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AN INFEASIBLE FULL-NEWTON STEP ALGORITHM FOR LINEAR OPTIMIZATION WITH ONE CENTERING STEP IN MAJOR ITERATION

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Abstract. Recently, Roos proposed a full-Newton step infeasible interior-point method (IIPM) for solving linear optimization (LO) problems. Later on, more variants of this algorithm were published. However, each main step of these methods is composed of one feasibility step and several centering steps. The purpose of this paper is to prove that by using a new search direction it is enough to take only one centering step in order to obtain a polynomial-time method. This algorithm has the same complexity as the best known IIPMs.

1. Introduction

In this paper, we define a new interior-point algorithm (IPA) for LO, which approximates the optimal solution, starting from infeasible points. Karmarkar's publication [7] appeared in 1984 and meant a paradigm shift in the area of optimization algorithms. Following this, a large amount of IPAs has been published. These algorithms have many applications in different fields, such as engineering, economics, transportation, statistics, machine learning and data mining. The first infeasible methods were developed by Lustig [9] and Tanabe [18]. The complexity of IPAs was analysed at first by Kojima, Meggido, Mizuno [8] and Zhang [23]. Bonnans and Potra [3] defined infeasible algorithms for linear complementarity problems. The predictor-corrector method for LO problem was studied by Potra [14, 15]. We can read about new results on infeasible interior-point algorithms in books wrote by Wright [21], Ye [22] and Vanderbei [19].

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Roos [16] introduced a new algorithm, which uses only full-Newton steps and starts from infeasible points. Mansouri and Roos [10] proposed a simplified IIPM. Gu et al. [6] presented an improved variant of the algorithm. Darvay [4, 5] defined a new technique for finding search directions for LO problems. Achache [1] generalized this approach to convex quadratic optimization, and Wang and Bai [20] to symmetric optimization. Ahmadi, Hasani and Kheirfam [2] adapted this technique to IIPMs. Pan, Li and He [13] proposed an IIPM using a logarithmic equivalent transformation of the centering equations. Mansouri, Siyavash and Zangiabadi [11] extended the algorithm introduced by Roos to semidefinite optimization problems using the method proposed in [5].

In the full-Newton step IIPMs defined in these papers two types of steps are used, one feasibility step and a few centering steps. In this paper, we present a new full-Newton step IIPM based on the technique introduced in [4, 5] and we prove that it suffices to take only one centering step in order to get a well-defined algorithm.

We introduce some notations used throughout the paper. Let x and s be two n-dimensional vectors. Then, xs denotes the componentwise product of the vectors x and s. Similarly, we define $\frac{x}{s} = \left[\frac{x_1}{s_1}, \frac{x_2}{s_2}, ..., \frac{x_n}{s_n}\right]^T$, where $s_i \neq 0$ for all $1 \leq i \leq n$. If $x \geq 0$, then $\sqrt{x}$ is the vector obtained by taking square roots of the components of x. Let e be the n-dimensional all-one vector. Furthermore, $diag(x)$ is a diagonal matrix, which contains on his main diagonal the elements of x in the original order. Besides these, $\|x\|$ denotes the Euclidean norm, $\|x\|_\infty$ the Chebyshev norm, $\|x\|_1$ the 1-norm, and $min(x)$ the minimal component of x. Finally, if $f(t) \geq 0$ and $g(t) \geq 0$ are real valued functions, then $f(t) = O(g(t))$ means that there exists a positive constant γ so that $f(t) \leq \gamma g(t)$.

The paper is organized in the following way. Firstly, we present the LO problem. In the next section the feasible primal-dual algorithm and its complexity analysis are revisited. Then, we introduce the perturbed problems and the new infeasible primal-dual algorithm. The purpose of the next sections is to provide the complexity analysis of the algorithm and to prove its polynomiality. Finally, the paper ends up with a conclusion.

2. The Linear Optimization Problem

Let us consider the following primal problem

$$(P) \qquad \begin{aligned} &min\ c^T x, \\ &Ax = b, \\ &x \geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $rank(A) = m$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.
The dual of this problem is

$$(D) \qquad \begin{aligned} &max\ b^T y,\\ &A^T y + s = c,\\ &s \geq 0. \end{aligned}$$

In case of the feasible LO algorithms we assume that the *interior-point condition* (IPC) holds for the primal and dual problems, i.e., there exists (x^0, y^0, s^0) so that

$$(IPC) \qquad \begin{aligned} Ax^0 &= b, \qquad & x^0 > 0,\\ A^T y^0 + s^0 &= c, \qquad & s^0 > 0. \end{aligned}$$

Using the self-dual embedding technique we can always construct a LO problem in such a way that the IPC holds. So, the IPC can be assumed without loss of generality. Furthermore, the self-dual embedding model yields $x^0 = s^0 = e$. Denote $\mu^0 = \frac{(x^0)^T s^0}{n} = 1$.
The optimal solution of the primal-dual pair is characterized by the following system of equations:

$$(1) \qquad \begin{aligned} Ax &= b, \qquad & x \geq 0,\\ A^T y + s &= c, \qquad & s \geq 0,\\ xs &= 0. \end{aligned}$$

The first and the second equation of system (1) are called *feasibility conditions*. They serve for maintaining feasibility. The last equation is named *complementarity condition*. Primal-dual interior-point methods replace the complementarity condition with a parameterized equation. Hence we obtain:

$$(2) \qquad \begin{aligned} Ax &= b, \qquad & x \geq 0,\\ A^T y + s &= c, \qquad & s \geq 0,\\ xs &= \mu e, \end{aligned}$$

where $\mu > 0$. If the IPC holds, then for a fixed $\mu > 0$ the system (2) has a unique solution, called the *μ-center* or *analytic center* (Sonnevend [17]). The set of μ-centers for $\mu > 0$ formes a well-behaved curve, called *central path*. As μ tends to zero, the central path converges to the optimal solutions of (P) and (D).

