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ON THE EIGENVALUE PROBLEM FOR A GENERALIZED HEMIVARIATIONAL INEQUALITY

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Abstract. In this paper the eigenvalue problem for a generalized hemivariational inequality is studied. Some general existence results are obtained. Applications from Engineering illustrate the theory.

1. **Introduction**

The mathematical theory of hemivariational inequalities and their applications in Mechanics, Engineering or Economics, were introduced and developed by P.D. Panagiotopoulos ([38], [39], [40], [41], [42], [44]). This theory has been developed in order to fill the gap existing in the variational formulations of boundary value problems (B. V. P.s) when nonsmooth and generally nonconvex energy functions are involved in the formulations of the problem. In fact, this theory of hemivariational inequalities may be considered as an extension of the theory of variational inequalities ([16], [23], [27], [26]). For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs ([39], [44], [36], [35]).

Until now many hemivariational inequalities have been formulated and studied ([36], [37], [43], [39], [14], [2], [17], [45], [35], [48], [31], [19], [3], [29], [30], [15], [1], [28], [18], [8]), and eigenvalue problems for hemivariational inequalities have been presented ([22], [33], [34], [20], [46], [7], [10], [6], [21]).

The study of eigenvalue problems for hemivariational inequalities has a deep practical motivation. For instance, the loading-unloading problems and thus also the hysteresis problems are typical examples for the theory of hemivariational inequalities and can be reduced to the study of the eigenvalue problem. Indeed, D. Motreanu

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and P. D. Panagiotopoulos ([44], [32]) proved that the global behaviour of a loading-unloading problem of a deformable body is governed by a sequence of hemivariational inequality expressions, one for each branch. They proved that the changing of branch leads to an eigenvalue problem. The stability of a Von Karman plate in adhesive contact with a rigid support or of Von Karman plates adhesively connected in sandwich form is another motivation for the study of eigenvalue problems for hemivariational inequalities ([24], [25]). Recent papers deal with eigenvalue hemivariational inequalities on a sphere-like type manifold ([6], [7]), with nonsymmetric perturbed eigenvalue hemivariational inequalities ([10], [46]), which imply applications in adhesively connected plates, etc.

In this paper we deal with a type of eigenvalue problem for a hemivariational inequality governed by two variable operators. The hemivariational inequality, which gave rise to the problem studied here, was introduced in [12], [11] as an extension to several hemivariational-variational problems. The aim of the present paper is to provide general existence results of the solutions on real Banach spaces and real reflexive Banach spaces. Finally, we illustrate our theoretical results by an application to Engineering.

2. The abstract framework

We assume that the following statements are valid:

(H1) V is a real Banach space endowed with the norm topology, and V^* is its dual endowed with the weak*-topology. Throughout the paper the duality pairing between a Banach space and its dual is denoted by $< .,. >$;

(H2) $T : V \to L^p\left(\Omega, \Re^k\right)$ is a linear and continuous operator, where $1 \leq p < \infty, k \geq 1$ and $\Omega \subseteq \Re^n$ is a bounded open set in n-dimensional Euclidean space;

(H3) $A : V \times V \rightsquigarrow V^*$ is a set-valued mapping;

The properties of the set-valued mapping A will be given later.

(H4) $j = j\left(x, y\right) : \Omega \times \Re^k \to \Re$ is a Caratheodory function, which is locally Lipschitz with respect to the second variable and satisfies the following assumption:

$$\exists h_1 \in L^{\frac{p}{p-1}}\left(\Omega, \Re\right) \text{ and } h_2 \in L^{\infty}\left(\Omega, \Re\right)$$

such that

$$|z| \leq h_1(x) + h_2(x)|y|^{p-1} \text{ a.e. } x \in \Omega, \forall y \in \Re^k, \forall z \in \partial j(x,y)$$

where,

$$j^0(x,y)(h) = \lim \sup_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y' + th) - j(x, y')}{t}$$

is the (partial) Clarke derivative of the locally Lipschitz mapping $j(x,.)$, $x \in \Omega$ fixed, at the point $y \in \Re^k$ with respect to the direction $h \in \Re^k$, and

$$\partial j(x,y) = \left\{ z \in \Re^k : \langle z, h \rangle \leq j^0(x,y)(h) \quad , \forall h \in \Re^k \right\}$$

is the Clarke generalized gradient of the mapping $j(x,.)$ at the point $y \in \Re^k$.

