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A MAXIMUM PRINCIPLE FOR A MULTIOBJECTIVE OPTIMAL CONTROL PROBLEM

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Rezumat. Un principiu de maxim pentru o problemă vectorială de control optimal. Ca aplicaţie a unei reguli abstracte a multiplicatorilor s-a stabilit în lucrarea [1] un principiu de maxim pentru o problemă vectorială de control optimal guvernată de o ecuaţie integrală de tip Fredholm. Pentru a nu mări excesiv lungimea lucrării [1], demonstraţia acestui principiu a fost acolo doar schiţată. În prezenta lucrare se dă acum demonstraţia completă.

1. **Introduction**

In the paper [1] we have established multiplier rules for so-called weak dynamic multiobjective optimization problems by using a suitable generalization of the derived sets introduced by M. R. Hestenes [2], [3], [4] for scalar optimization problems. Also in that paper we have used the obtained multiplier rules to state necessary conditions for the local solutions of an abstract multiobjective optimal control problem. Furthermore, we have noticed that these very general optimality conditions can yield a maximum principle for a multiobjective optimal control problem governed by an integral equation of Fredholm type (Theorem 5.1 in [1]). But, in order to avoid an excessive length of the paper, in [1] we have limited ourselves only to a sketch of this application. The goal of the present paper is to give the complete proof of this specific maximum principle.

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2. Notations

Throughout this paper, N is the set of all positive integers, R is the set of all real numbers, and for every $m \in N$, R^m is the usual m-dimensional Euclidean space of all m-tuples $v = (v_1, \ldots, v_m)$ of real numbers. The subset of R^m, consisting of all vectors $v = (v_1, \ldots, v_m)$ with $v_j \geq 0$ for each $j \in \{1, \ldots, m\}$, is denoted by R^m_+. The inner product of two vectors $v, w \in R^m$ is denoted by $\langle v, w \rangle$. If $v \in R^m$, then $\|v\|$ marks its Euclidean norm. Given any number $r > 0$, we put

$$B^m_+(r) = \big\{ v \in R^m_+ \;\big|\; \|v\| \leq r \big\}.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces over the same field, then $(\mathcal{X}, \mathcal{Y})^*$ denotes the normed linear space of all continuous linear mappings $A : \mathcal{X} \to \mathcal{Y}$. Given a point x_0 in a normed linear space and a number $r > 0$, we denote by $B(x_0, r)$ the closed ball centered at x_0 with radius r.

If M is a subset of a normed linear space, then int M designates the interior of M and cl M the closure of M.

Finally, we mention some notations regarding functions. The Fréchet derivative of a function f of a single variable is denoted by df, while the partial Fréchet derivative with respect to the nth variable of a function f of several variables is denoted by $d_n f$. If x is a point in a linear space $\mathcal{X}$ and A is a linear mapping from $\mathcal{X}$ into another linear space, then Ax denotes the value of A at x.

3. A Necessary Optimality Condition

Let $\mathcal{X}$ be a Banach space, which does not reduce to its zero-vector, let X be a nonempty open subset of $\mathcal{X}$, let U be a nonempty set, let m_1, m_2 and m_3 be positive integers, and let

$$f_1 : X \times U \to R^{m_1}, \quad f_2 : X \times U \to R^{m_2}, \quad f_3 : X \times U \to R^{m_3}$$

be vector-valued functions which are Fréchet differentiable at each point (x, u) in $X \times U$ with respect to the first variable. Further, let K_1, K_2 and K_3 be convex cones in the spaces R^{m_1}, R^{m_2} and R^{m_3}, respectively, satisfying the following assumptions:

$$\text{int } K_1 \neq \emptyset, \ \text{int } K_2 \neq \emptyset, \ K_2 \text{ and } K_3 \text{ are closed.} \tag{1}$$

For each $i \in \{1, 2, 3\}$, we define by

$$K_i^* = \{w \in R^{m_i} \,|\, \forall\, v \in K_i : \ \langle v, w\rangle \geq 0\}$$

the dual cone of K_i.

Let $F : X \times U \to \mathcal{X}$ be a function which is Fréchet differentiable at each point $(x, u) \in X \times U$ with respect to the first variable, and let S be the set defined by

$$S = \{(x, u) \in X \times U \mid F(x, u) = 0,\ f_2(x, u) \in K_2,\ f_3(x, u) \in K_3\}.$$

A point $(x_0, u_0) \in \mathcal{X} \times U$ is said to be a:

(i) *weakly K_1-maximal point* of f_1 over S if $(x_0, u_0) \in S$ and

$$[f_1(x_0, u_0) + \text{int } K_1] \cap f_1(S) = \emptyset;$$

(ii) *local weakly K_1-maximal point* of f_1 over S if $(x_0, u_0) \in S$ and if there is a neighbourhood V of x_0 such that

$$[f_1(x_0, u_0) + \text{int } K_1] \cap f_1(S \cap (V \times U)) = \emptyset.$$

The problem of finding the weakly K_1-maximal points of f_1 over S is called a *weak multiobjective optimal control problem* and is expressed in short as

(CP) $\quad f_1(x, u) \ \longrightarrow_{K_1}\ $ max weakly

subject to $(x, u) \in X \times U,\ F(x, u) = 0,\ f_2(x, u) \in K_2,\ f_3(x, u) \in K_3$.

The introduction of problem (CP) allows one to call the weakly K_1-maximal points of f_1 over S *solutions* to problem (CP). By analogy, the local weakly K_1-maximal points of f_1 over S can be named *local solutions* to problem (CP).

As an application of multiplier rules stated for arbitrary weak dynamic multiobjective optimization problems, in Section 4 of the paper [1] we have derived necessary optimality conditions for the local solutions to problem (CP). One of the theorems given there will be recalled here. In order to formulate shorter this theorem, we put $m = m_1 + m_2 + m_3$ and conceive the corresponding space R^m as the product space $R^{m_1} \times R^{m_2} \times R^{m_3}$, i.e. any vector $v \in R^m$ is identified with a certain triple $(v_1, v_2, v_3) \in R^{m_1} \times R^{m_2} \times R^{m_3}$. In particular, the zero-vector in R^m

is $0 = (0_1, 0_2, 0_3)$, where 0_i $(i \in \{1,2,3\})$ is the zero-vector in R^{m_i}. Further, we consider the vector-valued function $f : X \times U \to R^m$ defined by

$$f(x,u) = (f_1(x,u), f_2(x,u), f_3(x,u)).$$

By using these notations, the following theorem is valid.

