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A NOTE ON THE DIVISIBILITY OF SOME COMPRESSION SEMIGROUPS IN $\mathrm{Sl}(2,\mathbb{R})$

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Dedicated to my father Wolfgang W. Breckner on the occasion of his 60th birthday

Abstract. We give elementary proofs (avoiding, as much as possible, any machinery of Lie theory) for the divisibility of those compression semigroups in $\mathrm{Sl}(2,\mathbb{R})^+$ who are known to be the prototypes of the three dimensional exponential Lie subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$.

Why this note has been written. The natural nonabelian analogues of cones in real vector spaces are the divisible closed subsemigroups of connected Lie groups, these are exactly the exponential Lie semigroups. In [4] K.H. Hofmann and W.A.F. Ruppert classify the reduced exponential Lie semigroups and show that these semigroups are built up from a few building blocks, the so-called *Master Examples*. In 1999 B.E. Breckner and W.A.F. Ruppert started a project devoted to the study of the topological semigroup compactifications of divisible subsemigroups of Lie groups. A first step for carrying out this project is to investigate the topological semigroup compactifications of the *Master Examples*. So, Breckner and Ruppert focused for the beginning on one of the *Master Examples*, namely the exponential Lie subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$. It has turned out, however, that for the study of the compactifications of these semigroups one needs a very detailed knowledge of general structural features of $\mathrm{Sl}(2,\mathbb{R})$ (see [1]). We remark in passing that, using the tools introduced in [1], Breckner and Ruppert offer in [2] a fairly comprehensive study of the topological semigroup compactifications of certain subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ (including the exponential ones). A main result of [1], with important consequences for the investigations

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in [2], is the determination of the conjugacy classes of exponential subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ (see 7.14 of [1]):

Let S be a three dimensional exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. Then S is conjugate to exactly one of the following semigroups:

1. $\mathrm{Sl}(2,\mathbb{R})^+$,
2. $S_1 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+b \geq c+d\}$,
3. $S^1 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d\}$,
4. $S^1_\lambda = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d$ *and* $a+\frac{1}{\lambda}b \geq \lambda c + d\}$, *for some real* $\lambda > 0$.

In [1] the exponentiality of the semigroups S_1, S^1, and S^1_λ is shown by a typical Lie theoretical argument, involving the determination of the Lie wedges of the semigroups. Nevertheless, the exponentiality of these semigroups is of interest also from a pure algebraical point of view. To see this, recall that a closed submonoid of a connected Lie group is divisible if and only if it is an exponential Lie semigroup (cf, eg, 2.7 of [4]). Thus, a problem of own interest is to prove the divisibility of the semigroups S_1, S^1, and S^1_λ by a direct, algebraical argument. The present paper offers such a proof.

Divisible semigroups. A semigroup S is called *divisible* if $\forall s \in S$, $\forall n \in \mathbb{N}^*$ $\exists x \in S$ such that $x^n = s$.

Notations. Following [3], we write $\mathrm{Sl}(2,\mathbb{R})^+$ for the semigroup of matrices with nonnegative entries in $\mathrm{Sl}(2,\mathbb{R})$. For fixed positive reals $\lambda, \mu > 0$ we define the following subsets of $\mathrm{Sl}(2,\mathbb{R})^+$:

$$S_\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \,\middle|\, a + \frac{1}{\lambda} b \geq \lambda c + d \right\},$$

$$S^\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \,\middle|\, a + \frac{1}{\lambda} c \geq \lambda b + d \right\}, \text{ and } S^\mu_\lambda = S_\lambda \cap S^\mu.$$

The main statement. *The sets S_λ, S^λ, and S^μ_λ are divisible semigroups for every* $\lambda, \mu > 0$.

Remark. The first step in the proof of the main statement is to show that S_λ, S^λ, and S_λ^μ are indeed semigroups. For this it suffices to show that S_λ is a semigroup, because S^λ is the image of S_λ under the anti-isomorphism sending every matrix to its transpose. That S_λ is a semigroup is not obvious, since it cannot be seen immediately that the product of two arbitrary elements of S_λ belongs to S_λ. So, it turned out to be very convenient to follow [1] and to represent S_λ as a compression semigroup.

Compression semigroups. Let S be a semigroup which acts on some space X. Then for every subset M of X, we define the *compression semigroup of M in S* as the set

$$\mathrm{compr}_S(M) = \{s \in S \mid sM \subseteq M\}.$$

It is obvious that $\mathrm{compr}_S(M)$ is either empty or a subsemigroup of S.

The set S_λ as a compression semigroup. (cf 6.8 of [1]) Consider the natural action of $\mathrm{Sl}(2,\mathbb{R})^+$ (as a semigroup of endomorphisms of $\mathbb{R}^2$) on $\mathbb{R}^2$ and define for a fixed real $\lambda > 0$ the cone

$$C_\lambda = \{(x,y) \in \mathbb{R}^2 \mid x \geq \lambda y \geq 0\}.$$

The reader is invited to check by a straightforward computation that S_λ is the compression semigroup of C_λ in $\mathrm{Sl}(2,\mathbb{R})^+$ (see also 6.8 of [1]).