We recall some basic concepts, which are needed to formulate the problem under consideration.

Definition 1. *We say that the set-valued mapping $A : V \rightsquigarrow V^*$* ***is monotone*** *if it satisfies the relation*

$$\langle f - g, u - v \rangle \geq 0 \quad , \forall u, v \in V, \forall f \in A(u), \forall g \in A(v).$$

Definition 2. *We say that the set-valued mapping $A(.,v) : V \rightsquigarrow V^*$, where $v \in V$ fixed,* ***has the monotone property (M)*** *if it verifies the relation*

$$\sup_{f \in A(u,v)} \langle f, u - v \rangle \geq \sup_{g \in A(v,v)} \langle g, u - v \rangle \quad , \forall u \in V. \tag{M}$$

Remark 1. *Every set-valued mapping $A(.,v) : V \rightsquigarrow V^*$ (where $v \in V$ is fixed) which is monotone has the monotone property (M), but the inverse is not always true.*

Definition 3. *The set-valued mapping $A : V \rightsquigarrow V^*$ is said to be* ***concave*** *if*

$$(1-\alpha) A(x_1) + \alpha A(x_2) \supseteq A((1-\alpha) x_1 + \alpha x_2) \quad , \forall \alpha \in [0,1], \forall x_1, x_2 \in V.$$

Definition 4. *The set-valued mapping $\Im : V \rightsquigarrow V^*$ defined by*

$$\Im u := \left\{ f \in V^* : \|f\| = \|u\|, \langle f, u \rangle = \|u\|^2 \right\}, \forall u \in V$$

is called the ***duality map of V.***

The duality map has the following representation:

Proposition 1. *(see [4]) For every $u \in V$, $\Im u = \partial \left(\frac{1}{2} \|u\|^2 \right)$.*

Because the Banach space V is endowed with the norm topology and its dual V^* is endowed with the weak*-topology then, according to [35], [9], [5], [13], we can state some properties of the duality map:

Theorem 2. *Duality map $\Im$ has the following properties:*

(i) for every $u \in V$, the set $\Im u$ is convex and

for every $\lambda \in \Re$, for every $u \in V$, $\Im(\lambda u) = \lambda \Im(u)$;

(ii) the set $\Im(u)$ is weakly-compact, for every $u \in V$;*

(iii) the duality map $\Im$ is weakly-upper semicontinuous.*

The duality map $\Im$ is successfully involved in the representation of the semi-inner products.

The **semi-inner products** $(.,.)_{\pm} : V \times V \to \Re$ are defined (according to [13]) by

$$\begin{aligned} (x,y)_+ &= \|y\| \lim_{t \to 0^+} \frac{\|y+tx\| - \|y\|}{t} \\ (x,y)_- &= \|y\| \lim_{t \to 0^+} \frac{\|y\| - \|y-tx\|}{t}. \end{aligned}$$

Remark 2. *If V is a Hilbert space endowed with the inner product $(.,.)_V$, then*

$$(x,y)_+ = (x,y)_- = (x,y)_V \ , \forall x,y \in V.$$

Thus, let us note the representations of the semi-inner products:

Proposition 3. *(see [13]): The following estimations hold:*

$$\begin{aligned} (x,y)_+ &= \max \{ \langle f,x \rangle : f \in \Im y \} \\ (x,y)_- &= \min \{ \langle f,x \rangle : f \in \Im y \}. \end{aligned}$$

Our goal is to study the following problem (EP):

Find $u \in V, \lambda \in \Re \setminus \{0\}$ such that

$$\sup_{f \in A(u,u)} \langle f, v-u \rangle + \int_\Omega j^0 (x, Tu(x)) (Tv(x) - Tu(x)) \, dx \geq \lambda (v-u,u)_+ , \forall v \in V \tag{EP}$$

which is the eigenvalue problem corresponding to the hemivariational inequality problem (P):

Find $u \in V$ such that

$$\sup_{f\in A(u,u)} \langle f, v-u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right)dx \geq 0, \forall v \in V. \tag{P}$$

Remark 3. *In fact, the eigenvalue problem (EP) is equivalent with the following problem:*

Find $u \in V, \lambda \in \Re \setminus \{0\}$ *such that*

$$\sup_{f\in A(u,u)} \langle f, v\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right)\right)dx \geq \lambda\left(v,u\right)_+ \ , \forall v \in V.$$

Because our approach is based on the results obtained for the problem (P),we will take into account the earliest formulation of the eigenvalue problem (EP).