THEOREM 1 [1, Theorem 4.6]. Let $(x_0, u_0) \in \mathcal{X} \times U$ be a local solution to problem (CP) for which the operator $A = d_1F(x_0,u_0)$ is bijective, and let $D \subseteq R^m$ be a non-empty set such that, for all $n \in N$ and all n-tuples $(d^1, \dots, d^n)$ of points belonging to D, there exist a number $r_0 > 0$ and a function $\omega_2 : B_+^n(r_0) \to U$ satisfying the following conditions:

(i) $\omega_2(0) = u_0$;

(ii) for each $x \in X$, the function $t \in B_+^n(r_0) \longmapsto F(x, \omega_2(t)) \in \mathcal{X}$ is continuous on $B_+^n(r_0)$;

(iii) the function $t \in B_+^n(r_0) \longmapsto d_1F(x_0, \omega_2(t)) \in (\mathcal{X}, \mathcal{X})^*$ is continuous at 0;

(iv) $\lim_{x\to x_0} \sup \left\{\|d_1F(x,\omega_2(t)) - d_1F(x_0,\omega_2(t))\| \;\middle|\; t \in B_+^n(r_0)\right\} = 0$;

(v) for each $x \in X$, the function $t \in B_+^n(r_0) \longmapsto f(x,\omega_2(t)) \in R^m$ is continuous on $B_+^n(r_0)$;

(vi) there is a number $a > 0$ such that $B(x_0,a) \subseteq X$ and such that

$$\sup\left\{\|d_1f(x,\omega_2(t))\| \;\middle|\; x \in B(x_0,a),\ t \in B_+^n(r_0)\right\} < \infty;$$

(vii) $\sup \left\{\|F(x_0,\omega_2(t))\| \,/\, \|t\| \;\middle|\; t \in B_+^n(r_0),\ t \neq 0\right\} < \infty$;

(viii) $\sup \left\{\|d_1f(x_0,\omega_2(t)) - d_1f(x_0,u_0)\| \,/\, \|t\| \;\middle|\; t \in B_+^n(r_0),\ t \neq 0\right\} < \infty$;

(ix) $\lim_{x\to x_0} \sup \left\{\|d_1f(x,\omega_2(t)) - d_1f(x_0,\omega_2(t))\| \;\middle|\; t \in B_+^n(r_0)\right\} = 0$;

(x) $\lim_{t\to 0} \frac{1}{\|t\|}\left[f(x_0,\omega_2(t)) - f(x_0,u_0) - Pt - d_1f(x_0,u_0)\omega_0(t)\right] = 0$, where

$$Pt = t_1d^1 + \dots + t_nd^n \quad \text{for all } t = (t_1,\dots,t_n) \in R^n$$

and

$$\omega_0(t) = A^{-1}F(x_0,\omega_2(t)) \quad \text{for all } t \in B_+^n(r_0).$$

Then there exists a vector

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

such that

$$\sup\big\{\langle d_1, \lambda_1^*\rangle + \langle d_2, \lambda_2^*\rangle + \langle d_3, \lambda_3^*\rangle \;\big|\; (d_1, d_2, d_3) \in D\big\} \leq 0$$

and

$$\langle f_2(x_0, u_0), \lambda_2^*\rangle = 0.$$

hold.

4. **The Maximum Principle**

In this section we apply Theorem 1 to derive a maximum principle for a multiobjective optimal control problem governed by an integral equation of Fredholm type.

In what follows we suppose that T is a positive number, V is a non-empty subset of a real Banach space $\mathcal{V}$, and $\mathcal{W}$ is a real Banach space which does not reduce to its zero-vector. Let I denote the interval $[0, T]$, let $C(I, \mathcal{W})$ be the linear space of all continuous functions $x : I \to \mathcal{W}$ endowed with the norm

$$\|x\| \;=\; \max\;\{\|x(\tau)\| \mid \tau \in I\},$$

and let $PC(I, V)$ be the set of all piecewise continuous functions $u : I \to V$ that are continuous at 0 and continuous on the left at each point belonging to the interval $]0, T]$.

Further, let

$$\varphi_i : I \times \mathcal{W} \times \operatorname{cl} V \to R^{m_i} \quad (i \in \{1, 2, 3\})$$

be functions that are continuous, Fréchet differentiable with respect to the second variable and such that the mappings

$$d_2\varphi_i : I \times \mathcal{W} \times \operatorname{cl} V \to (\mathcal{W}, R^{m_i})^* \quad (i \in \{1, 2, 3\})$$

are continuous, and let

$$\phi : I \times I \times \mathcal{W} \times \operatorname{cl} V \to \mathcal{W}$$

be a function which is continuous, Fréchet differentiable with respect to the third variable, and for which the mapping

$$d_3\phi : I \times I \times \mathcal{W} \times \operatorname{cl} V \to (\mathcal{W}, \mathcal{W})^*$$

is continuous and has the property that the family

$$\left\{ d_3\phi(\sigma, \tau, \cdot, v) : \mathcal{W} \to (\mathcal{W}, \mathcal{W})^* \;\middle|\; (\sigma, \tau, v) \in I \times I \times V \right\}$$

is uniformly equicontinuous on each closed bounded subset of $\mathcal{W}$.

As in Section 3, let K_1, K_2 and K_3 be convex cones in the spaces R^{m_1}, R^{m_2} and R^{m_3}, respectively, satisfying the assumptions specified in (1).

The problem we will discuss in this section is:

(ECP) $$\int_0^T \varphi_1(\tau, x(\tau), u(\tau))\, d\tau \;\longrightarrow_{K_1}\; \text{max weakly}$$

subject to

$$x \in C(I, \mathcal{W}),\;\; u \in PC(I, V),$$

$$x(\sigma) \;=\; \int_0^T \phi(\sigma, \tau, x(\tau), u(\tau))\, d\tau \quad (\sigma \in I),$$

$$\int_0^T \varphi_2(\tau, x(\tau), u(\tau))\, d\tau \in K_2, \quad \int_0^T \varphi_3(\tau, x(\tau), u(\tau))\, d\tau \in K_3.$$

This problem is a special case of the problem (CP) investigated in the preceding section. To see this, it suffices to define the functions

$$f_i : C(I, \mathcal{W}) \times PC(I, V) \to R^{m_i} \quad (i \in \{1, 2, 3\})$$

by

$$f_i(x, u) = \int_0^T \varphi_i(\tau, x(\tau), u(\tau))\, d\tau \quad (i \in \{1, 2, 3\}),$$

on the one hand, and

$$F : C(I, \mathcal{W}) \times PC(I, V) \to C(I, \mathcal{W})$$

by

$$F(x,u)(\sigma) = x(\sigma) - \int_0^T \phi(\sigma,\tau,x(\tau),u(\tau))\,d\tau \quad (\sigma \in I),$$

on the other hand, as well as to take $\mathcal{X} = X = C(I,\mathcal{W})$ and $U = PC(I,V)$.