3. Feasible Primal-Dual Algorithm

In this section we present the new technique for finding search directions introduced in [5]. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable and invertible function. The system of equations, which defines the central path (2) can be written in the following equivalent form:

$$
\begin{aligned}
Ax &= b, \qquad x \geq 0, \\
A^T y + s &= c, \qquad s \geq 0, \\
\varphi\left(\frac{x_i s_i}{\mu}\right) &= \varphi(1), \text{ for all } 1 \leq i \leq n.
\end{aligned} \tag{3}
$$

Applying Newton's method to (3) we can obtain new search directions. If $\varphi(t) = \sqrt{t}$, then we get the following system:

$$
\begin{aligned}
A\Delta x &= 0, \\
A^T \Delta y + \Delta s &= 0, \\
s\Delta x + x\Delta s &= 2(\sqrt{\mu x s} - xs).
\end{aligned} \tag{4}
$$

We can give a proximity measure to the central path [5]:

$$
\sigma(xs, \mu) = \left\| e - \sqrt{\frac{xs}{\mu}} \right\|. \tag{5}
$$

The feasible primal-dual algorithm can be described as in Figure 1.

Feasible primal-dual algorithm [5]

Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter (default $\theta = \frac{1}{2\sqrt{n}}$) and $0 < \tau < 1$ the proximity parameter (default $\tau = \frac{1}{2}$). Assume that for (x^0, y^0, s^0) the IPC holds, and $\mu^0 = \frac{(x^0)^T s^0}{n}$. Furthermore, suppose that $\sigma(x^0 s^0, \mu^0) < \tau$.

begin
  $(x, y, s) := (x^0, y^0, s^0)$;
  $\mu := \mu^0$;
  while $x^T s > \epsilon$ **do begin**
    $\mu := (1 - \theta)\mu$;
    calculate $(\Delta x, \Delta y, \Delta s)$ *from (4)*
    $x := x + \Delta x$;
    $y := y + \Delta y$;
    $s := s + \Delta s$;
  end
end.

FIGURE 1. Feasible primal-dual algorithm

The following lemmas (cf. [5]) are meant to prove the polynomiality of the algorithm. Let $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$ be the vectors we get after a full-Newton step.

Lemma 3.1 *Let $\sigma = \sigma(xs, \mu) < 1$. Then $x_+ > 0$ and $s_+ > 0$, so the full-Newton step is strictly feasible.*

Lemma 3.2 *Let $\sigma = \sigma(xs, \mu) < 1$. Then*

$$\sigma(x_+ s_+, \mu) \leq \frac{\sigma^2}{1 + \sqrt{1 - \sigma^2}},$$

which means that the full-Newton step ensures local quadratic convergence of the proximity measure.

Lemma 3.3 *Let $\sigma = \sigma(xs, \mu)$. Then*

$$(x_+)^T s_+ = \mu(n - \sigma^2),$$

thus $(x_+)^T s_+ \leq \mu n$.

Lemma 3.4 *Let $\sigma = \sigma(xs, \mu) < 1$ and $\mu_+ = (1-\theta)\mu$, where $0 < \theta < 1$. Then*

$$\sigma(x_+ s_+, \mu_+) \leq \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}}.$$

Furthermore, if $\sigma < \frac{1}{2}$, $\theta = \frac{1}{2\sqrt{n}}$ and $n \geq 4$, then $\sigma(x_+ s_+, \mu_+) < \frac{1}{2}$.

Lemma 3.5 *Suppose that (x^0, s^0) are strictly feasible, $\mu^0 = \frac{(x^0)^T s^0}{n}$ and $\sigma(x^0 s^0, \mu^0) < \frac{1}{2}$. Let x^k and s^k be the vectors obtained after k iterations. Then, for every*

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil$$

we get $(x^k)^T s^k \leq \epsilon$.

Lemma 3.6 *Assume that $x^0 = s^0 = e$. Then, Algorithm 1 demands no more than*

$$\left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil$$

interior-point iterations.

Theorem 3.7 *Suppose that $x^0 = s^0 = e$. Using the default values for θ and τ we get that Algorithm 1 requires at most*

$$O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$$

interior-point iterations. The resulting vectors satisfy $x^T s \leq \epsilon$.

4. The Perturbed Problems

From now on we don't assume that the initial points are feasible solutions of the primal and dual problems, but we suppose that an optimal solution exists. Let $\zeta > 0$ be given so that

$$\|\bar{x} + \bar{s}\|_\infty \leq \zeta, \tag{6}$$

where $\bar{x}$ and $(\bar{y}, \bar{s})$ are optimal solutions of (P) and (D). Hence, the algorithm will start with the following initial iterates:

$$x^0 = s^0 = \zeta e, \qquad y^0 = 0, \qquad \mu^0 = \zeta^2. \tag{7}$$

Instead of the original problem, we consider the following perturbed problem, which was studied by many researchers (see for example Ye [22] and Roos [16]):

$$(P_\nu) \qquad \begin{array}{c} min\ (c - \nu(c - A^T y^0 - s^0))^T x, \\ Ax = b - \nu(b - Ax^0), \\ x \geq 0, \end{array}$$

and its dual problem:

$$(D_\nu) \qquad \begin{array}{c} max\ (b - \nu(b - Ax^0))^T y, \\ A^T y + s = c - \nu(c - A^T y^0 - s^0), \\ s \geq 0, \end{array}$$

where $0 < \nu \leq 1$. The following lemma holds.