For the general study of this eigenvalue problem (EP) we need some to recall results about the existence of solutions of the problem (P).

Theorem 4. (see [12]) *Assume that all the hypotheses (H1)-(H4) are satisfied. Moreover, the following assumptions hold:*

(i) for each $v \in V$, *the set-valued mapping* $A\left(.,v\right) : V \rightsquigarrow V^*$ *has the monotone property (M) and it is weakly*-upper semicontinuous from the line segments of* V *in* V^* *;*

(ii) for each $u \in V$, *the set-valued mapping* $A\left(u,.\right) : V \rightsquigarrow V^*$ *is weakly*-upper semicontinuous;*

(iii) there exists a compact subset $K \subseteq V$, *and an element* $u_0 \in V$ *such that the coercivity condition*

$$\sup_{f\in A(u,u)} \langle f, u_0-u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tu_0\left(x\right) - Tu\left(x\right)\right)dx < 0 \quad , \forall u \in V\diagdown K$$

holds;

(iv) for each $u, v \in V$, *the set* $A\left(u,v\right)$ *is weakly*-compact.*

Then the problem (P) admits a solution $u \in V$.

If in addition $A\left(u,u\right)$ *is a convex set, then u is also a solution of the following problem (Pc):*

Find $u \in V,\ f \in A\left(u,u\right)$ *such that*

$$\langle f, v-u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right)dx \geq 0 \quad , \quad \forall v \in V. \tag{Pc}$$

We define the set R(A,j,V) of **asymptotic directions** by

$$R(A,j,V) = \left\{ \begin{array}{c} w \in V \ \mid \exists (u_n) \subseteq V, t_n := \|u_n\| \to \infty, w_n := \frac{u_n}{\|u_n\|} \rightharpoonup w, \\ \\ \inf_{f \in A(u_n,u_n)} \langle f, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \le 0 \end{array} \right\}.$$

Theorem 5. *(see* [11]*) Assume that all the hypotheses (H1)-(H4) are satisfied, and V is a real reflexive Banach space. Moreover,*

(i) for each $v \in V$, the set-valued mapping $A(.,v) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous from the line segments of V into V^*, concave and monotone;

(ii) for each $u \in V$, the set-valued mapping $A(u,.) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous;

(iii) $R(A,j,V) = \emptyset$;

(iv) for each $u, v \in V$, the set $A(u,v)$ is weakly-compact.

Then the problem (P) admits a solution.

If in addition the set A(u,u) is convex, then the problem (Pc) admits solution also.

3. **The main results**

The aim of our study is to provide verifiable conditions ensuring the existence of solutions to problem (EP). Our existence results concerning problem (EP) are the following.

Theorem 6. *Assume that all the hypotheses (H1)-(H4) are satisfied. Moreover, the following assumptions hold:*

(i) for each $v \in V$, the set-valued mapping $A(.,v) : V \rightsquigarrow V^*$ has the monotone property (M) and it is weakly-upper semicontinuous from the line segments of V in V^*;*

(ii) for each $u \in V$, the set-valued mapping $A(u,.) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous;*

(iii) there exists a compact subset $K \subseteq V$, and an element $u_0 \in V$ such that

$$\|u_0\| \le \|u\|, \ \forall u \in V \backslash K$$

and

$$\sup_{f\in A(u,u)} \langle f, u_0 - u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tu_0\left(x\right) - Tu\left(x\right)\right) dx < 0, \forall u \in V \diagdown K;$$

(iv) for each $u, v \in V$, *the set* $A(u, v)$ *is weakly*-compact.*

Then for every $\lambda < 0$, *the problem (EP) admits a solution* $u \in V$.

