^{-1}\right\| \mathbf{e}^3$$

and thus the inequality of hypothesis **v**) of **Theorem 2.1** become:

$$\rho_0 < \frac{1}{4}.$$

As $\alpha\rho_0 = \dfrac{LM^2h_0}{4LB_0^2} = \dfrac{\mathbf{e}^9h_0}{4}$, we need the condition $h_0 < \dfrac{1}{\mathbf{e}^9}$ or $B_0^2\rho_0 < \dfrac{1}{2\mathbf{e}^9L}$, condition that evidently also implies $h_0 \leq \dfrac{1}{2}$.

In what the radius of the ball on which the properties take place is concerned, it verifies the inequality:

$$\delta \geq \frac{M\rho_0}{1 - \alpha\rho_0}.$$

As $\alpha\rho_0 < \dfrac{1}{4}$ we deduce that $\dfrac{1}{1-\alpha\rho_0} < \dfrac{4}{3}$ and so if $\delta \geq \dfrac{3M\rho_0}{4}$ the requirement is fulfilled. Also:

$$M = \left\|[f'(x_0)]^{-1}\right\| \mathbf{e}^3.$$

In this way we have the following:

Corollary 3.1. *We consider the same elements as in* ***Theorem** 2.1. If:*

i) *X is a Banach space, and $S(x_0, \delta) \subseteq D$;*

ii) *for any* $x \in D$*, there exists* $f'(x) \in (X,Y)^{*}$*, representing the Fréchet derivative of* f *in* x *and there exists* $L > 0$ *so that:*

$$\|f'(x) - f'(y)\| \leq L\|x - y\|_X$$

for any $x, y \in D$*;*

iii)*the sequence verifies the equality:*

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y,$$

iv) *the mapping* $f'(x_0) \in (X,Y)^{*}$ *is invertible;*

v) *the initial point* $x_0 \in D$ *verifies the inequalities:*

$$\left(\left\|[f'(x_0)]^{-1}\right\|\right)^2 \|f(x_0)\|_Y < \frac{1}{2\mathbf{e}^9 L},\quad \delta \geq \frac{3\mathbf{e}^3}{4}\left\|[f'(x_0)]^{-1}\right\| \cdot \left\|[f'(x_0)]^{-1}\right\|,$$

then:

j)$x_n \in S(x_0,\delta)$ *and* $[f'(x_n)]^{-1} \in (Y,X)^{*}$ *exists, having the relations:*

$$\left\|[f'(x_n)]^{-1}\right\| \leq \left\|[f'(x_0)]^{-1}\right\|\mathbf{e}^3$$

and:

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n)$$

for any $n \in \mathbb{N}$.

jj) *the equation* (1)*admits a solution* $x^{*} \in S(x_0,\delta)$*;*

jjj) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is a approximant sequence of the second order of the solution* x^{*} *of this equation;*

jv) *the following estimates hold:*

$$\max\left\{\|f(x_n)\|_Y,\ \frac{1}{M}\|x_{n+1} - x_n\|_X\right\} \leq \left(\frac{LM^2}{2}\right)^{2^n-1}\|f(x_n)\|_Y^{2^n},$$

$$\|x^{*} - x_n\|_X \leq \frac{M\rho_0\left(\dfrac{\rho_0 LM^2}{2}\right)^{2^n}}{1 - \left(\dfrac{\rho_0 LM^2}{2}\right)^{2^n}}$$

where $M = \left\|[f'(x_0)]^{-1}\right\|\mathbf{e}^3$ *and* $\|f(x_0)\|_Y$.

Let us now consider the case of **Chebyshev's method**. In this case $f : D \to Y$ admits, for any $x \in X$, Fréchet derivatives of the first and the second order, and in addition to the main sequence $(x_n)_{n\in N} \subseteq D$, we consider another sequence $(y_n)_{n\in\mathbb{N}} \subseteq D$ so that for any $n \in \mathbb{N}$ the following is verified:

$$\begin{cases} f'(x_n)(x_{n+1}-x_n)+f(x_n)+\dfrac{1}{2}f''(x_n)y_n^2, \\ f'(x_n)y_n+f(x_n)=\theta_Y \end{cases} \tag{21}$$

If for any $n \in N$, $[f'(x_n)]^{-1}$ exists, we can deduce from the relation (21) that:

$$x_{n+1}=x_n-\left[f'\left(x_n\right)\right]^{-1}f\left(x_n\right)-\frac{1}{2}[f'(x_n)]^{-1}f''(x_n)\{[f'(x_n)]^{-1}f(x_n)\}^2 \tag{22}$$

the form under which **Chebychev's method** is known. We will show that in this case the conditions of **Theorem 2.1** will be verified for $p=2$.