The following notion, similar to that of a compression semigroup, will be crucial for the proof of the main statement.

Almost compression semigroups. Let S be a semigroup which acts on some space X and consider M, M' subsets of X such that $M' \subseteq M$. We define the *almost compression semigroup of the pair (M, M') in S* to be the set

$$\mathrm{alcompr}_S(M, M') = \{s \in S \mid sM \subseteq M'\}.$$

It follows readily from its definition that $\mathrm{alcompr}_S(M, M')$ is either empty or a subsemigroup of S.

We collect now some facts needed for the proof of the main statement.

Fact 1: *The semigroup* $\mathrm{Sl}(2,\mathbb{R})^+$ *is divisible.*

For those who are familiar with Lie theory this is a well-known result. It can be proved by direct calculation involving the formula for the exponential function $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{Sl}(2,\mathbb{R})$ (cf, eg, p. 416 ff. of [3]).

Fact 2: *Let S be a divisible semigroup and $(S_i)_{i\in I}$ a family of subsemigroups of S such that $S \setminus S_i$ is a semigroup for every $i \in I$. Then the intersection $\cap_{i\in I} S_i$ is either empty or a divisible semigroup.*

Proof: Put $T = \cap_{i\in I} S_i$ and choose $s \in T$ and $n \in \mathbb{N}^*$ arbitrarily. Since S is divisible there exists $x \in S$ such that $x^n = s$. Then x belongs to T. Otherwise the fact that $x \notin S_i$ for some $i \in I$ would imply that $s = x^n \in S \setminus S_i$, a contradiction. Thus T is a divisible subsemigroup of S, if it is not empty. □

Fact 3: *Let $\lambda > 0$. The set*

$$\mathrm{Sl}(2,\mathbb{R})^+ \setminus S_\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a + \frac{1}{\lambda} b < \lambda c + d \right\}$$

is a semigroup.

Proof: Put $\tilde{S}_\lambda = \mathrm{Sl}(2,\mathbb{R})^+ \setminus S_\lambda$. We prove that $\tilde{S}_\lambda$ is an almost compression semigroup. For this consider again the natural action of $\mathrm{Sl}(2,\mathbb{R})^+$ on $\mathbb{R}^2$ and define the sets

$$\tilde{C}_\lambda = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x \le \lambda y\} \setminus \{(0,0)\}, \quad \tilde{W}_\lambda = \{(x,y) \in \tilde{C}_\lambda \mid x < \lambda y\}.$$

We show that

$$(*) \qquad \tilde{S}_\lambda = \mathrm{alcompr}_{\mathrm{Sl}(2,\mathbb{R})^+}(\tilde{C}_\lambda, \tilde{W}_\lambda).$$

If $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ is such that $s\tilde{C}_\lambda \subseteq \tilde{W}_\lambda$ then $s \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \in \tilde{W}_\lambda$. Hence $a\lambda + b < \lambda(\lambda c + d)$ or, equivalently, $a + \frac{1}{\lambda} b < \lambda c + d$.

Conversely, if $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ is such that $a + \frac{1}{\lambda} b < \lambda c + d$ then we observe first that $\lambda d > b$, since multiplying the first inequality with $d > 0$ yields (note that $ad = 1 + bc$)

$$ad + \frac{1}{\lambda} bd < \lambda cd + d^2 \implies 1 + bc + \frac{1}{\lambda} bd < \lambda cd + d^2 \implies 1 < (\lambda d - b)(c + \frac{1}{\lambda} d).$$

Pick an arbitrary $(x,y) \in \tilde{C}_\lambda$. Then there exists $\alpha, \beta \in \mathbb{R}_+$ with $\alpha^2+\beta^2 \neq 0$ such that $(x,y) = \alpha(0,1)+\beta(\lambda,1)$. Now

$$s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \in \tilde{W}_\lambda \quad \text{as well as} \quad s\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} a\lambda+b \\ c\lambda+d \end{pmatrix} \in \tilde{W}_\lambda.$$

Since $\alpha^2+\beta^2 \neq 0$ we conclude that $s\begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{W}_\lambda$. This proves $(*)$, so $\tilde{S}_\lambda$ is a semigroup. □

Proof of the main statement: Fact 1, Fact 2, and Fact 3 imply that S_λ is divisible. Since the anti-isomorphism sending every matrix to its transpose maps S_λ onto S^λ, it follows that S^λ is also divisible and that $\mathrm{Sl}(2,\mathbb{R})^+ \setminus S^\lambda$ is a semigroup. Using once again Fact 2, it finally follows that $S_\lambda^\mu = S_\lambda \cap S^\mu$ is divisible. □

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