If in addition $A(u, u)$ *is a convex set, then the following problem (EPc):*

Find $u \in V, \lambda \in \Re \setminus \{0\}$, $f \in A(u, u)$ *such that*

$$\langle f, v - u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right) dx \geq \lambda\left(v - u, u\right)_+ \quad , \quad \forall v \in V \tag{EPc}$$

admits a solution $u \in V, f \in A(u, u)$ *for every* $\lambda < 0$.

Theorem 7. *Assume that all the hypotheses (H1)-(H4) are satisfied, and* V *is a real reflexive Banach space. Moreover,*

(i) for each $v \in V$, *the set-valued mapping* $A(., v) : V \rightsquigarrow V^*$ *is weakly-upper semicontinuous from the line segments of* V *into* V^* , *concave and monotone;*

(ii) for each $u \in V$, *the set-valued mapping* $A(u, .) : V \rightsquigarrow V^*$ *is weakly-upper semicontinuous;*

(iii)$R(A, j, V) = \emptyset$*;*

(iv) for each $u, v \in V$, *the set* $A(u, v)$ *is weakly-compact.*

Then the problem (EP) admits a solution.

If in addition the set A(u,u) is convex, then the problem (EPc) admits solution also.

Remark 4. *Under the assumptions of the Theorems 6, 7 not only the eigenvalue problem (EP) but also the hemivariational inequality (P) admits solution.*

4. Proofs of the theorems

4.1. Proof of the first theorem.

The assumptions of the Theorem 6 allow to apply Theorem 4.

First, let us note that the eigenvalue inequality of problem (Ep) can be rewritten, according to the Proposition 3, as

$$\sup_{f\in A(u,u)} \langle f, v - u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right) dx \quad \geq \quad \lambda \sup_{g\in\Im u} \langle g, v - u\rangle, \quad \forall v \in V.$$

So,

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle-\lambda\sup_{g\in\Im u}\langle g,v-u\rangle$$
$$+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V.$$

Consider $\lambda<0$. Hence, $(-\lambda)>0$ and in this case we can obtain

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle+\sup_{g\in\Im u}\langle(-\lambda)\,g,v-u\rangle$$
$$+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V.$$

By the Theorem 2(i), we can note that

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle+\sup_{g\in\Im(-\lambda u)}\langle g,v-u\rangle$$
$$+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V. \tag{1}$$

Knowing that $\sup_{a\in A,b\in B}\left(\phi\left(a\right)+\psi\left(b\right)\right)=\sup_{a\in A}\phi\left(a\right)+\sup_{b\in B}\psi\left(b\right)$, problem (EP) and the inequality (1) lead us to the following problem:

Find $u\in V$ such that

$$\sup_{f\in F(u,u)}\langle f,v-u\rangle+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V \tag{EPn}$$

where, we denoted by F the set-valued mapping defined by $F:V\times V\rightsquigarrow V^{*},F\left(u,v\right)=A\left(u,v\right)+\Im\left(-\lambda v\right)$.

We show that all the hypotheses of the Theorem 4 are verified in the case of the problem (EPn).

'**Hypothesis (i)'**:

Let $v\in V$ be a fixed element. Then, using the monotone property (M) of $A\left(.,v\right)$, we have

$$\begin{aligned}\sup_{f\in F(u,v)}\langle f,u-v\rangle &= \sup_{f\in A(u,v)+\Im(-\lambda v)}\langle f,u-v\rangle=\sup_{\substack{f\in A(u,v)\\ g\in\Im(-\lambda v)}}\left(\langle f,u-v\rangle+\langle g,u-v\rangle\right)\\ &= \sup_{f\in A(u,v)}\langle f,u-v\rangle+\sup_{g\in\Im(-\lambda v)}\langle g,u-v\rangle\\ &\geq \sup_{f\in A(v,v)}\langle f,u-v\rangle+\sup_{g\in\Im(-\lambda v)}\langle g,u-v\rangle=\sup_{f\in F(v.v)}\langle f,u-v\rangle.\end{aligned}$$

This proves that the set-valued mapping $F\left(.,v\right)$ has the monotone property (M).

Moreover, the definition of the mapping F and the assumption (i) on the operator A(.,v) imply that the mapping $F(.,v)$ is weakly*-upper semicontinuous from the line segments of V in V^*.

