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TWO INTEGRAL OPERATORS

DANIEL BREAZ, NICOLETA BREAZ

Abstract. The aim of this work is to prove the univalence criteria for some integral operators.

1. **Introduction**

In this paper an equivalence criterion obtained by V. Pescar on integral operators, see [5], is extended to the case of more $S-$class functions.

Theorem A [2]. If the function $f(z)$ belongs to the class S then, for any complex number $\gamma, |\gamma| \leq \frac{1}{4}$ the function

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in S.

Theorem B [3]. If the function f is regular in unit disc U, $f(z) = z + a_2 z^2 + ...$ and

$$\left(1 - |z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then the function f is univalent in U.

Theorem C [1]. Let α be a complex number, $\mathrm{Re}\alpha > 0$, and $f(z) = z + a_2 z^2 + ...$ be a regular function in U. If

$$\frac{1 - |z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then for any complex number $\beta, \mathrm{Re}\,\beta \geq \mathrm{Re}\,\alpha$,the function

$$F_\beta(z) = \left[\int_0^z t^{\beta-1} f'(t)\,dt\right]^{\frac{1}{\beta}}$$

is in the class S.

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Theorem D [6]. If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| \tag{1}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

the equalities hold in case $g(z) = \varepsilon \frac{z+u}{1+\overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark E [7]. For $z = 0$, from inequality (1) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|}$$

for all $z \in U$.

Theorem F [5]. Let $\gamma \in C, f \in S, f(z) = z + a_2 z^2 + \ldots$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, (\forall)\, z \in U$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right]}$$

then

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt \in S$$

Theorem G [5]. Let $\alpha, \beta, \gamma \in C, f \in S, f(z) = z + a_2 z^2 + \ldots$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, (\forall)\, z \in U$$

$$Re\gamma \geq \operatorname{Re}\delta > 0$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right]}$$

then

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\gamma} dt\right]^{\frac{1}{\beta}} \in S$$

2. Main results

Theorem 1. Let $\alpha_n \in C, f_n \in S, f_n(z) = z + a_2^n z^2 + ..., n \in N^*$.

If

$$\left|\frac{zf_n'(z) - f_n(z)}{zf_n(z)}\right| \leq 1, (\forall)\, n \in N^*, (\forall)\, z \in U \tag{2}$$

$$\frac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} < 1 \tag{3}$$

$$|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|}\right]} \tag{4}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_n a_2^n\right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|}$$

then

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \in S$$

Proof. We have $f_n \in S, (\forall)\, n \in N^*$ and $\frac{f_n(z)}{z} \neq 0, (\forall)\, n \in N^*$.

For $z = 0$ we have $\left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n} = 1$.

Consider the function

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \frac{F''(z)}{F'(z)}$$

The function $h(z)$ has the form:

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot \frac{zf_1'(z) - f_1(z)}{zf_1(z)} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot \frac{zf_n'(z) - f_n(z)}{zf_n(z)}$$

We have:

$$h(0) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot a_2^1 + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot a_2^n$$

By using the relations (2) and (3) we obtain

$$|h(z)| < 1$$

and

$$|h(0)| = \frac{\left|\alpha_1 \cdot a_2^1 + ... + \alpha_n \cdot a_2^n\right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} = |c|$$

Applying Remark E for the function h we obtain

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \left|\frac{F''(z)}{F'(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}\ (\forall)\, z \in U \Leftrightarrow$$

$$\Leftrightarrow \left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''(z)}{F'(z)}\right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|}, (\forall)\, z \in U \quad (5)$$

Let′s consider the function $H : [0,1] \to R$

$$H(x) = \left(1 - x^2\right) x \frac{x + |c|}{1 + |c|\,x}; x = |z|.$$

$$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + |c|}{2 + |c|} > 0 \Rightarrow \max_{x \in [0,1]} H(x) > 0$$

Using this result and the form (5) we have:

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''(z)}{F'(z)}\right| \leq$$

$$\leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \max_{|z|<1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|}\right], (\forall)\, z \in U \quad (6)$$

Applying the condition (4) in the form (6) we obtain:

$$\left(1 - |z|^2\right) \left|\frac{zF''(z)}{F'(z)}\right| \leq 1, (\forall)\, z \in U,$$

and from Theorem B $F \in S$.

