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# AN ALGORITHM FOR DETERMINATION OF NASH EQUILIBRIA IN THE INFORMATIONAL EXTENDED TWO-MATRIX GAMES

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Abstract. In this article the informational extended games ${}_1\Gamma$ and ${}_2\Gamma$ are defined. For these informational extended two-matrix games we present two modes for construction of the extended matrices and an algorithm for determination of Nash equilibria. For this algorithm we make some modifications and present an algorithm for determination Nash equilibria in the informational extended two-matrix games in the case, in which the dimensions of the matrices are too big. Using this algorithm we can also determine the number of Nash equilibria in informational extended game, without using of the extended matrices.

Last years the informational aspect represents a real fillip for the elaboration of the new study methods for the non-cooperative game theory. The informational aspect in the game theory is manifested by: the devise of possession information about strategy's choice, the payoff functions, the order of moves, and optimal principles of players; the using methods of possessed information in the strategy's choice by players. The inclusion of information as an important element of game have imposed a new structure to the game theory: the games in complete information (the games in extended form), the games in not complete information and the games in imperfect information (the Bayes games). The player's possession of supplementary information about unfolding of the game can influence appreciably the player's payoffs.

An important element for the players represents the possession of information about the behaviour of his opponents. Thus for the same sets of strategies and the same payoff functions it is possible to obtain different results, if the players have supplementary information. So the information for the players about the strategy's choice by the others players have a significant role for the unfolding of the game.

Let us consider the two-matrix game in the normal form $\Gamma = \langle N, X_1, X_2, A, B\rangle$, where $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $i = \overline{1,m}$, $j = \overline{1,n}$ ($A$ and $B$ are the payoff matrices for the first and the second player respectively. Each player can choose one of his strategies and his purpose is to maximize his payoff. The player can choose his strategy independently of his opponent and the player does not know the chosen strategy of his opponent.

According to [1] we will define the Nash equilibrium.

**Definition 1.** *The pair* $(i^*, j^*)$, $i^* \in X_1, j^* \in X_2$ *is called Nash equilibrium (NE) for the game* $\Gamma$, *if the next relations hold*

$$\begin{cases} a_{i^*j^*} \geqslant a_{ij^*}, \forall i \in X_1, \\ b_{i^*j^*} \geqslant a_{i^*j}, \forall j \in X_2. \end{cases}$$

*Notation:* $(i^*, j^*) \in NE(\Gamma)$.

There are two-matrix games for which the set of the Nash equilibria is empty: $NE(\Gamma) = \emptyset$ (solutions do not exist in pure strategies).

For every two-matrix game we can construct some informational extended games. If one of the players knows the strategy chosen by the other, we consider that it is one form of the informational extended two-matrix game for the initial game. Even if the initial two-matrix game has no solutions in pure strategies, for the informational extended games at least one solution in pure strategies always exists (Nash equilibria). Proof of this assertion see in [2], [3]. In the case of informational extended games the player which knows the chosen strategy of his opponent has one advantage and he will obtain one of his greater payoff.

According to [1], let us define two forms of informational extended games ${}_1\Gamma$ and ${}_2\Gamma$. We consider that for the game ${}_1\Gamma$ the first player knows the chosen strategy of the second player, and for the game ${}_2\Gamma$ the second player knows the chosen strategy of the first player.

If one of the players knows the chosen strategy of the other, then the set of the strategies for this player can be represented by a set of mappings defined on the set of strategies of his opponent.

**Definition 2.** *(The game* ${}_1\Gamma$ according to [1]*) The informational extended two-matrix game* ${}_1\Gamma$ *can be defined in the normal form by:* ${}_1\Gamma = \left\langle N, \overline{X_1}, X_2, \overline{A}, \overline{B} \right\rangle$, *where* $N = \{1, 2\}$, $\overline{X_1} = \{\varphi_1 : X_2 \longrightarrow X_1\}$, $\overline{A} = \{\overline{a}_{ij}\}$, $\overline{B} = \left\{\overline{b}_{ij}\right\}$, $i = \overline{1, m^n}$, $j = \overline{1, n}$.