Furthermore, it should be emphasized that the functions f_i ($i \in \{1,2,3\}$) and F introduced above are Fréchet differentiable with respect to the first variable. The corresponding partial Fréchet derivatives are given by

$$d_1 f_i(x,u)y \;=\; \int_0^T d_2\varphi_i(\tau,x(\tau),u(\tau))y(\tau)\,d\tau \quad (i \in \{1,2,3\}),$$

$$(d_1F(x,u)y)(\sigma) \;=\; y(\sigma) - \int_0^T d_3\phi(\sigma,\tau,x(\tau),u(\tau))y(\tau)\,d\tau \quad (\sigma \in I),$$

for all $(x,u) \in C(I,\mathcal{W}) \times PC(I,V)$ and all $y \in C(I,\mathcal{W})$. Thus it makes sense to try to specialize Theorem 1 to problem (ECP).

To this end we define the functions

$$\varphi : I \times \mathcal{W} \times \operatorname{cl} V \to R^m \quad \text{and} \quad f : C(I,\mathcal{W}) \times PC(I,V) \to R^m$$

by

$$\begin{aligned}\varphi(\tau,w,v) &= (\varphi_1(\tau,w,v),\varphi_2(\tau,w,v),\varphi_3(\tau,w,v)),\\ f(x,u) &= (f_1(x,u),f_2(x,u),f_3(x,u)),\end{aligned}$$

respectively. Then we have

$$f(x,u) \;=\; \int_0^T \varphi(\tau,x(\tau),u(\tau))\,d\tau, \quad d_1f(x,u)y \;=\; \int_0^T d_2\varphi(\tau,x(\tau),u(\tau))y(\tau)\,d\tau$$

for all $(x,u) \in C(I,\mathcal{W}) \times PC(I,V)$ and all $y \in C(I,\mathcal{W})$.

Taking into account all these assumptions and considerations concerning the problem (ECP), we get from Theorem 1 the following result.

THEOREM 2 [1, Theorem 5.1]. Let $(x_0,u_0) \in C(I,\mathcal{W}) \times PC(I,V)$ be a local solution to problem (ECP) satisfying the following conditions:

(j) for each $y \in C(I, \mathcal{W})$ the integral equation

$$x = y + \int_0^T d_3\phi(\cdot, \tau, x_0(\tau), u_0(\tau))x(\tau)\, d\tau$$

has a unique solution $x \in C(I, \mathcal{W})$;

(jj) there is a number $a > 0$ such that

$$\sup \left\{ \|d_2\varphi(\tau, x(\tau), v)\| \;\middle|\; (\tau, x, v) \in I \times C(I, \mathcal{W}) \times V,\ \|x - x_0\| \le a \right\} < \infty.$$

Then there exists a vector

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

such that

$$\max \{H(\tau, v) \mid v \in V\} = H(\tau, u_0(\tau)) \ \text{ for all } \tau \in I_0 \tag{2}$$

and

$$\langle \int_0^T \varphi_2(\tau, x_0(\tau), u_0(\tau))\, d\tau, \lambda_2^* \rangle = 0, \tag{3}$$

where I_0 is the set of all points $\tau \in]0, T]$ at which u_0 is continuous, $H(\tau, \cdot) : V \to R$ is the function defined by

$$H(\tau, v) = \langle \ \varphi(\tau, x_0(\tau), v) + \int_0^T d_2\varphi(\sigma, x_0(\sigma), u_0(\sigma))h(\sigma; \tau, v)\, d\sigma, \lambda^* \ \rangle,$$

and $h(\cdot\ ; \tau, v) : I \to \mathcal{W}$ denotes the solution of the variational equation

$$x = \phi(\cdot, \tau, x_0(\tau), v) + \int_0^T d_3\phi(\cdot, t, x_0(t), u_0(t))x(t)\, dt.$$

Proof. At first we notice that the operator $A = d_1F(x_0, u_0)$ is bijective because of condition (j). Next we construct a subset D of the space R^m which satisfies the hypotheses of Theorem 1. For this purpose we associate with each pair $(\tau, v) \in I_0 \times V$ the following expressions:

$$\alpha(\tau, v) = \varphi(\tau, x_0(\tau), v) - \varphi(\tau, x_0(\tau), u_0(\tau)),$$
$$\beta(\tau, v) = \phi(\cdot, \tau, x_0(\tau), v) - \phi(\cdot, \tau, x_0(\tau), u_0(\tau)),$$
$$d(\tau, v) = \alpha(\tau, v) + d_1f(x_0, u_0) \circ A^{-1}\beta(\tau, v).$$

After that we put

$$D = \{d(\tau, v) \mid (\tau, v) \in I_0 \times V\}.$$

Now, let n be any positive integer, and let $d^j = d(\tau_j, v_j)$ $(j \in \{1, \dots, n\})$ be points belonging to D. For each $j \in \{1, \dots, n\}$ we set $\alpha^j = \alpha(\tau_j, v_j)$ and $\beta^j = \beta(\tau_j, v_j)$. Then we have

$$d^j = \alpha^j + d_1 f(x_0, u_0) \circ A^{-1}\beta^j \quad \text{for all } j \in \{1, \dots, n\}.$$

Without loss of the generality we can assume that the points $d^1, \dots, d^n$ are in such a manner numbered that

$$0 < \tau_1 \le \tau_2 \le \ldots \le \tau_n \le T.$$

Put $\tau_0 = 0$. Then choose a number $r > 0$ satisfying

$$r < \tau_{j+1} - \tau_j \quad \text{whenever } j \in \{0, \dots, n-1\} \text{ and } \tau_j < \tau_{j+1} \tag{4}$$

and

$$[\tau_j - r, \tau_j] \subseteq I_0 \quad \text{for all } j \in \{1, \dots, n\}.$$

Set $r_0 = r/n$.

Next we define a function $\omega_2 : B^n_+(r_0) \to PC(I, V)$. Fix any point $t = (t_1, \dots, t_n)$ in $B^n_+(r_0)$, Then we have

$$t_1 + \ldots + t_n \le n\|t\| \le r. \tag{5}$$

For each $j \in \{1, \dots, n\}$ we denote

$$N_j = \left\{k \in N \;\middle|\; j < k \le n \text{ and } \tau_k = \tau_j\right\}$$

and

$$a_j = \begin{cases} t_j & \text{if } N_j = \emptyset \\ t_j + \sum_{k \in N_j} t_k & \text{if } N_j \ne \emptyset. \end{cases}$$

It is easily seen that (4) and (5) imply

$$0 < \tau_j - a_j \le \tau_j - a_j + t_j \le T \quad \text{for all } j \in \{1, \dots, n\}. \tag{6}$$

When $n > 1$, then we additionally have

$$\tau_j - a_j + t_j \le \tau_{j+1} - a_{j+1} \quad \text{for all } j \in \{1, \dots, n-1\}. \tag{7}$$

From (6) and (7) it follows that the intervals I_j $(j \in \{1, \ldots, n\})$, defined by

$$I_j =]\tau_j - a_j, \tau_j - a_j + t_j] \text{ for every } j \in \{1, \ldots, n\},$$

satisfy

$$I_j \subseteq I \text{ for all } j \in \{1, \ldots, n\},$$

and

$$I_j \cap I_k = \emptyset \text{ for all } j, k \in \{1, \ldots, n\}, j \neq k.$$

These properties of the intervals I_j $(j \in \{1, \ldots, n\})$ enable us to define the function $\omega_2(t) : I \to V$ by

$$\omega_2(t)(\tau) = \begin{cases} v_j & \text{if } \tau \in I_j \text{ for some } j \in \{1, \ldots, n\} \\ u_0(\tau) & \text{if } \tau \in I \setminus (I_1 \cup \ldots \cup I_n). \end{cases}$$

In view of this definition we obviously have $\omega_2(t) \in PC(I, V)$.