Lemma 4.1 (*cf. [22], Theorem 5.13). The problems (P) and (D), are feasible if and only if for each ν satisfying $0 < \nu \leq 1$ the perturbed problems (P_ν) and (D_ν) satisfy the (IPC).*

The system of equations, which defines the central path of the perturbed problems can be written in the following form:

$$\begin{array}{rcll} b - Ax &=& \nu(b - Ax^0), & x \geq 0, \\ c - A^T y - s &=& \nu(c - A^T y^0 - s^0), & s \geq 0, \\ xs &=& \mu e. & \end{array} \tag{8}$$

Let us consider the function φ defined in Section 3. Then the system (8) is equivalent to

$$\begin{array}{rcll} b - Ax &=& \nu(b - Ax^0), & x \geq 0, \\ c - A^T y - s &=& \nu(c - A^T y^0 - s^0), & s \geq 0, \\ \varphi\left(\dfrac{x_i s_i}{\mu}\right) &=& \varphi(1), \text{ for all } 1 \leq i \leq n. & \end{array} \tag{9}$$

Now we apply Newton's method for system (9). Assuming that $\varphi(t) = \sqrt{t}$, and x and (y, s) are strictly feasible solutions of (P_ν) and (D_ν), we obtain system (4).

5. A New Primal-Dual Algorithm

Let $\nu_+ = (1-\theta)\nu$, where $0 < \theta < 1$. Let us introduce the following notations:

$$r_b^0 = b - Ax^0, \quad r_c^0 = c - A^T y^0 - s^0.$$

Assuming that x and (y, s) are strictly feasible solutions of (P_ν) and (D_ν), we define the $(\Delta^f x, \Delta^f y, \Delta^f s)$ step in order to get feasible solutions of (P_{ν_+}) and (D_{ν_+}). Thus, using $\varphi(t) = \sqrt{t}$, we obtain the following system:

$$\begin{aligned} A\Delta^f x &= \theta\nu r_b^0, \\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_c^0, \\ s\Delta^f x + x\Delta^f s &= 2(\sqrt{\mu x s} - xs). \end{aligned} \tag{10}$$

We introduce the following notations:

$$v = \sqrt{\frac{xs}{\mu}}, \qquad d_x = \frac{v\Delta^f x}{x}, \qquad d_s = \frac{v\Delta^f s}{s},$$

we obtain

$$\mu v(d_x + d_s) = s\Delta^f x + x\Delta^f s \tag{11}$$

and

$$d_x d_s = \frac{\Delta^f x \Delta^f s}{\mu}. \tag{12}$$

Using these notations we get the scaled form of system (10):

$$\begin{aligned} \bar{A}d_x &= \frac{\theta\nu}{\mu} r_b^0, \\ \bar{A}^T\Delta^f y + d_s &= \theta\nu \frac{r_c^0 v}{s}, \\ d_x + d_s &= p_v, \end{aligned} \tag{13}$$

where $p_v = 2(e - v)$ and $\bar{A} = \frac{1}{\mu} A diag\left(\frac{x}{v}\right)$. The proximity measure defined by (5) can be written as follows:

$$\sigma(v) = \sigma(xs, \mu) = \frac{\|p_v\|}{2} = \|e - v\|.$$

Let $q_v = d_x - d_s$. Then

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2}.$$

Multiplying the two equalities, we get

$$\frac{q_v^2}{4} = \frac{p_v^2}{4} - d_x d_s. \tag{14}$$

It follows that

$$\frac{\|q_v\|^2}{4} = \frac{\|p_v\|^2}{4} - d_x^T d_s. \tag{15}$$

The algorithm is defined in Figure 2.

Infeasible primal-dual algorithm

Let $\epsilon > 0$ be the accuracy parameter and $0 < \theta < 1$ the update parameter (default $\theta = \frac{1}{8n}$). We assume that the initial points are (x^0, y^0, s^0), $x^0 > 0$, $s^0 > 0$ and $x^0 s^0 = \mu^0 e$ (default $x^0 = \zeta e$, $y^0 = 0$, $s^0 = \zeta e$, $\mu^0 = \zeta^2$, where $\zeta > 0$).

begin
  $(x, y, s) := (x^0, y^0, s^0)$;
  $\mu := \mu^0$; $\nu := 1$;
  while $max(x^T s, \|b - Ax\|, \|c - A^T y - s\|) \geq \epsilon$ **do begin**
    $(x, y, s) := (x, y, s) + (\Delta^f x, \Delta^f y, \Delta^f s)$;
    $\mu := (1 - \theta)\mu$;
    $\nu := (1 - \theta)\nu$;
    $(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$;
  end
end.

FIGURE 2. Infeasible primal-dual algorithm

In the following sections we analyse the complexity of the algorithm.