For the game ${}_1\Gamma$ we have $\overline{X_1} = \{1, 2, \ldots, m^n\}$, $X_2 = \{1, 2, \ldots, n\}$, $\left|\overline{X_1}\right| = m^n$, and the matrices $\overline{A}$ and $\overline{B}$ have dimension $[m^n \times n]$ and are formed from elements of initial matrices $A$ and $B$ respectively.

The matrices $\overline{A}$ and $\overline{B}$ will be constructed in the next mode:

Let us denote by $A_{i\cdot} = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$, $B_{i\cdot} = \{b_{i1}, b_{i2}, \ldots, b_{in}\}$, $i = \overline{1, m}$ the rows $i$ in the matrices $A$ and $B$, respectively).

Choosing one element from each of these rows $A_{1\cdot}, A_{2\cdot}, \ldots, A_{m\cdot}$, we will build one column in the matrix $\overline{A}$. The columns from the matrix $\overline{B}$ are built in the same mode, choosing one element from each of the rows $B_{1\cdot}, B_{2\cdot}, \ldots, B_{m\cdot}$.

Thus, the matrices $\overline{A}$ and $\overline{B}$ have the dimension $[m^n \times n]$.

**Definition 3.** *(The game* ${}_2\Gamma$ according to [1]*) The informational extended two-matrix game* ${}_2\Gamma$ *can be defined in the normal form by:* ${}_2\Gamma = \left\langle N, X_1, \overline{X_2}, \widetilde{A}, \widetilde{B} \right\rangle$, *where* $\overline{X_2} = \{\varphi_2 : X_1 \longrightarrow X_2\}$, $|\overline{X_2}| = n^m$, $\widetilde{A} = \{\widetilde{a}_{ij}\}$, $\widetilde{B} = \left\{\widetilde{b}_{ij}\right\}$, $i = \overline{1,m}$, $j = \overline{1,n^m}$.

For the game ${}_2\Gamma$ we have $X_1 = \{1, 2, \ldots, m\}$, $\overline{X_2} = \{1, 2, \ldots, n^m\}$ and the matrices $\widetilde{A}$ and $\widetilde{B}$ have dimension $[m \times n^m]$ and are formed from elements of initial matrices $A$ and $B$ respectively.

The extended matrices $\widetilde{A}$ *and* $\widetilde{B}$ will be built in analogical mode as in the case of the game ${}_2\Gamma$.

Let us denote by $A_{\cdot j} = \{a_{1j}, a_{2j}, \ldots, a_{mj}\}$, $B_{\cdot j} = \{b_{1j}, b_{2j}, \ldots, b_{mj}\}$, $j = \overline{1,n}$ the columns $j$ in the initial matrices $A$ and $B$, respectively). Each of rows in the matrices $\widetilde{A}$ (or in the matrix $\widetilde{B}$, respectively) will be built choosing one element from each of the columns $A_{\cdot j}$ (or from the columns $B_{\cdot j}$, respectively).

The next theorem represents the condition of the Nash equilibria existence for the informational extended two-matrix games ${}_1\Gamma$ and ${}_2\Gamma$.

**Theorem 1.** *For every two-matrix game* $\Gamma$ *we have the following*

$$NE({}_1\Gamma) \neq \emptyset,\ NE({}_2\Gamma) \neq \emptyset;\ \text{and}\ NE(\Gamma) \subset NE({}_1\Gamma),\ NE(\Gamma) \subset NE({}_2\Gamma).$$

For proof see [2], [3].

For the informational extended games ${}_1\Gamma$ and ${}_2\Gamma$ we can proof the following statements.