In what follows we prove that the number r_0 and the function ω_2 defined above satisfy the conditions (i) – (x) of Theorem 1. In the proofs of some of these conditions we shall use the compact set

$$L = [\tau_1 - r, \tau_1] \cup \ldots \cup [\tau_n - r, \tau_n],$$

which is enclosed in I_0. Besides, given any $t = (t_1, \ldots, t_n) \in B_+^n(r_0)$, we shall need the intervals

$$L_j = [\tau_j - a_j, \tau_j - a_j + t_j], \text{ where } j \in \{1, \ldots, n\}.$$

They satisfy

$$L_j \subseteq [\tau_j - r, \tau_j] \subseteq L \text{ for all } j \in \{1, \ldots, n\}.$$

Indeed, let j be any index in $\{1, \ldots, n\}$. Since $t_j \leq a_j$, we have $L_j \subseteq [\tau_j - a_j, \tau_j]$. On the other hand, the inequality $a_j \leq t_1 + \ldots + t_n$ holds. Consequently, (5) implies $a_j \leq r$, whence $[\tau_j - a_j, \tau_j] \subseteq [\tau_j - r, \tau_j]$. Thus we have $L_j \subseteq [\tau_j - r, \tau_j] \subseteq L$, as claimed.

Now we consecutively prove that the conditions (i) – (x) occurring in Theorem 1 are satisfied.

Condition (i): If $t = 0$, then $I_j = \emptyset$ for every $j \in \{1, \ldots, n\}$. Thus we have $\omega_2(0) = u_0$.

Condition (ii): We fix a function $x \in C(I, \mathcal{W})$. Since the functions

$$(\sigma, \tau) \in I \times L \longmapsto \phi(\sigma, \tau, x(\tau), v_j) \in \mathcal{W} \quad (j \in \{1, \ldots, n\})$$

and

$$(\sigma, \tau) \in I \times L \longmapsto \phi(\sigma, \tau, x(\tau), u_0(\tau)) \in \mathcal{W}$$

are continuous on the compact set $I \times L$, there exists a number $c > 0$ such that

$$\|\phi(\sigma, \tau, x(\tau), v_j)\| + \|\phi(\sigma, \tau, x(\tau), u_0(\tau))\| \leq c \tag{8}$$

for all $(\sigma, \tau) \in I \times L$ and all $j \in \{1, \ldots, n\}$.

Let $t^1 = (t^1_1, \ldots, t^1_n)$ and $t^2 = (t^2_1, \ldots, t^2_n)$ be points in $B^n_+(r_0)$. For every $j \in \{1, \ldots, n\}$ we put

$$L_{j1} = [\tau_j - a_{j1}, \tau_j - a_{j1} + t^1_j], \quad L_{j2} = [\tau_j - a_{j2}, \tau_j - a_{j2} + t^2_j],$$

$$M_j = \{\tau_j - (1 - \tau) a_{j1} - \tau a_{j2} \mid \tau \in [0, 1]\},$$

where a_{j1} and a_{j2} are the numbers used in the definition of the function $\omega_2(t^1)$ and $\omega_2(t^2)$, respectively. Obviously, we have

$$|a_{j1} - a_{j2}| \leq |t^1_j - t^2_j| + \sum_{k \in N_j} |t^1_k - t^2_k| \leq n \, \|t^1 - t^2\| \tag{9}$$

for every $j \in \{1, \ldots, n\}$. Fix any $\sigma \in I$. In virtue of (8) and (9) it follows that

$$\begin{aligned} \Big\| \int_{L_{j1}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau - \int_{L_{j2}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau \Big\| &\leq c(2|a_{j1} - a_{j2}| + |t^1_j - t^2_j|) \\ &\leq c(2n + 1) \, \|t^1 - t^2\| \end{aligned}$$

and

$$\Big\| \int_{M_j} \phi(\sigma, \tau, x(\tau), u_0(\tau)) \, d\tau \Big\| \leq c|a_{j1} - a_{j2}| \leq cn \, \|t^1 - t^2\|$$

for all $j \in \{1, \ldots, n\}$. Accordingly, we have

$$\Big\| \int_{\tau_{j-1}}^{\tau_j} \phi(\sigma, \tau, x(\tau), \omega_2(t^1)(\tau)) \, d\tau - \int_{\tau_{j-1}}^{\tau_j} \phi(\sigma, \tau, x(\tau), \omega_2(t^2)(\tau)) \, d\tau \Big\|$$

$$\leq \Big\| \int_{M_j} \phi(\sigma, \tau, x(\tau), u_0(\tau)) \, d\tau \Big\| + \Big\| \int_{L_{j1}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau - \int_{L_{j2}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau \Big\|$$

$$+\sum_{k\in N_j}\Big\|\int\limits_{L_{k1}}\phi(\sigma,\tau,x(\tau),v_k)\,d\tau-\int\limits_{L_{k2}}\phi(\sigma,\tau,x(\tau),v_k)\,d\tau\Big\|\le 2cn(n+1)\,\|t^1-t^2\|$$

for every $j\in\{1,\dots,n\}$ such that $\tau_{j-1}<\tau_j$. Taking into account that

$$\Big\|\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^1)(\tau))\,d\tau-\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^2)(\tau))\,d\tau\Big\|$$

$$\le\sum_{j=1}^n\Big\|\int\limits_{\tau_{j-1}}^{\tau_j}\phi(\sigma,\tau,x(\tau),\omega_2(t^1)(\tau))\,d\tau-\int\limits_{\tau_{j-1}}^{\tau_j}\phi(\sigma,\tau,x(\tau),\omega_2(t^2)(\tau))\,d\tau\Big\|,$$

we obtain

$$\Big\|\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^1)(\tau))\,d\tau-\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^2)(\tau))\,d\tau\Big\|$$
$$\le 2cn^2(n+1)\,\|t^1-t^2\|.$$

Since $\sigma\in I$ was arbitrarily chosen, this result implies

$$\|F(x,\omega_2(t^1))-F(x,\omega_2(t^2))\|\le 2cn^2(n+1)\,\|t^1-t^2\|.$$

Thus the function $t\in B_+^n(r_0)\longmapsto F(x,\omega_2(t))\in C(I,\mathcal{W})$ is continuous on $B_+^n(r_0)$.