6. Analysis of the Algorithm

Let $x^f = x + \Delta^f x$ and $s^f = s + \Delta^f s$ be the vectors obtained after the feasibility step. In the next lemma we give a condition, which guarantees the feasibility of x^f and s^f. Let $0 < \theta < 1$ and denote

$$\omega(v) = \frac{1}{2}\sqrt{\|d_x\|^2 + \|d_s\|^2},$$

$$v^f = \sqrt{\frac{x^f s^f}{\mu}}, \quad v^+ = \sqrt{\frac{x^f s^f}{\mu_+}}, \quad \mu_+ = (1 - \theta)\mu.$$

Lemma 6.1 *Let $x > 0$ be a feasible solution of (P_ν) and $s > 0$ a feasible solution of (D_ν), and $\sigma(v) = \sigma(xs, \mu)$, which satisfies $\sigma(v)^2 + 2\omega(v)^2 < 1$. Then we have*

$$x^f > 0 \quad \text{and} \quad s^f > 0, \tag{16}$$

thus x^f and s^f are strictly feasible solutions of (P_{ν_+}) and (D_{ν_+}).

Proof. From the definition of x^f and s^f and system (10) we deduce that we have to prove (16). For each $0 \le \alpha \le 1$ denote $x^f(\alpha) = x + \alpha\Delta^f x$ and $s^f(\alpha) = s + \alpha\Delta^f s$. Thus

$$x^f(\alpha)s^f(\alpha) = xs + \alpha(s\Delta^f x + x\Delta^f s) + \alpha^2\Delta^f x\Delta^f s.$$

Using (11) and (12) we may write

$$\frac{1}{\mu}x^f(\alpha)s^f(\alpha) = v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s. \tag{17}$$

By (14) we have

$$\frac{1}{\mu}x^f(\alpha)s^f(\alpha) = (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right).$$

Moreover, from $p_v = 2(e - v)$ we get

$$v^2 + vp_v = 2v - v^2 = e - (e-v)^2 = e - \frac{p_v^2}{4}, \tag{18}$$

so

$$\frac{1}{\mu}x^f(\alpha)s^f(\alpha) = (1-\alpha)v^2 + \alpha\left(e - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right). \tag{19}$$

The inequality $x^f(\alpha)s^f(\alpha) > 0$ holds if

$$\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty < 1.$$

Using (15) we obtain

$$\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty \le (1-\alpha)\frac{\|p_v^2\|_\infty}{4} + \alpha\frac{\|q_v^2\|_\infty}{4} \le$$

$$\le (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4} = \sigma(v)^2 - \alpha d_x^T d_s.$$

Moreover,

$$-d_x^T d_s \le |d_x^T d_s| \le \|d_x\|\,\|d_s\| \le \frac{1}{2}\left(\|d_x\|^2 + \|d_s\|^2\right) = 2\omega(v)^2. \tag{20}$$

Using this inequality we may write

$$\sigma(v)^2 - \alpha d_x^T d_s \le \sigma(v)^2 + 2\omega(v)^2 < 1,$$

thus we obtain that for each $0 \leq \alpha \leq 1$ the $x^f(\alpha)s^f(\alpha) > 0$ inequality holds. Therefore, the linear functions of α, $x^f(\alpha)$ and $s^f(\alpha)$ do not change sign on the interval $[0,1]$. Consequently, $x^f(0) = x > 0$ and $s^f(0) = s > 0$ yield $x^f(1) = x^f > 0$ and $s^f(1) = s^f > 0$. □

In the following lemmas we analyse how far are the vectors obtained after the feasibility step from the next points of the central path of the perturbed problems.

Lemma 6.2 *Let $x > 0$ be a feasible solution of (P_ν) and $s > 0$ a feasible solution of (D_ν), and $\sigma(v) = \sigma(xs, \mu)$ such that $\sigma(v)^2 + 2\omega(v)^2 < 1$. Then*

$$\sigma(v^+) = \sigma(x^f s^f, \mu_+) \leq \frac{\theta\sqrt{n} + \sigma(v)^2 + 2\omega(v)^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma(v)^2 - 2\omega(v)^2)}}.$$

Proof. From Lemma 6.1 we get $x^f > 0$ and $s^f > 0$. Using (18) and the last equation of (13), from (17) we obtain

$$\frac{1}{\mu} x^f(\alpha)s^f(\alpha) = (1-\alpha)v^2 + \alpha(2v - v^2) + \alpha^2 d_x d_s. \tag{21}$$

Substituting $\alpha = 1$ into (21) and using (14) we get

$$\frac{1}{\mu} x^f s^f = (v^f)^2 = 2v - v^2 + d_x d_s = e - (e-v)^2 + d_x d_s = e - \frac{p_v^2}{4} + d_x d_s = e - \frac{q_v^2}{4}.$$

From this we obtain

$$e - (v^f)^2 = \frac{q_v^2}{4}. \tag{22}$$

Using $\sigma(v^+) = \left\| e - \sqrt{\frac{x^f s^f}{\mu_+}} \right\|$ we get

$$\sigma(v^+) = \frac{1}{\sqrt{1-\theta}} \left\| \sqrt{1-\theta}\, e - v^f \right\| = \frac{1}{\sqrt{1-\theta}} \left\| \frac{(1-\theta)e - (v^f)^2}{\sqrt{1-\theta}\, e + v^f} \right\|.$$

From (22) it follows that

$$\sigma(v^+) = \frac{1}{\sqrt{1-\theta}} \left\| \frac{-\theta e + \frac{q_v^2}{4}}{\sqrt{1-\theta}\, e + v^f} \right\|$$

and

$$min(v^f) \geq \sqrt{1 - \frac{\|q_v^2\|_\infty}{4}} \geq \sqrt{1 - \frac{\|q_v^2\|}{4}} \geq \sqrt{1 - \frac{\|q_v\|^2}{4}}.$$