So we will have:

Theorem 3.2. *We consider the same data as in* ***theorem 2.1****. If:*

i) *X is a Banach space and $S(x_0,\delta) \subseteq D$, $S(x_0,\delta)$ representing the ball with the centre x_0 and the radius δ;*

ii) *the function admits Fréchet derivatives up to the second order included, and for $f'' : D \to (X^2,Y)^*$, the number $L>0$ exists, so that for any $x,y \in D$ the following inequality is verified:*

$$\left\|f''\left(x\right)-f''\left(y\right)\right\| \leq L\left\|x-y\right\|_X; \tag{23}$$

iii) *the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$, together with an auxiliary sequence $(y_n)_{n\in\mathbb{N}} \subseteq D$, verifies the relations (21) for any $n \in \mathbb{N}$;*

iv) *the mapping $f'\left(x_0\right) \in \left(X,Y\right)^*$ is invertible;*

v) *if we note:*

$$\begin{gathered} \rho_0=\left\|f\left(x_0\right)\right\|_Y,\ B_0=\left\|\left[f'\left(x_0\right)\right]^{-1}\right\|,\ M=B_0\mathbf{e}^{\frac{3}{2}},\ b=\frac{L_2M^2\rho_0}{2}, \\ a=(b+1)\frac{\left(L_2M^2\right)^2}{2},\ \alpha=a+L\frac{\left(bM\right)^3}{6}; \end{gathered} \tag{24}$$

the following inequalities are verified:

$$\alpha^{\frac{1}{2}}\rho_0<\frac{1}{4},\ \ \frac{bM\rho_0}{1-\alpha\rho_0^2} \leq \delta \leq \frac{1}{L}\left(\frac{1}{2bB_0^2\rho_0}-\left\|f''\left(x_0\right)\right\|\right); \tag{25}$$

then:

j) *$x_n \in S(x_0,\delta)$, the mapping $[f'(x_n)]^{-1} \in (Y,X)^*$ exists, we have the inequality $\left\|[f'(x_n)]^{-1}\right\| \leq M$ for any $n\in\mathbb{N}$ and the sequence $(x_n)_{n\in\mathbb{N}}$ is generated by the relation of recurrence* (21) *or* (22) *is convergent;*

jj) *the equation* (1) *are the solution $x^* \in S(x_0,\delta)$;*

jjj) *the sequence$(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the third order of this solution of the equation* (1);

jv) *the following estimates hold:*

$$\max\left\{\|f(x_n)\|_Y,\ \frac{1}{Mb}\|x_{n+1}-x_n\|_X\right\} \leq \alpha^{\frac{3^n-1}{2}}\|f(x_0)\|_Y^{3^n}, \tag{26}$$

and:

$$\|x^*-x_n\|_X \leq Mb\frac{\alpha^{\frac{3^n-1}{2}}\|f(x_0)\|_Y^{3^n}}{1-\left(\alpha\|f(x_0)\|_Y^2\right)^{3^n}}, \tag{27}$$

for any $n\in\mathbb{N}$.

Proof. From the condition:

$$\delta \leq \frac{1}{L}\left(\frac{1}{2bB_0^2\rho_0} - \|f''(x_0)\|\right),$$

if we keep in mind that $L_2 = \| f''(x_0) \| + L\delta$, we deduce that:

$$h_0 = bL_2B_0^2\rho_0 \leq \frac{1}{2}.$$

We will introduce the same sequences as in the proof of **theorem 2.1.** We will show that for any $n\in\mathbb{N}$ the following properties are verified:

a) $x_n \in S(x_0,\delta)$;

b) $[f'(x_n)]^{-1} \in (Y,X)^*$ exists and $\left\|[f'(x_n)]^{-1}\right\| \leq B_n$;

c) $\|f(x_n)\|_Y \leq \rho_n \leq \dfrac{(\sqrt{\alpha}\rho_0)^{3^n}}{\sqrt{\alpha}}$;

d) $h_n \leq \min\left\{\dfrac{1}{2}, \dfrac{(\beta h_0)^{3^n}}{\sqrt{\beta}}\right\}$, where $\beta = \dfrac{1}{4h_0^2}$;

e) $B_n \leq B_0 \leq M$;

f) $\|f'(x_n)(x_{n+1}-x_n)\|_Y \leq b\|f(x_n)\|_Y$;

g) $\left\|f(x_n)+f'(x_n)(x_{n+1}-x_n)+\dfrac{1}{2}f''(x_n)(x_{n+1}-x_n)^2\right\|_Y \leq a\|f(x_n)\|_Y^3$.

To start with , let us suppose that the properties **a)-e)** are true for a certain number $n \in \mathbb{N}$. We will show that, for that number, the properties **f)** and **g**) are also verified.