Condition (iii): Since the functions

$$(\sigma,\tau)\in I\times L\longmapsto d_3\phi(\sigma,\tau,x_0(\tau),v_j)\in(\mathcal{W},\mathcal{W})^*\quad(j\in\{1,\dots,n\})$$

and

$$(\sigma,\tau)\in I\times L\longmapsto d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\in(\mathcal{W},\mathcal{W})^*$$

are continuous on the compact set $I\times L$, there exists a number $c>0$ such that

$$\|d_3\phi(\sigma,\tau,x_0(\tau),v_j)-d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\|\le c \tag{10}$$

for all $(\sigma,\tau)\in I\times L$ and all $j\in\{1,\dots,n\}$.

Let the number $\varepsilon>0$ be arbitrarily given. Let $t\in B_+^n(r_0)$ be such that $\|t\|<\varepsilon/(cn)$. Fix any function $y\in C(I,\mathcal{W})$ for which $\|y\|\le 1$. In virtue of (10), the expression

$$g(\sigma)=\Big\|\int\limits_0^T\big[d_3\phi(\sigma,\tau,x_0(\tau),\omega_2(t)(\tau))-d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\big]\,y(\tau)\,d\tau\Big\|$$

satisfies for all $\sigma \in I$

$$g(\sigma) \leq \sum_{j=1}^{n} \int_{L_j} \|d_3\phi(\sigma,\tau,x_0(\tau),v_j) - d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\| \cdot \|y(\tau)\| \, d\tau$$

$$\leq c(t_1 + \ldots + t_n) \leq cn \, \|t\| < \varepsilon.$$

On the other hand we have

$$\big\| \big[d_1F(x_0,\omega_2(t)) - d_1F(x_0,u_0) \big] \, y \big\| = \max \, \{ g(\sigma) \ \ | \ \sigma \in I \}.$$

Consequently, it follows that

$$\big\| \big[d_1F(x_0,\omega_2(t)) - d_1F(x_0,u_0) \big] \, y \big\| < \varepsilon.$$

Since y was arbitrarily chosen in $C(I,\mathcal{W})$ such that $\|y\| \leq 1$, we get

$$\|d_1F(x_0,\omega_2(t)) - d_1F(x_0,u_0)\| \leq \varepsilon.$$

So we have shown that the function

$$t \in B_+^n(r_0) \longmapsto d_1F(x_0,\omega_2(t)) \in (C(I,\mathcal{W}), C(I,\mathcal{W}))^*$$

is continuous at 0.

Condition (iv): Let the number $\varepsilon > 0$ be arbitrarily given. Since the family

$$\big\{ d_3\phi(\sigma,\tau,\cdot,v) : \mathcal{W} \to (\mathcal{W},\mathcal{W})^* \ \big| \ (\sigma,\tau,v) \in I \times I \times V \big\}$$

is uniformly equicontinuous on the set

$$W = \big\{ w \in \mathcal{W} \ \big| \ \|w\| \leq \|x_0\| + 1 \big\},$$

there is a number $\delta > 0$ such that

$$\|d_3\phi(\sigma,\tau,w_1,v) - d_3\phi(\sigma,\tau,w_2,v)\| < \varepsilon/T \tag{11}$$

for all $w_1, w_2 \in W$ with $\|w_1 - w_2\| < \delta$ and all $(\sigma,\tau,v) \in I \times I \times V$. Now fix any $x \in C(I,\mathcal{W})$ such that $\|x - x_0\| < \min \, \{1, \delta\}$. Then we have $x(\tau), x_0(\tau) \in W$ and $\|x(\tau) - x_0(\tau)\| < \delta$ for all $\tau \in I$. Next fix a point $t \in B_+^n(r_0)$ and, for short, denote $u = \omega_2(t)$. Then (11) implies

$$\big\| \big[d_1F(x,u) - d_1F(x_0,u) \big] \, y \big\| = \max \, \Big\{ \Big\| \int_0^T G(\sigma,\tau) y(\tau) \, d\tau \Big\| \ \Big| \ \sigma \in I \Big\}$$

$$\leq \max\left\{\int_0^T \left\|G(\sigma,\tau)\,d\tau\right\| \;\Big|\; \sigma\in I\right\}\leq\varepsilon$$

for all $y \in C(I,\mathcal{W})$ satisfying $\|y\| \leq 1$, where

$$G(\sigma,\tau) = d_3\phi(\sigma,\tau,x(\tau),u(\tau)) - d_3\phi(\sigma,\tau,x_0(\tau),u(\tau)).$$

Consequently, we have

$$\|d_1F(x,u) - d_1F(x_0,u)\| \leq \varepsilon.$$

Since t was arbitrarily chosen in $B^n_+(r_0)$, the following inequality is true:

$$\sup\left\{\|d_1F(x,\omega_2(t)) - d_1F(x_0,\omega_2(t))\| \;\Big|\; t\in B^n_+(r_0)\right\}\leq\varepsilon.$$

Thus we have

$$\lim_{x\to x_0}\ \sup\left\{\|d_1F(x,\omega_2(t)) - d_1F(x_0,\omega_2(t))\| \;\Big|\; t\in B^n_+(r_0)\right\}=0.$$

Condition (v): We fix a function $x \in C(I,\mathcal{W})$. Since the functions

$$\tau\in L\longmapsto \varphi(\tau,x(\tau),v_j)\in R^m \quad (j\in\{1,\dots,n\})$$

and

$$\tau\in L\longmapsto \varphi(\tau,x(\tau),u_0(\tau))\in R^m$$

are continuous on the compact set L, there exists a number $c > 0$ such that

$$\|\varphi(\tau,x(\tau),v_j)\| + \|\varphi(\tau,x(\tau),u_0(\tau))\| \leq c \tag{12}$$

for all $\tau \in L$ and all $j \in \{1,\dots,n\}$.