Using $\frac{\|p_v\|^2}{4} = \sigma(v)^2$, we get from (15)

$$\frac{\|q_v\|^2}{4} = \sigma(v)^2 - d_x^T d_s, \tag{23}$$

and hence

$$min(v^f) \geq \sqrt{1 - \frac{\|q_v\|^2}{4}} = \sqrt{1 - \sigma(v)^2 + d_x^T d_s}. \tag{24}$$

Using (23), (24) and (20) we obtain

$$\begin{aligned}
\sigma(v^+) &= \frac{1}{\sqrt{1-\theta}} \left\| \frac{-\theta e + \frac{q_v^2}{4}}{\sqrt{1-\theta} e + v^f} \right\| \leq \frac{\left\| -\theta e + \frac{q_v^2}{4} \right\|}{\sqrt{1-\theta}(\sqrt{1-\theta} + \sqrt{1 - \sigma(v)^2 + d_x^T d_s})} \\
&\leq \frac{\theta\sqrt{n} + \sigma(v)^2 - d_x^T d_s}{1 - \theta + \sqrt{(1-\theta)(1 - \sigma(v)^2 + d_x^T d_s)}} \\
&\leq \frac{\theta\sqrt{n} + \sigma(v)^2 + 2\omega(v)^2}{1 - \theta + \sqrt{(1-\theta)(1 - \sigma(v)^2 - 2\omega(v)^2)}}.
\end{aligned}$$

This proves the lemma. □

As a consequence we get that Lemma 7 of [2] holds under different assumptions. The next lemma gives an upper bound for $\omega(v)$. In order to accomplish this, we introduce the following notations as in [16]. Let

$$\mathcal{L} = \left\{ \xi \in \mathbb{R}^n : \bar{A}\xi = 0 \right\}$$

be the null space of the matrix $\bar{A}$ and

$$\mathcal{L}^\perp = \left\{ \bar{A}^T y : y \in \mathbb{R}^m \right\}$$

the row space of $\bar{A}$. Note that $\mathcal{L} \perp \mathcal{L}^\perp$, $\mathcal{L} + \mathcal{L}^\perp = \mathbb{R}^n$ and $\mathcal{L} \cap \mathcal{L}^\perp = \{0\}$. The $\left\{ \xi \in \mathbb{R}^n : \bar{A}\xi = \theta\nu r_b^0 \right\}$ affine space is identical with the space $d_x + \mathcal{L}$. We have $d_s \in \theta\nu v s^{-1} r_c^0 + \mathcal{L}^\perp$, hence $d_x + \mathcal{L}$ and $d_s + \mathcal{L}^\perp$ meet in a unique point, which is denoted by q. The following two lemmas can be proved as in [16]. They determine upper bounds for $\omega(v)$ and $\|q\|$.

Lemma 6.3 (cf. Roos [16], Lemma 4.6) *If* $\{q\} = (d_x + \mathcal{L}) \cap (d_s + \mathcal{L}^\perp)$, *then*

$$2\omega(v) \leq \sqrt{\|q\|^2 + (\|q\| + 2\sigma(v))^2}.$$

Lemma 6.4 (cf. Roos [16], Lemma 4.7) *The inequality*

$$\sqrt{\mu}\,\|q\| \leq \theta\nu\zeta\sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)}$$

holds.

In the next lemma we give lower and upper bounds for the components of $\frac{xs}{\mu}$.

Lemma 6.5 *Let* $\sigma(xs, \mu) < \tau < 1$. *Then*

$$\left(1 - \sqrt{\tau}\right)^2 < \frac{x_i s_i}{\mu} < \left(1 + \sqrt{\tau}\right)^2, \quad \textit{for all } 1 \leq i \leq n.$$

Proof. *From $\sigma(v) < \tau < 1$ we obtain $\|v - e\| < \tau$. Thus,*

$$(v_i - 1)^2 \leq \sum_{i=1}^{n} (v_i - 1)^2 < \tau, \quad \textit{for all } 1 \leq i \leq n,$$

and this yields

$$1 - \sqrt{\tau} < v_i < 1 + \sqrt{\tau}.$$

This proves the lemma. □

Lemma 6.6 *One has*

$$\sqrt{e^T \left(\frac{x}{s} + \frac{s}{x} \right)} < \frac{1}{\sqrt{\mu}\,(1 - \sqrt{\tau})} \sqrt{\|x\|^2 + \|s\|^2}.$$

Proof. *We know that*

$$\frac{x_i}{s_i} + \frac{s_i}{x_i} = \frac{x_i^2 + s_i^2}{x_i s_i}, \quad \textit{for all } 1 \leq i \leq n.$$

From Lemma 6.5 we get $x_i s_i > \mu\,(1 - \sqrt{\tau})^2$, thus

$$\frac{1}{x_i s_i} < \frac{1}{\mu\,(1 - \sqrt{\tau})^2}.$$

Using this we obtain

$$\frac{x_i^2 + s_i^2}{x_i s_i} < \frac{1}{\mu\,(1 - \sqrt{\tau})^2} \left(x_i^2 + s_i^2 \right), \quad \textit{for all } 1 \leq i \leq n,$$

hence

$$e^T \left(\frac{x}{s} + \frac{s}{x} \right) < \frac{1}{\mu\,(1 - \sqrt{\tau})^2} \left(\|x\|^2 + \|s\|^2 \right),$$

and this implies the lemma. □

Lemma 6.7 *If x and s are strictly feasible solutions of (P_ν) and (D_ν) and $\sigma(xs, \mu) < \tau$, then the following inequality holds:*

$$\sqrt{\|x\|^2 + \|s\|^2} \leq \zeta n \left(2 + 2\sqrt{\tau} + \tau \right).$$

Proof. *Using that $\bar{x}$ and $(\bar{y}, \bar{s})$ are optimal solutions of the original primal-dual pair, from (6) we get the following system:*

$$\begin{aligned} A\bar{x} &= b, & 0 \leq \bar{x} \leq \zeta e, \\ A^T \bar{y} + \bar{s} &= c, & 0 \leq \bar{s} \leq \zeta e, \\ \bar{x}\bar{s} &= 0. & \end{aligned} \tag{25}$$