'Hypothesis (ii)':

Because $A(u,.)$ is weakly*-upper semicontinuous, by the assumption (ii), and because $\Im(.)$ is weakly*-upper semicontinuous, according to the Theorem 2(iii), it follows that $F(u,.)$ is weakly*-upper semicontinuous.

'Hypothesis (iii)':

Let both $K \subseteq V$ and $u_0 \in V$ be the elements from the assumption (iii). The question we need to ask is if:

$$\sup_{f \in F(u,u)} \langle f, u_0 - u \rangle + \int_\Omega j^0(x, Tu(x))(Tu_0(x) - Tu(x))\,dx < 0 \quad, \forall u \in V \diagdown K$$

i.e.

$$\sup_{f \in A(u,u)} \langle f, u_0 - u \rangle + \sup_{g \in \Im(-\lambda u)} \langle g, u_0 - u \rangle$$
$$+ \int_\Omega j^0(x, Tu(x))(Tu_0(x) - Tu(x))\,dx < 0 \quad, \forall u \in V \diagdown K$$

which leads us to the

$$\sup_{f \in A(u,u)} \langle f, u_0 - u \rangle + \int_\Omega j^0(x, Tu(x))(Tu_0(x) - Tu(x))\,dx$$
$$< \lambda \sup_{g \in \Im u} \langle g, u_0 - u \rangle, \forall u \in V \diagdown K. \tag{2}$$

We note that the left hand side of the relation (2) is less than zero, by the assumption (iii). Moreover, the right hand side of the relation (2) is greater than zero, for $\lambda < 0$, because of the Proposition 1 and assumption (iii). Precisely, for $\forall g \in \Im u = \partial\left(\frac{1}{2}\|u\|^2\right)$, $\forall u \in V \backslash K$,

$$\langle g, u_0 - u \rangle \leq \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u\|^2 \leq 0,$$

which implies that

$$\sup_{g \in \Im u} \langle g, u_0 - u \rangle \leq 0. \tag{3}$$

If we multiply the inequality (3) by λ ($\lambda < 0$), we obtain

$$\lambda \sup_{g \in \Im u} \langle g, u_0 - u \rangle \geq 0.$$

As a conclusion, the 'hypothesis (iii)' is verified.

'Hypothesis (iv)':

By the assumption (iv), as well as by the Theorem 2(ii), we can infer that the set $F(u,v)$ is weakly*-compact, for every $u, v \in V$.

Finally, according to the Theorem 4 the eigenvalue problem (EP) admits a solution $u \in V$, when $\lambda < 0$.

In addition, if $A(u,u)$ is convex, it follows from the Theorem 2(i) that $F(u,u)$ is also convex. So, by the second part of the Theorem 4, we infer that the eigenvalue problem (EPc) admits solution for every $\lambda < 0$.

4.2. **Proof of the second theorem.** For the proof of the Theorem 7, we proceed in the same way. Again, for $\lambda < 0$, we note that the eigenvalue problem (Ep) is equivalent to the hemivariational inequality problem

Find $u \in V$ such that

$$\sup_{f \in F(u,u)} \langle f, v-u \rangle + \int_\Omega j^0(x, Tu(x))(Tv(x) - Tu(x))\,dx \geq 0 \quad, \forall v \in V \qquad \text{(EPn)}$$

where, we denoted by F the set-valued mapping defined by $F : V \times V \rightsquigarrow V^*$, $F(u,v) = A(u,v) + \Im(-\lambda v)$.

We show that all the hypotheses of the Theorem 5 are verified in the case of the problem (EPn).

'Hypothesis (i)':

First, let us emphasize that, because V is a reflexive Banach space, there exists an equivalent norm on V, such that under this new norm, the duality map is a single-valued monotone demicontinuous function. Having this, let $v \in V$ be a fixed element. Then using the fact that $A(.,v)$ is monotone, we have

$$\langle f_1 + \Im(-\lambda v) - f_2 - \Im(-\lambda v), u_1 - u_2 \rangle = \langle f_1 - f_2, u_1 - u_2 \rangle \geq 0,$$

for every $f_1 \in A(u_1, v), f_2 \in A(u_2, v)$.