Let $t^1 = (t^1_1,\dots,t^1_n)$ and $t^2 = (t^2_1,\dots,t^2_n)$ be points in $B^n_+(r_0)$. By using the intervals L_{j1}, L_{j2} and M_j ($j \in \{1,\dots,n\}$) that we previously employed to show that condition (ii) is satisfied, it follows from (9) and (12) that

$$\left\|\int_{L_{j1}}\varphi(\tau,x(\tau),v_j)\,d\tau - \int_{L_{j2}}\varphi(\tau,x(\tau),v_j)\,d\tau\right\|$$

$$\leq c(2|a_{j1}-a_{j2}| + |t^1_j - t^2_j|)\leq c(2n+1)\,\|t^1-t^2\|$$

and that

$$\left\|\int_{M_j}\varphi(\tau,x(\tau),u_0(\tau))\,d\tau\right\|\leq c|a_{j1}-a_{j2}|\leq cn\,\|t^1-t^2\|$$

for all $j \in \{1, \ldots, n\}$. Accordingly, we have

$$\Big\| \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^1)(\tau))\, d\tau - \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^2)(\tau))\, d\tau \Big\|$$

$$\leq \Big\| \int_{M_j} \varphi(\tau, x(\tau), u_0(\tau))\, d\tau \Big\| + \Big\| \int_{L_{j1}} \varphi(\tau, x(\tau), v_j)\, d\tau - \int_{L_{j2}} \varphi(\tau, x(\tau), v_j)\, d\tau \Big\|$$

$$+ \sum_{k \in N_j} \Big\| \int_{L_{k1}} \varphi(\tau, x(\tau), v_k)\, d\tau - \int_{L_{k2}} \varphi(\tau, x(\tau), v_k)\, d\tau \Big\| \leq 2cn(n+1)\, \|t^1 - t^2\|$$

for every $j \in \{1, \ldots, n\}$ such that $\tau_{j-1} < \tau_j$. Taking into account that

$$\|f(x, \omega_2(t^1)) - f(x, \omega_2(t^2))\|$$

$$= \Big\| \int_0^T \varphi(\tau, x(\tau), \omega_2(t^1)(\tau))\, d\tau - \int_0^T \varphi(\tau, x(\tau), \omega_2(t^2)(\tau))\, d\tau \Big\|$$

$$\leq \sum_{j=1}^{n} \Big\| \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^1)(\tau))\, d\tau - \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^2)(\tau))\, d\tau \Big\|,$$

we obtain

$$\|f(x, \omega_2(t^1)) - f(x, \omega_2(t^2))\| \leq 2cn^2(n+1)\, \|t^1 - t^2\|.$$

Thus the function $t \in B_+^n(r_0) \longmapsto f(x, \omega_2(t)) \in R^m$ is continuous on $B_+^n(r_0)$.

Condition (vi): Set

$$B(x_0, a) = \big\{ x \in C(I, \mathcal{W}) \;\big|\; \|x - x_0\| \leq a \big\}$$

and

$$c = \sup \big\{ \|d_2\varphi(\tau, x(\tau), v)\| \;\big|\; \tau \in I, x \in B(x_0, a), v \in V \big\}.$$

Let x be in $B(x_0, a)$, and let t be in $B_+^n(r_0)$. Since the function $\omega_2(t)$ takes its values in V, we have

$$\|d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\, y(\tau)\| \leq \|d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\| \cdot \|y(\tau)\| \leq c\, \|y\|$$

for all $\tau \in I$ and all $y \in C(I, \mathcal{W})$. This result implies

$$\|d_1 f(x, \omega_2(t))y\| = \Big\| \int_0^T d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\, y(\tau)\, d\tau \Big\|$$

$$\leq T \sup \big\{ \|d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\, y(\tau)\| \;\big|\; \tau \in I \big\} \leq cT\, \|y\|$$

for all $y \in C(I,\mathcal{W})$. Hence, we have $\|d_1 f(x,\omega_2(t))\| \leq cT$. Since x and t were arbitrarily chosen in $B(x_0,a)$ and $B_+^n(r_0)$, respectively, it is true that

$$\sup \left\{\|d_1 f(x,\omega_2(t))\| \;\middle|\; x \in B(x_0,a), t \in B_+^n(r_0)\right\} \leq cT.$$

Condition (vii): Since the functions

$$(\sigma,\tau) \in I \times L \longmapsto \phi(\sigma,\tau,x_0(\tau),u_0(\tau)) \in \mathcal{W}$$

and

$$(\sigma,\tau) \in I \times L \longmapsto \phi(\sigma,\tau,x_0(\tau),v_j) \in \mathcal{W} \quad (j \in \{1,\ldots,n\})$$

are continuous on the compact set $I \times L$, there exists a number $c > 0$ such that

$$\|\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau,x_0(\tau),v_j)\| \leq c$$

for all $(\sigma,\tau) \in I \times L$ and all $j \in \{1,\ldots,n\}$. Then we have

$$\begin{aligned} &\|F(x_0,\omega_2(t))\| \\ &= \max \left\{\left\|\int_0^T \left[\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau,x_0(\tau),\omega_2(t)(\tau))\right] d\tau\right\| \;\middle|\; \sigma \in I\right\} \\ &\leq \max \left\{\sum_{j=1}^{n} \int_{L_j} \left\|\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau,x_0(\tau),v_j)\right\| d\tau \;\middle|\; \sigma \in I\right\} \\ &\leq c(t_1 + \ldots + t_n) \leq cn\,\|t\| \end{aligned}$$

for all $t \in B_+^n(r_0)$, and thus

$$\sup \left\{\|F(x_0,\omega_2(t))\| \;/\; \|t\| \;\middle|\; t \in B_+^n(r_0), t \neq 0\right\} \leq cn.$$

Condition (viii): Since the functions

$$\tau \in L \longmapsto d_2\varphi(\tau,x_0(\tau),v_j) \in (\mathcal{W},R^m)^* \quad (j \in \{1,\ldots,n\})$$

and

$$\tau \in L \longmapsto d_2\varphi(\tau,x_0(\tau),u_0(\tau)) \in (\mathcal{W},R^m)^*$$

are continuous on the compact set L, there exists a number $c > 0$ such that

$$\|d_2\varphi(\tau,x_0(\tau),v_j) - d_2\varphi(\tau,x_0(\tau),u_0(\tau))\| \leq c$$

for all $\tau \in L$ and all $j \in \{1, \dots, n\}$. Fix any $t \in B_+^n(r_0)$. Then we have

$$
\begin{aligned}
&\|[d_1 f(x_0, \omega_2(t)) - d_1 f(x_0, u_0)]\, y\| \\
&= \Big\| \int_0^T \big[d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\big]\, y(\tau)\, d\tau \Big\| \\
&\le \int_0^T \big\|d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\big\|\, d\tau \\
&= \sum_{j=1}^n \int_{L_j} \big\|d_2\varphi(\tau, x_0(\tau), v_j) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\big\|\, d\tau \\
&\le c(t_1 + \ldots + t_n) \le cn\,\|t\|
\end{aligned}
$$

for all $y \in C(I, \mathcal{W})$ satisfying $\|y\| \le 1$. This result implies

$$\|d_1 f(x_0, \omega_2(t)) - d_1 f(x_0, u_0)\| \le cn\,\|t\|.$$

Since t was arbitrarily chosen in $B_+^n(r_0)$, we get

$$\sup\big\{\|d_1 f(x_0, \omega_2(t)) - d_1 f(x_0, u_0)\| \,/\, \|t\| \;\big|\; t \in B_+^n(r_0),\ t \neq 0\big\} \le cn.$$