Let y be the vector such that x and (y, s) are the feasible solutions of (P_ν) and (D_ν). Then, using (7) we may write

$$\begin{aligned} Ax &= b - \nu(b - A\zeta e), & x \geq 0, \\ A^T y + s &= c - \nu(c - \zeta e), & s \geq 0. \end{aligned} \tag{26}$$

We follow the method introduced in [16] using the characteristics of the search direction specified by us. However, our approach differs from the one proposed in [16] in the sense that we don't assume that x and (y, s) are perfectly centered. Hence, we have

$$\begin{aligned} A\bar{x} - Ax &= \nu(A\bar{x} - A\zeta e), & x \geq 0, \\ A^T\bar{y} + \bar{s} - A^T y - s &= \nu(A^T\bar{y} + \bar{s} - \zeta e), & s \geq 0. \end{aligned} \tag{27}$$

Therefore

$$\begin{aligned} A\left(\bar{x} - x - \nu\bar{x} + \nu\zeta e\right) &= 0, & x \geq 0, \\ A^T\left(\bar{y} - y - \nu\bar{y}\right) &= s - \bar{s} + \nu\bar{s} - \nu\zeta e, & s \geq 0. \end{aligned} \tag{28}$$

Since the null space and the row space of a matrix are orthogonal we get

$$\left(\bar{x} - x - \nu\bar{x} + \nu\zeta e\right)^T\left(s - \bar{s} + \nu\bar{s} - \nu\zeta e\right) = 0.$$

Let

$$a := (1-\nu)\bar{x} + \nu\zeta e, \quad d := (1-\nu)\bar{s} + \nu\zeta e.$$

Then $(a - x)^T(d - s) = 0$, which implies

$$a^T d + x^T s = a^T s + d^T x.$$

From Lemma 6.5 we get $x_i s_i < \mu\left(1 + \sqrt{\tau}\right)^2$, and this yields

$$x^T s < \mu n\left(1 + \sqrt{\tau}\right)^2.$$

In addition, $\bar{x}^T\bar{s} = 0$, $\bar{x} + \bar{s} \leq \zeta e$ and $\mu = \nu\mu^0 = \nu\zeta^2$. Thus, we may write

$$\begin{aligned} a^T d + x^T s &= \left((1-\nu)\bar{x} + \nu\zeta e\right)^T\left((1-\nu)\bar{s} + \nu\zeta e\right) + x^T s \\ &= \nu(1-\nu)(\bar{x} + \bar{s})^T\zeta e + \nu^2\zeta^2 n + x^T s \\ &\leq \nu(1-\nu)(\zeta e)^T\zeta e + \nu^2\zeta^2 n + \mu n\left(1 + \sqrt{\tau}\right)^2 \\ &= \nu(1-\nu)\zeta^2 n + \nu^2\zeta^2 n + \mu n\left(1 + \sqrt{\tau}\right)^2 \\ &= \nu\zeta^2 n + \mu n\left(1 + \sqrt{\tau}\right)^2 \\ &= \nu\zeta^2 n(2 + 2\sqrt{\tau} + \tau). \end{aligned} \tag{29}$$

Using $a \geq \nu\zeta e$ and $d \geq \nu\zeta e$ it follows that

$$a^T s + d^T x \geq \nu\zeta e^T(x + s) = \nu\zeta(\|x\|_1 + \|s\|_1). \tag{30}$$

Moreover,

$$\|x\|_1 + \|s\|_1 \leq \frac{a^T s + d^T x}{\nu\zeta} = \frac{a^T d + x^T s}{\nu\zeta} \leq \zeta n(2 + 2\sqrt{\tau} + \tau).$$

Since

$$\|x\|^2 + \|s\|^2 \leq (\|x\|_1 + \|s\|_1)^2 \leq \zeta^2 n^2 (2 + 2\sqrt{\tau} + \tau)^2,$$

it follows that $\sqrt{\|x\|^2 + \|s\|^2} \leq \zeta n\,(2 + 2\sqrt{\tau} + \tau)$. *This proves the lemma.* □

In the next lemma we give an upper bound for $\|q\|$, which depends only on θ, n and τ.

Lemma 6.8 *One has*

$$\|q\| < \theta n \frac{2 + 2\sqrt{\tau} + \tau}{1 - \sqrt{\tau}}.$$

Proof. From Lemma 6.7, Lemma 6.6 and $\sqrt{\mu} = \sqrt{\nu}\zeta$ *we get*

$$\sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)} < \frac{\zeta n\,(2 + 2\sqrt{\tau} + \tau)}{\sqrt{\mu}(1 - \sqrt{\tau})} = \frac{n}{\sqrt{\nu}} \frac{(2 + 2\sqrt{\tau} + \tau)}{(1 - \sqrt{\tau})}.$$

From the previous inequality and Lemma 6.4 we obtain

$$\sqrt{\mu}\,\|q\| \leq \theta\nu\zeta\sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)} < \frac{\theta n\sqrt{\nu}\zeta\,(2 + 2\sqrt{\tau} + \tau)}{1 - \sqrt{\tau}}.$$

Since $\sqrt{\mu} = \sqrt{\nu}\zeta$, *the proof of the lemma is complete.* □

Let x^f and s^f be strictly feasible solutions of (P_ν) and (D_ν). Assuming that the value of μ_+ does not change, the vectors x_+ and s_+ can be determined by a full-Newton step at x^f and s^f. Suppose that $\sigma^+ = \sigma(x_+ s_+, \mu_+)$ and we want to show that the algorithm is well defined. We have to specify the values of θ and τ such that after a main iteration the inequality $\sigma^+ < \tau$ holds.