Corollary 2. Let $\alpha, \beta \in C, f, g \in S, f(z) = z + a_2 z^2 + ..., g(z) = z + b_2 z^2 + ...,$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, (\forall)\, z \in U$$

$$\left|\frac{zg'(z) - g(z)}{zg(z)}\right| \leq 1, (\forall)\, z \in U$$

$$\frac{1}{\alpha} + \frac{1}{\beta} < 1$$

$$|\alpha\beta| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|}\right]}$$

where

$$|c| = \frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}$$

then

$$F_{\alpha\beta}(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} \cdot \left(\frac{g(t)}{t}\right)^{\beta} dt \in S$$

Proof. In Theorem 1, we consider $n = 2, f_1 = f, f_2 = g, \alpha_1 = \alpha, \alpha_2 = \beta$.

Remark. If in Theorem 1, we consider $n = 1, f_1 = f, \alpha_1 = \gamma$, we obtained Theorem F.

Theorem 3. Let $\alpha_n, \gamma, \delta \in C, f_n \in S, f_n(z) = z + a_2^n z^2 + ..., n \in N^*$.

If

$$\left|\frac{zf_n'(z) - f_n(z)}{zf_n(z)}\right| \leq 1, (\forall)\, n \in N^*, (\forall)\, z \in U \tag{7}$$

$$\frac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} < 1 \tag{8}$$

$$Re\gamma \geq \operatorname{Re}\delta > 0$$

$$|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|}\right]} \tag{9}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_n a_2^n\right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|}$$

then

$$G(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt\right]^{\frac{1}{\gamma}} \in S$$

Proof. We consider the function

$$h(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n}$$

$$p(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \frac{h''(z)}{h'(z)}$$

$$p(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot \frac{zf_1'(z) - f_1(z)}{zf_1(z)} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot \frac{zf_n'(z) - f_n(z)}{zf_n(z)}$$

By using the relations (7) and (8) we obtain

$$|p(z)| < 1$$

and

$$|p(0)| = \frac{\left|\alpha_1 \cdot a_2^1 + ... + \alpha_n \cdot a_2^n\right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} = |c|$$

Applying Remark E for the function p we obtain

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \left|\frac{h''(z)}{h'(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}\, (\forall)\, z \in U \Leftrightarrow$$

$$\Leftrightarrow \left|\frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot z \cdot \frac{h''(z)}{h'(z)}\right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|}, (\forall)\, z \in U \tag{10}$$

Let′s consider the function $Q : [0,1] \to R$

$$Q(x) = \frac{1 - x^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} x \frac{x + |a_2|}{1 + |a_2| x}; x = |z|.$$

$$Q\left(\frac{1}{2}\right) > 0 \Rightarrow \max_{x\in[0,1]} Q(x) > 0$$

Using this result and the relation (10) we have:

$$\frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \left| \frac{zh''(z)}{h'(z)} \right| \leq$$

$$\leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \max_{|z|<1} \left[\frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right], (\forall)\, z \in U \qquad (11)$$

Applying the condition (9) in the relation (11) we obtain:

$$\left(1 - |z|^2\right) \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, (\forall)\, z \in U,$$

and from Theorem C, $G \in S$.

Remark. If we consider $\gamma = 1, \operatorname{Re}\delta = 1$ we obtain Theorem 1.

Corollary 4. Let $\alpha, \beta, \gamma, \delta \in C, f, g \in S, f(z) = z + a_2 z^2 + ..., g(z) = z + b_2 z^2 + ...,$.

If

$$\left| \frac{zf'(z) - f(z)}{zf(z)} \right| \leq 1, (\forall)\, z \in U$$

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| \leq 1, (\forall)\, z \in U$$

$$Re\gamma \geq \operatorname{Re}\delta > 0$$

$$\frac{1}{\alpha} + \frac{1}{\beta} < 1$$

$$|\alpha\beta| \leq \frac{1}{\max\limits_{|z|\leq 1} \left[\frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right]}$$

where

$$|c| = \frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}$$

then

$$G_{\alpha\beta,\gamma}(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(\frac{f(t)}{t}\right)^{\alpha} \cdot \left(\frac{g(t)}{t}\right)^{\beta} dt \right]^{\frac{1}{\gamma}} \in S$$

Proof. In Theorem 3, we consider $n = 2, f_1 = f, f_2 = g, \alpha_1 = \alpha, \alpha_2 = \beta$.

Remark. If in Theorem 3, we consider $n = 1, f_1 = f, \gamma = \beta$, we obtained Theorem F.

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1 Decembrie 1918 University of Alba Iulia, Departament of Mathematics