Condition (ix): Let the number $\varepsilon > 0$ be arbitrarily given. For each $(\tau, w) \in I \times \mathcal{W}$ we denote

$$g_j(\tau, w) = \|d_2\varphi(\tau, x_0(\tau) + w, v_j) - d_2\varphi(\tau, x_0(\tau), v_j)\| \quad (j \in \{1, \dots, n\}),$$

and

$$g(\tau, w) = \|d_2\varphi(\tau, x_0(\tau) + w, u_0(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\|.$$

Since the function

$$(\tau, w, v) \in I \times \mathcal{W} \times \operatorname{cl} V \longmapsto d_2\varphi(\tau, w, v) \in (\mathcal{W}, R^m)^*$$

is continuous, we can apply Lemma 2, given in [5], and conclude that

$$\lim_{s\to 0}\ \sup\big\{g_j(\tau, w) \;\big|\; \tau \in I, w \in \mathcal{W}, \|w\| \le s\big\} = 0 \text{ for each } \ j \in \{1, \dots, n\}$$

and that

$$\lim_{s\to 0}\ \sup\big\{g(\tau, w) \;\big|\; \tau \in I, w \in \mathcal{W}, \|w\| \le s\big\} = 0.$$

Consequently, there is a number $\delta > 0$ such that

$$\sum_{j=1}^{n} \sup \left\{ g_j(\tau, w) \;\middle|\; \tau \in I, w \in \mathcal{W}, \|w\| \le \delta \right\} < \varepsilon/(2T)$$

and

$$\sup \left\{ g(\tau, w) \;\middle|\; \tau \in I, w \in \mathcal{W}, \|w\| \le \delta \right\} < \varepsilon/(2T).$$

Now let $x \in C(I, \mathcal{W})$ be any function satisfying $\|x - x_0\| < \delta$. Fix any $t \in B^n_+(r_0)$. Then we have

$$\begin{aligned}
&\left\| \left[d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t)) \right] y \right\| \\
&= \left\| \int_0^T \left[d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) \right] y(\tau)\, d\tau \right\| \\
&\le \int_0^T \left\| d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) \right\| d\tau \\
&\le \sum_{j=1}^{n} \int_0^T \left\| d_2\varphi(\tau, x(\tau), v_j) - d_2\varphi(\tau, x_0(\tau), v_j) \right\| d\tau \\
&\quad + \int_0^T \left\| d_2\varphi(\tau, x(\tau), u_0(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau)) \right\| d\tau \\
&\le T \sum_{j=1}^{n} \sup \left\{ g_j(\tau, x(\tau) - x_0(\tau)) \;\middle|\; \tau \in I \right\} + T \sup \left\{ g(\tau, x(\tau) - x_0(\tau)) \;\middle|\; \tau \in I \right\} < \varepsilon
\end{aligned}$$

for all $y \in C(I, \mathcal{W})$ satisfying $\|y\| \le 1$. This result implies

$$\|d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t))\| \le \varepsilon.$$

Since t was arbitrarily chosen in $B^n_+(r_0)$, we have

$$\sup \left\{ \|d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t))\| \;\middle|\; t \in B^n_+(r_0) \right\} \le \varepsilon.$$

Consequently, it is true that

$$\lim_{x \to x_0} \sup \left\{ \|d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t))\| \;\middle|\; t \in B^n_+(r_0) \right\} = 0.$$

Condition (x): We denote

$$P_\alpha t = t_1 \alpha^1 + \ldots + t_n \alpha^n \quad \text{for all } t = (t_1, \ldots, t_n) \in B^n_+(r_0).$$

We claim that the function

$$t \in B_+^n(r_0) \longmapsto f(x_0, \omega_2(t)) - f(x_0, u_0) - P_\alpha t \in R^m$$

satisfies

$$\lim_{t \to 0} \frac{1}{\|t\|} \big[f(x_0, \omega_2(t)) - f(x_0, u_0) - P_\alpha t \big] = 0. \tag{13}$$

To prove this, let the number $\varepsilon > 0$ be arbitrarily given. Since the functions

$$\tau \in L \longmapsto \varphi(\tau, x_0(\tau), v_j) \in R^m \quad (j \in \{1, \ldots, n\})$$

and

$$\tau \in L \longmapsto \varphi(\tau, x_0(\tau), u_0(\tau)) \in R^m$$

are continuous on the compact set L, they are uniformly continuous on this set. Thus there exists a number $\delta > 0$ such that for all $j \in \{1, \ldots, n\}$ and all $\tau \in L$ satisfying $|\tau - \tau_j| < \delta$ the following inequalities hold:

$$\|\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau_j, x_0(\tau_j), v_j)\| < \varepsilon/(2n);$$

$$\|\varphi(\tau, x_0(\tau), u_0(\tau)) - \varphi(\tau_j, x_0(\tau_j), u_0(\tau_j))\| < \varepsilon/(2n).$$

These inequalities imply

$$\|\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau, x_0(\tau), u_0(\tau)) - \alpha^j\| < \varepsilon/n \tag{14}$$

for all $j \in \{1, \ldots, n\}$ and all $\tau \in L$ satisfying $|\tau - \tau_j| < \delta$.

Now let $t \in B_+^n(r_0) \setminus \{0\}$ be any point such that $\|t\| < \delta/n$. Then we have

$$\begin{aligned} &\|f(x_0, \omega_2(t)) - f(x_0, u_0) - P_\alpha t\| \\ &= \Big\| \sum_{j=1}^{n} \int\limits_{L_j} \big[\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau, x_0(\tau), u_0(\tau)) - \alpha^j\big] \, d\tau \Big\| \\ &\le t_1 A_1 + \ldots + t_n A_n, \end{aligned} \tag{15}$$

where

$$A_j = \max \big\{ \|\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau, x_0(\tau), u_0(\tau)) - \alpha^j\| \;\big|\; \tau \in L_j \big\}$$

for $j \in \{1, \ldots, n\}$. Next take into consideration that, if $\tau \in L_j$ for some $j \in \{1, \ldots, n\}$, then τ lies in L and satisfies

$$|\tau - \tau_j| \le a_j \le t_1 + \ldots + t_n \le n \, \|t\| < \delta. \tag{16}$$

Consequently, (14) implies $A_j < \varepsilon/n$ for all $j \in \{1,\dots,n\}$. In view of this result, we get from (15) that

$$\|f(x_0,\omega_2(t)) - f(x_0,u_0) - P_\alpha t\| < \varepsilon\,(t_1+\dots+t_n)/n \le \varepsilon\,\|t\|,$$

and hence

$$\Big\|\frac{1}{\|t\|}\big[f(x_0,\omega_2(t)) - f(x_0,u_0) - P_\alpha t\big]\Big\| < \varepsilon.$$

Thus (13) is true, as claimed.