7. Polynomiality of the Algorithm

Now we analyse the consequences of the previous lemmas when $\tau = \frac{1}{16}$ and $\theta = \frac{1}{8n}$.

Corollary 7.1 *If* $\sigma(v) < \frac{1}{16}$ *and* $\theta = \frac{1}{8n}$, *then* $\omega(v) < \frac{1}{2\sqrt{2}}$.

Proof. From Lemma 6.3 and Lemma 6.8 it follows that

$$\begin{aligned} 4\omega(v)^2 &\leq \|q\|^2 + (\|q\| + 2\sigma(v))^2 \\ &< \left(\theta n \frac{2 + 2\sqrt{\tau} + \tau}{1 - \sqrt{\tau}}\right)^2 + \left(\theta n \frac{2 + 2\sqrt{\tau} + \tau}{1 - \sqrt{\tau}} + 2\tau\right)^2. \end{aligned}$$

Using $\tau = \frac{1}{16}$ *and* $\theta n = \frac{1}{8}$ *we get* $4\omega(v)^2 < \frac{1}{2}$, *which implies* $\omega(v) < \frac{1}{2\sqrt{2}}$. □

Corollary 7.2 *If* $\sigma(v) < \frac{1}{16}$ *and* $\theta = \frac{1}{8n}$, *then* x^f *and* s^f *are strictly feasible.*

Proof. *Using Lemma 6.1 and Corollary 7.1 we may write*

$$\sigma(v)^2 + 2\omega(v)^2 < \frac{1}{16^2} + \frac{1}{4} < 1,$$

so $x^f > 0$ *and* $s^f > 0$, *thus* x^f *and* s^f *are strictly feasible.* □

Corollary 7.3 *We can define an upper bound for* $\sigma(v^+)$. *The following inequality holds:*

$$\sigma(v^+) < \frac{1}{4}.$$

Proof. *Using Lemma 6.2 we get*

$$\begin{aligned}\sigma(v^+) &\leq \frac{\theta\sqrt{n} + \sigma(v)^2 + 2\omega(v)^2}{1-\theta+\sqrt{(1-\theta)(1-\sigma(v)^2-2\omega(v)^2)}} \\ &\leq \frac{\theta\sqrt{n} + \tau^2 + 2\omega(v)^2}{1-\theta+\sqrt{(1-\theta)(1-\tau^2-2\omega(v)^2)}}.\end{aligned}$$

From $\tau = \frac{1}{16}$ *and* $\theta = \frac{1}{8n}$, *using* $n \geq 1$, *we obtain* $\sigma(v^+) < \frac{1}{4}$. □

The following corollary defines an upper bound for the proximity measure.

Corollary 7.4 *Let* x_+ *and* s_+ *be the vectors obtained by a full-Newton step at* x^f *and* s^f. *Then* $\sigma^+ = \sigma(x_+ s_+, \mu_+) < \frac{1}{16}$.

Proof. *We use Lemma 3.2 for* (P_ν) *and* (D_ν), *when the initial points are* x^f *and* s^f. □

We call (x, y, s) an ϵ-solution if the following inequality holds

$$max(x^T s, \|b - Ax\|, \|c - A^T y - s\|) < \epsilon.$$

Corollary 7.5 *The algorithm requires at most*

$$8nlog\frac{max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$$

iterations.

Proof. *Since* μ *and* ν *are multiplied by* $1-\theta$ *at each iteration, we can prove that after at most*

$$\frac{1}{\theta}\frac{max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$$

inner iterations the algorithm finds an ϵ*-solution. The proof is similar to the one of Lemma 3.6. Using* $\theta = \frac{1}{8n}$ *we obtain the upper bound given in the corollary.* □

Since every main iteration contains two inner iterations, we get the following theorem.

Theorem 7.6 *If (P) and (D) are feasible and* $\zeta > 0$ *is defined such that* $\|\bar{x} + \bar{s}\|_\infty \leq \zeta$, *where* $\bar{x}$ *and* $(\bar{y}, \bar{s})$ *are the optimal solutions of (P) and (D),* *then after at most*

$$16nlog\frac{max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$$

interior-point iterations the algorithm finds an ϵ-solution of (P) and (D).

8. Numerical experiment

In order to compare the efficiency of infeasible interior-point algorithms with different number of centering steps we have implemented in the C++ programming language a short-step IIPM in the following way. We have used Mehrotra's heuristic [12] followed by a few standard primal-dual steps to define the starting points x^0, y^0 and s^0. In each major iteration the algorithm performs a feasibility step using (10) and one or more centering steps using (4). We calculate the maximum step sizes in order to maintain the nonnegativity of the variables x and s, and we reduce these step sizes with a factor ρ, where $0 < \rho < 1$. As in the case of the algorithm in Figure 2, after the feasibility step we reduce the value of μ and ν by a factor $1 - \theta$, where $0 < \theta < 1$.