Indeed, we first notice that from $x_n \in S(x_0, \delta)$ we deduce that:

$$\|f''(x_n)\| \leq L_2.$$

Then it is obvious that:

$$\|y_n\|_X \leq \left\|[f'(x_n)]^{-1}\right\| \cdot \|f(x_n)\|_Y$$

and:

$$\|x_{n+1} - x_n - y_n\|_X = \left\|[f'(x_n)]^{-1}[f'(x_n)(x_{n+1} - x_n) - f'(x_n)y_n]\right\|_X \leq$$

$$\leq \left\|[f'(x_n)]^{-1}\right\|_Y \cdot \left\|-f(x_n) - \frac{1}{2}f''(x_n)y_n^2 + f(x_n)\right\|_Y \leq \frac{1}{2}B_n L_2 \|y_n\|_X^2 \leq$$

$$\leq \frac{1}{2}M^3 L_2 \|f(x_n)\|_Y^2 .$$

So:

$$\|f'(x_n)(x_{n+1} - x_n)\|_Y = \left\|-f(x_n) - \frac{1}{2}f''(x_n)y_n^2\right\|_Y \leq$$

$$\leq \left(1 + \frac{1}{2}M^2 L_2 \|f(x_n)\|_Y\right)\|f(x_n)\|_Y .$$

As $\sqrt{\alpha}\rho_0 < 1$ we deduce that:

$$(\sqrt{\alpha}\rho_0)^{3^n} \leq \sqrt{\alpha}\rho_0$$

and:

$$\|f(x_n)\|_Y \leq \rho_n \leq \frac{(\sqrt{\alpha}\rho_0)^{3^n}}{\sqrt{\alpha}} \leq \frac{\sqrt{\alpha}\rho_0}{\sqrt{\alpha}} = \rho_0$$

and thus:

$$\|f'(x_n)(x_{n+1} - x_n)\|_Y \leq \left(1 + \frac{1}{2}M^2 L_2 \rho_0\right)\|f(x_n)\|_Y = b\|f(x_n)\|_Y .$$

But, from the symmetry of $f''(x) \in \mathcal{L}_2(X, Y)$ for any $x \in D$, we have:

$$f''(x_n)(x_{n+1} - x_n)^2 - f''(x_n)y_n^2 = f''(x_n)(x_{n+1} - x_n)^2 -$$

$$-f''(x_n)(x_{n+1} - x_n, y_n) + f''(x_n)(y_n, x_{n+1} - x_n) - f''(x_n)y_n^2 =$$

$$= f''(x_n)(x_{n+1} - x_n, x_{n+1} - x_n - y_n) + f''(x_n)(y_n, x_{n+1} - x_n - y_n) =$$

$$= [f''(x_n)(x_{n+1} - x_n) + f''(x_n)y_n](x_{n+1} - x_n - y_n),$$

then it is obvious that:

$$\left\| f\left(x_n\right)+f'\left(x_n\right)\left(x_{n+1}-x_n\right)+\frac{1}{2}f''\left(x_n\right)\left(x_{n+1}-x_n\right)^2\right\|_Y=$$

$$=\frac{1}{2}\left\| f''\left(x_n\right)\left(x_{n+1}-x_n\right)^2-f''\left(x_n\right)y_n^2\right\|_Y\leq$$

$$\leq\frac{1}{2}\left[\left\| f''\left(x_n\right)\left(x_{n+1}-x_n\right)\right\|+\left\| f''\left(x_n\right)y_n\right\|\right]\cdot\left\| x_{n+1}-x_n-y_n\right\|_X\leq$$

$$\leq\frac{1}{2}\left\| f''\left(x_n\right)\right\|\cdot\left\| x_{n+1}-x_n-y_n\right\|_X\cdot\left(\left\| x_{n+1}-x_n\right\|_X+\left\| y_n\right\|_X\right).$$

It is obvious that:

$$\left\| x_{n+1}-x_n\right\|_X=\left\|\left[f'\left(x_n\right)\right]^{-1}f'\left(x_n\right)\left(x_{n+1}-x_n\right)\right\|_X\leq Mb\left\| f\left(x_n\right)\right\|_Y,$$

so:

$$\left\| f\left(x_n\right)+f'\left(x_n\right)\left(x_{n+1}-x_n\right)+\frac{1}{2}f''\left(x_n\right)\left(x_{n+1}-x_n\right)^2\right\|_Y\leq$$

$$\leq\frac{1}{2}M^3L_2^2\left(Mb+M\right)\left\| f\left(x_n\right)\right\|_Y^3=\frac{1}{2}\left(b+1\right)M^4L_2^2\left\| f\left(x_n\right)\right\|_Y^3=a\left\| f\left(x_n\right)\right\|_Y^3.$$

So indeed **f**) and **g**) are true for the $n\in\mathbb{N}$ we considered.

The statements **a)-e)** are proven similarly to the proof of **theorem 2.1**. This entitles us to assert that the properties **a)-g)** are true for any $n\in\mathbb{N}$. Also, the properties **f)** and **g)** , together with the hypothesis show that impossible to apply **theorem 2.1** with $p=2$. Using this theorem, we deduce that the conclusions of the theorem to be proved are true.

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