Next, we denote

$$P_\beta t = t_1\beta^1 + \dots + t_n\beta^n \quad \text{for all } t=(t_1,\dots,t_n)\in B^n_+(r_0).$$

A reasoning similar to that used in the proof of (13) reveals that the function

$$t \in B^n_+(r_0) \longmapsto F(x_0,\omega_2(t)) + P_\beta t \in C(I,\mathcal{W})$$

satisfies

$$\lim\nolimits_{t\to 0} \frac{1}{\|t\|}\big[F(x_0,\omega_2(t)) + P_\beta t\big] = 0. \tag{17}$$

Indeed, let the number $\varepsilon > 0$ be arbitrarily given. Since the functions

$$(\sigma,\tau)\in I\times L \longmapsto \phi(\sigma,\tau,x_0(\tau),v_j)\in\mathcal{W} \quad (j\in\{1,\dots,n\})$$

and

$$(\sigma,\tau)\in I\times L \longmapsto \phi(\sigma,\tau,x_0(\tau),u_0(\tau))\in\mathcal{W}$$

are continuous on the compact set $I\times L$, they are uniformly continuous on this set. Thus there exists a number $\delta>0$ such that for all $j\in\{1,\dots,n\}$, all $\sigma\in I$, and all $\tau\in L$ satisfying $|\tau-\tau_j|<\delta$ the following inequalities hold:

$$\|\phi(\sigma,\tau,x_0(\tau),v_j) - \phi(\sigma,\tau_j,x_0(\tau_j),v_j)\| < \varepsilon/(2n);$$

$$\|\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau_j,x_0(\tau_j),u_0(\tau_j))\| < \varepsilon/(2n).$$

These inequalities imply

$$\|\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau,x_0(\tau),v_j) + \beta^j(\sigma)\| < \varepsilon/n \tag{18}$$

for all $j\in\{1,\dots,n\}$, all $\sigma\in I$, and all $\tau\in L$ satisfying $|\tau-\tau_j|<\delta$.

Now, let $t \in B_+^n(r_0) \setminus \{0\}$ be any point such that $\|t\| < \delta/n$. Then we have

$$\|F(x_0, \omega_2(t))(\sigma) + (P_\beta t)(\sigma)\|$$
$$= \Big\| \sum_{j=1}^{n} \int_{L_j} \big[\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), v_j) + \beta^j(\sigma)\big]\, d\tau \Big\|$$
$$\le t_1 B_1(\sigma) + \ldots + t_n B_n(\sigma) \tag{19}$$

for every $\sigma \in I$, where

$$B_j(\sigma) = \max\, \big\{ \|\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), v_j) + \beta^j(\sigma)\| \;\big|\; \tau \in L_j \big\}$$

for $j \in \{1, \ldots, n\}$. As before, now take into consideration that if $\tau \in L_j$ for some index $j \in \{1, \ldots, n\}$, then τ lies in L and satisfies (16). Consequently, (18) implies

$$B_j(\sigma) < \varepsilon/n \;\text{ for all } j \in \{1, \ldots, n\} \text{ and all } \sigma \in I.$$

In view of this result, we get from (19) that

$$\|F(x_0, \omega_2(t))(\sigma) + (P_\beta t)(\sigma)\| < \varepsilon\,(t_1 + \ldots + t_n)/n \le \varepsilon\,\|t\|$$

for all $\sigma \in I$. From this it follows that

$$\|F(x_0, \omega_2(t)) + P_\beta t\| < \varepsilon\,\|t\|,$$

and hence

$$\Big\| \frac{1}{\|t\|} \big[F(x_0, \omega_2(t)) + P_\beta t\big] \Big\| < \varepsilon.$$

Thus (17) is true, as claimed.

From (17) we obtain

$$\lim_{t \to 0} \frac{1}{\|t\|} \big[\omega_0(t) + \sum_{j=1}^{n} t_j A^{-1} \beta^j\big] = 0, \tag{20}$$

where

$$\omega_0(t) = A^{-1} F(x_0, \omega_2(t)) \;\text{ for all } t \in B_+^n(r_0).$$

Obviously, (20) yields

$$\lim_{t \to 0} \frac{1}{\|t\|} \big[d_1 f(x_0, u_0)\omega_0(t) + \sum_{j=1}^{n} t_j\, d_1 f(x_0, u_0) \circ A^{-1} \beta^j\big] = 0. \tag{21}$$

Finally, note that the point Pt defined by

$$Pt = t_1 d^1 + \ldots + t_n d^n \;\text{ for all } t = (t_1, \ldots, t_n) \in R^n,$$

in our case can be written under the form

$$Pt = P_\alpha t + \sum_{j=1}^{n} t_j \, d_1 f(x_0, u_0) \circ A^{-1} \beta^j .$$

Accordingly, we conclude from (13) and (21) that

$$\lim_{t \to 0} \frac{1}{\|t\|} \big[f(x_0, \omega_2(t)) - f(x_0, u_0) - Pt - d_1 f(x_0, u_0) \omega_0(t) \big] = 0.$$

Summing up, all the hypotheses of Theorem 1 are fulfilled. By applying this theorem, it follows that there is a vector

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

satisfying the inequality

$$\langle d(\tau, v), \lambda^* \rangle \leq 0 \quad \text{whenever } (\tau, v) \in I_0 \times V \tag{22}$$

as well as the equality (3).

From (22) we obtain (2). Indeed, to see this, we fix any $\tau \in I_0$. Since we have

$$A^{-1} \phi(\cdot, \tau, x_0(\tau), v) = h(\cdot; \tau, v) \quad \text{for all } v \in V,$$

it follows that

$$d_1 f(x_0, u_0) \circ A^{-1} \phi(\cdot, \tau, x_0(\tau), v) = \int_0^T d_2 \varphi(\sigma, x_0(\sigma), u_0(\sigma)) h(\sigma; \tau, v) \, d\sigma.$$

In view of this result, $H(\tau, \cdot)$ can be rewritten as follows:

$$H(\tau, v) = \langle \varphi(\tau, x_0(\tau), v) + d_1 f(x_0, u_0) \circ A^{-1} \phi(\cdot, \tau, x_0(\tau), v), \lambda^* \rangle$$

for every $v \in V$. Therefore we have

$$H(\tau, v) - H(\tau, u_0(\tau)) = \langle d(\tau, v), \lambda^* \rangle \quad \text{for all } v \in V.$$

In virtue of (22) it follows that

$$H(\tau, v) \leq H(\tau, u_0(\tau)) \quad \text{for all } v \in V.$$

Consequently, the equality (2) holds, which completes the proof.

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