The algorithm of Roos [16] performs at most three centering steps in each major iteration. We have solved the problem *afiro*, given in the standard MPS form in the Netlib test collection, using the following parameters: $\epsilon = 0.0001$, $\theta = 0.5$ and $\rho = 0.9999$. The number of major iterations was the same in the case of one centering step and three centering steps. Thus, the algorithm performed 15 major iterations, and a total number of 30 interior-point iterations for one centering step, and 60 iterations for three centering steps. In the algorithm proposed by Roos in each major iteration the centering steps are performed until the proximity measure becomes smaller than the parameter τ. Taking $\tau = 0.25$, the first main iteration of the algorithm performed three centering steps, followed by 4 major iterations with two centering steps and 10 iterations with one centering step. This yields a total number of 36 inner iterations.

9. Conclusion

We have defined a full-Newton step infeasible interior-point algorithm with new search directions. We have applied the square root function for the centering equations and we have used Newton's method in order to get the new search directions. We have proved that the algorithm finds an ϵ-solution in polynomial time. We have shown that in a major iteration it is enough to take only one centering step in order to prove that the algorithm is well defined.

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References

[1] M. Achache. A new primal-dual path-following method for convex quadratic programming. *Computational and Applied Mathematics*, 25(1): 97–110, 2006.

[2] K. Ahmadi, F. Hasani, and B. Kheirfam. A full-Newton step infeasible interior-point algorithm based on Darvay directions for linear optimization. *Journal of Mathematical Modelling and Algorithms in Operations Research*, 2013. doi: 10.1007/s10852-013-9227-7.

[3] J.F. Bonnans and F.A. Potra. Infeasible path-following algorithms for linear complementarity problems. *Mathematics of Operations Research*, 22(2):378–407, 1997.

[4] Zs. Darvay. A new algorithm for solving self-dual linear optimization problems. *Studia Universitatis Babeş-Bolyai, Series Informatica*, 47(1): 15–26, 2002.

[5] Zs. Darvay. New interior point algorithms in linear programming. *Advanced Modeling and Optimization*, 5(1):51–92, 2003.

[6] G. Gu, H. Mansouri, M. Zangiabadi, Y.Q. Bai, and C. Roos. Improved full-Newton step O(nL) infeasible interior-point method for linear optimization. *Journal of Optimization Theory and Applications*, 145(2): 271–288, 2010.

[7] N.K. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4:373–395, 1984.

[8] M. Kojima, N. Megiddo, and S. Mizuno. A primal-dual infeasible-interior-point algorithm for linear programming. *Mathematical Programming*, 61 (1-3):263–280, 1993.

[9] I.J. Lustig. Feasibility issues in a primal-dual interior-point method for linear programming. *Mathematical Programming*, 49(1-3):145–162, 1990.

[10] H. Mansouri and C. Roos. Simplified O(nL) infeasible interior-point algorithm for linear optimization using full-Newton steps. *Optimization Methods and Software*, 22(3):519–530, June 2007.

[11] H. Mansouri, T. Siyavash, and M. Zangiabadi. A path-following infeasible interior-point algorithm for semidefinite programming. *Iranian Journal of Operations Research*, 3(1):11–30, 2012.

[12] S. Mehrotra. On the implementation of a primal-dual interior point method. *SIAM Journal on Optimization*, 2(4):575–601, 1992.

[13] S. Pan, X. Li, and S. He. An infeasible primal-dual interior-point algorithm for linear programs based on logarithmic equivalent transformation. *Journal of Mathematical Analysis and Applications*, 314(2):644–660, 2006.

[14] F. A. Potra. A quadratically convergent predictor-corrector method for solving linear programs from infeasible starting points. *Mathematical Programming*, 67(1-3):383–406, 1994.

[15] F. A. Potra. An infeasible-interior-point predictor-corrector algorithm for linear programming. *SIAM Journal on Optimization*, 6(1):19–32, 1996.

[16] C. Roos. A full-Newton step O(n) infeasible interior-point algorithm for linear optimization. *SIAM Journal on Optimization*, 16(4):1110–1136, 2006.

[17] Gy. Sonnevend. An "analytic center" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming. In A. Prékopa, J. Szelezsán, and B. Strazicky, editors, *System Modelling and Optimization: Proceedings of the 12th IFIP-Conference held in Budapest, Hungay, September 1985*, volume 84 of *Lecture Notes in Control and Information Sciences*, pages 866–876. Springer Verlag, Berlin, West-Germany, 1986.

[18] K. Tanabe. Centered Newton method for linear programming: Interior and 'exterior' point method. In K. Tone, editor, *New Methods for Linear Programming*, volume 3, pages 98–100. 1990. In Japanese.

[19] R.J. Vanderbei. *Linear Programming: Foundations and Extensions.* Kluwer Academic Publishers, Boston, 1996.

[20] G. Q. Wang and Y. Q. Bai. A new full Nesterov-Todd step primal-dual path-following interior-point algorithm for symmetric optimization. *Journal of Optimization Theory and Applications*, 154(3):966–985, 2012.

[21] S.J. Wright. *Primal-Dual Interior-Point Methods.* SIAM, Philadelphia, USA, 1997.

[22] Y. Ye. *Interior Point Algorithms, Theory and Analysis.* John Wiley & Sons, Chichester, UK, 1997.

[23] Y. Zhang. On the convergence of a class of infeasible interior-point methods for the horizontal linear complementarity problem. *SIAM Journal on Optimization*, 4(1):208–227, 1994.

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