**Assertion 1.** *If* $\exists i^* \in X_1$, $\exists j^* \in X_2$ *for which* $a_{i^*j^*} = \max\limits_i \max\limits_j a_{ij}$, $b_{i^*j^*} = \min\limits_i \min\limits_j b_{ij}$ *and* $\forall i \in X_1$, $\forall j \in X_2 : (i,j) \neq (i^*, j^*)$ *so that* $a_{ij} < a_{i^*j^*}$, $b_{ij} > b_{i^*j^*}$; *then:*

*1) in the game* ${}_2\Gamma$ *all columns* $k$ *(from* $\widetilde{A}$ *which contain the element* $a_{i^*j^*}$, *and from* $\widetilde{B}$ *which contain the element* $b_{i^*j^*}$*) do not contain NE equilibria*;

*2) in the game* ${}_1\Gamma$ *the column* $j^*$ *(in the matrices* $\overline{A}$ *and* $\overline{B}$*) do not contains NE equilibria.*

**Assertion 2.** *If* $\exists i^* \in X_1$, $\exists j^* \in X_2$ *so that* $a_{i^*j^*} = \min\limits_i \min\limits_j a_{ij}$ *and* $b_{i^*j^*} = \max\limits_i \max\limits_j b_{ij}$, *and* $\forall i \in X_1$, $\forall j \in X_2 : (i,j) \neq (i^*, j^*)$ *so that* $a_{ij} > a_{i^*j^*}$, $b_{ij} < b_{i^*j^*}$; *then:*

*1) in the game* ${}_2\Gamma$ *the row* $i^*$ *(the* $\widetilde{A}_{i^*\cdot}$ *and the* $\widetilde{B}_{i^*\cdot}$*) does not contain NE equilibria*;

*2) in the game* ${}_1\Gamma$ *all rows* $k$ *(the* $\overline{A}_{k\cdot}$, *and the* $\overline{B}_{k\cdot}$ *which contain the elements* $a_{i^*j^*}$ *and* $b_{i^*j^*}$, *respectively) do not contain NE equilibria.*

From the assertions 1 and 2 the next two statements result.

**Assertion 3.** *Consider that* $\exists i^* \in X_1$, $\exists j^* \in X_2$ *so that* $a_{i^*j^*} = \max\limits_i \max\limits_j a_{ij}$ *and* $b_{i^*j^*} = \min\limits_i \min\limits_j b_{ij}$.

*1) If* $\forall i \in X_1 \backslash \{i^*\}$, $\forall j \in X_2 : a_{ij} < a_{i^*j^*}$, *and* $\forall j \in X_2 \backslash \{j^*\} : b_{i^*j} > b_{i^*j^*}$, *then in the game* ${}_2\Gamma$ *each of columns* $k$ *(*$\widetilde{A}_{\cdot k}$, $\widetilde{B}_{\cdot k}$ *which contains the elements* $a_{i^*j^*}$ *and* $b_{i^*j^*}$, *respectively) does not contain NE equilibria.*

*2) If* $\forall i \in X_1, \forall j \in X_2 \backslash \{j^*\} : b_{ij} > b_{i^*j^*}$ *and* $\forall i \in X_1 \backslash \{i^*\} : a_{ij^*} < a_{i^*j^*}$*, then in the game* ${}_1\Gamma$ *the column* $j^*$ *(*$\overline{A}_{\cdot j^*}$ *and* $\overline{B}_{\cdot j^*}$*) does not contain NE equilibria.*

**Assertion 4.** *Consider that* $\exists i^* \in X_1$, $\exists j^* \in X_2$ *so that* $a_{i^*j^*} = \min_i \min_j a_{ij}$ *and* $b_{i^*j^*} = \max_i \max_j b_{ij}$*.*

*1) If* $\forall i \in X_1 \backslash \{i^*\}, \forall j \in X_2 : a_{ij} > a_{i^*j^*}$*, and* $\forall j \in X_2 \backslash \{j^*\} : b_{i^*j} < b_{i^*j^*}$*, then in the game* ${}_2\Gamma$ *the row* $i^*$ *(*$\widetilde{A}_{i^*\cdot}$*,* $\widetilde{B}_{i^*\cdot}$*) does not contain NE equilibria.*