This proves that the set-valued mapping $F(.,v)$ is monotone.

By the definition of the operator F, and the assumption (i), it follows that $F(.,v)$ is concave.

Moreover, the definition of the mapping F and the assumption (i) on the operator A(.,v) imply that the mapping $F(.,v)$ is weakly-upper semicontinuous from the line segments of V in V^*.

'Hypothesis (ii)':

Because $A(u,.)$ is weakly-upper semicontinuous, by the assumption (ii), and because $\Im(.)$ is demicontinuous, it follows that $F(u,.)$ is weakly-upper semicontinuous.

'Hypothesis (iii)':

Assume, by contradiction, that there exists $w \in R(F,j,V)$. This means that

$$\exists (u_n) \subseteq V, t_n := \|u_n\| \to \infty, w_n := \frac{u_n}{\|u_n\|} \rightharpoonup w \text{ such that}$$

$$\inf_{f \in F(u_n,u_n)} \langle f, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \leq 0. \tag{4}$$

Taking into account the definition of the operator F, inequality (4) becomes:

$$\inf_{f \in A(u_n,u_n)} \langle f, u_n \rangle - \lambda \langle \Im u_n, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \leq 0. \tag{5}$$

Knowing that

$$-\lambda \langle \Im u_n, u_n \rangle > 0$$

the relation (5) may be true if and only if the next inequality holds:

$$\inf_{f \in A(u_n,u_n)} \langle f, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \leq 0.$$

We can conclude that $w \in R(A,j,V)$, which is a contradiction with our assumption (iii).

'Hypothesis (iv)':

By the assumption (iv), as well as by the definition of the operator F, we can infer that the set $F(u,v)$ is weakly-compct, for every $u, v \in V$.

Finally, according to the Theorem 7, the eigenvalue problem (EP) admits a solution $u \in V$, when $\lambda < 0$.

In addition, if $A(u,u)$ is convex, it follows that $F(u,u)$ is also convex. So, by the second part of the Theorem 5, we infer that the eigenvalue problem (EPc) admits solution, for every $\lambda < 0$.

5. Applications to Engineering

Our results can be applied directly to the study of B. V. P.s in Engineering. Let us analyze a very general situation which leads us to the hemivariational inequality problem (EP). For instance, let us consider an open, bounded, connected subset

$\Omega \subseteq \Re^3$ referred to a fixed Cartesian coordinate system $Ox_1x_2x_3$ and we formulate the problem

$$-\Delta u + h(u) + cu = f \text{ in } \Omega \tag{6}$$

$$u = 0 \text{ on } \Gamma. \tag{7}$$

Here Γ is the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient), c is a given constant, and h is a continuous function, which has the property

$$u(x)\, h(u(x)) \geq 0, \forall x \in \Omega. \tag{8}$$

In order to physically motivate problem (6),(7) in a simple way, we interpret u as the temperature of a medium in a region Ω. The differential equation in (6) describes a stationary temperature state with the heat source $f - h(u) - cu$ that depends on temperature (see [47]).

We seek a function u such that to verify (6), (7) with

$$-f \in \partial j(x, u) \tag{9}$$

where $j(x, .)$ is a locally Lipschitz function.

Let us consider the Sobolev space $V = H_0^1(\Omega)$, which can be viewed as a Hilbert space endowed with the inner-product

$$(u, v) = \int_\Omega uvdx, \ \forall u, v \in V.$$

Let us denote by $C(\Omega)$ the constant of the Poincaré-Friedrichs inequality

$$\int_\Omega v^2 dx \leq C(\Omega) \int_\Omega (\nabla v)^2 dx, \ \forall v \in V. \tag{10}$$

Moreover, let us assume that the following directional growth condition holds:

$$j^0(x, \xi)(-\xi) \leq \alpha(x) \mid \xi \mid, \ \forall x \in \Omega, \forall \xi \in \Re \tag{11}$$

for some nonnegative function $\alpha \in L^2(\Omega)$, with

$$\|\alpha\|_{L^2(\Omega)} \leq \frac{1}{C(\Omega)} \ . \tag{12}$$

Now, we multiply (6) by $(v - u)$ and integrate over Ω. This gives us the following relation

$$\int_\Omega -\Delta u\,(v-u)\,dx + \int_\Omega h(u)\,(v-u)\,dx + c\int_\Omega u\,(v-u)\,dx = \int_\Omega f\,(v-u)\,dx. \tag{13}$$

Then from the Gauss-Green Theorem applied to (13) we are led to the equality

$$\int_{\Omega} \nabla u \nabla (v-u)\, dx + \int_{\Omega} h(u)(v-u)\, dx + c\int_{\Omega} u(v-u)\, dx$$
$$= \int_{\Gamma} \frac{\partial u}{\partial n}(v-u)\, d\Gamma + \int_{\Omega} f(v-u)\, dx. \tag{14}$$

Because $u, v \in H_0^1(\Omega)$ the surface integral vanishes.

Relation (9) implies that

$$-f(v-u) \le j^0(x,u)(v-u). \tag{15}$$

If we introduce the notation

$$a(u,v) = \int_{\Omega} \nabla u \nabla v dx$$

then the relations (14) and (15) give us the inequality

$$a(u,v) + \int_{\Omega} h(u)(v-u)\, dx + c\int_{\Omega} u(v-u)\, dx$$
$$+ \int_{\Omega} j^0(x,u)(v-u)\, dx \ge 0, \forall v \in V. \tag{16}$$

Let us note that there exists a linear monotone continuous operator $B : V \to V^*$ such that

$$\langle B(u), v\rangle = a(u,v), \quad \forall u, v \in V.$$

Consider $\Im : V \to V^*$ the duality isomorphism

$$\langle \Im u, v\rangle = (u,v), \ \forall u, v \in V$$

Thus, if we consider the following multivalued mapping

$$\begin{aligned} A &: V \times V \rightsquigarrow V^* \\ A(u,v) &= B(u) + \Im(h(v)) \end{aligned}$$

then the hemivariational inequality (16) lead us to the following problem:

find $u \in V$ such that for any $v \in V$

$$\sup_{f \in A(u,u)} \langle f, v-u\rangle + \int_{\Omega} j^0(x,u)(v-u)\, dx \ge (-c)\langle \Im u, v-u\rangle \tag{EPeng}$$

First, let us remark that the operator A satisfies the assumptions (i), (ii), (iv) of the Theorem 7. All we have to do now is to verify if the assumption (iii) is satisfied. For this goal, let us assume that there exists $w \in R(A, j, V)$. So,

$$\exists (u_n) \subseteq V, t_n := \|u_n\|_{L^2(\Omega)} \to \infty, w_n := \frac{u_n}{\|u_n\|_{L^2(\Omega)}} \rightharpoonup w \text{ such that}$$

$$\int_\Omega (\nabla u_n)^2 dx + \int_\Omega h(u_n) u_n dx - \int_\Omega j^0(x, u_n(x))(-u_n(x)) dx \leq 0. \tag{17}$$

There exists a rank m such that $\|u_n\|_{L^2(\Omega)} > 1$, for every $n \geq m$. By the Holder inequality and because of the relations (10), (11), (12), the following evaluation holds for every $u_n, n \geq m$:

$$\int_\Omega (\nabla u_n)^2 dx \geq \frac{1}{C(\Omega)} \int_\Omega (u_n)^2 dx > \frac{1}{C(\Omega)} \left(\int_\Omega (u_n)^2 dx \right)^{\frac{1}{2}}$$

$$= \frac{\|u_n\|_{L^2(\Omega)}}{C(\Omega)} \geq \|\alpha\|_{L^2(\Omega)} \cdot \|u_n\|_{L^2(\Omega)} \geq \int_\Omega \alpha(x) \cdot \mid u(x) \mid dx$$

$$\geq \mid \int_\Omega j^0(x, u(x))(-u(x)) dx \mid \geq \int_\Omega j^0(x, u(x))(-u(x)) dx.$$

The last evaluation and the property (8) of the function h show us that the relation (17) is impossible. This contradiction guarantees that the assumption (iii) of the Theorem 7 is also satisfied.

Since all the assumptions of the Theorem 7 are ensured and the embedding $V \subseteq L^2(\Omega)$ is linear and continuous, we can prove the existence of solutions of (EPeng) for all $c > 0$.

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