*2) If* $\forall i \in X_1, \forall j \in X_2 \backslash \{j^*\} : b_{ij} < b_{i^*j^*}$ *and* $\forall i \in X_1 \backslash \{i^*\} : a_{ij^*} > a_{i^*j^*}$*, then in the game* ${}_1\Gamma$ *each of rows k (*$\overline{A}_{k\cdot}$*,* $\overline{B}_{k\cdot}$ *which contains the elements* $a_{i^*j^*}$ *and* $b_{i^*j^*}$*, respectively) does not contain NE equilibria.*

**Example 1.** (For Assertions 2 and 4).

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 5 & 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 7 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}.$$

For this game $NE(\Gamma) = \emptyset$.

For the game ${}_2\Gamma$ there are two Nash equilibria $(2,2),(2,8) \in NE\left({}_2\Gamma\right)$.

For the game ${}_1\Gamma$ there is only one Nash equilibrium $(6,2) \in NE\left({}_1\Gamma\right)$.

In this game, for $i = 1$, $j = 1 : \min_i \min_j a_{ij} = 0$, $\max_i \max_j b_{ij} = 7$. According to Assertion 2 and 4, it follows that: for the game ${}_2\Gamma$ the first row does not contain Nash equilibria and for the game ${}_1\Gamma$ the $1^{st}$, $2^{d}$, $3^{d}$, $4^{th}$ rows do not contain Nash equilibria.

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 3 & 1 & 1 & 1 \\ 5 & \underline{\mathbf{2}} & 4 & 5 & 2 & 4 & 5 & \underline{\mathbf{2}} & 4 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} 7 & 7 & 7 & 3 & 3 & 3 & 6 & 6 & 6 \\ 1 & \underline{\mathbf{5}} & 0 & 1 & 5 & 0 & 1 & \underline{\mathbf{5}} & 0 \end{pmatrix}.$$

$$\overline{A} = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 2 & 4 \\ 5 & 3 & 1 \\ 5 & \underline{\mathbf{3}} & 4 \\ 5 & 2 & 1 \\ 5 & 2 & 4 \end{pmatrix}, \overline{B} = \begin{pmatrix} 7 & 3 & 6 \\ 7 & 3 & 0 \\ 7 & 5 & 6 \\ 7 & 5 & 0 \\ 1 & 3 & 6 \\ 1 & \underline{\mathbf{3}} & 0 \\ 1 & 5 & 6 \\ 1 & 5 & 0 \end{pmatrix}.$$

**Example 2.** (For Assertions 1 and 3).

$$A = \begin{pmatrix} 7 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 3 & 1 \\ 4 & 2 & 5 \end{pmatrix}.$$

For this game $NE(\Gamma) = \emptyset$, and for the informational extended games there are some solutions $(1,4),(1,6) \in NE\left({}_2\Gamma\right), (3,2) \in NE\left({}_1\Gamma\right)$.

In this game, for $i = 1$, $j = 1 : \max_i \max_j a_{ij} = 7$, $\min_i \min_j b_{ij} = 0$. According to Assertions 1 and 3, it follows that: for the game ${}_2\Gamma$ the $1^{st}$, $2^{d}$, $3^{d}$ columns do not contain Nash equilibria and for the game ${}_1\Gamma$ the first column does not contain Nash equilibria.

$$\widetilde{A} = \begin{pmatrix} 7 & 7 & 7 & \underline{\mathbf{3}} & 3 & \underline{\mathbf{3}} & 6 & 6 & 6 \\ 1 & 5 & 0 & 1 & 5 & 0 & 1 & 5 & 0 \end{pmatrix} \quad \widetilde{B} = \begin{pmatrix} 0 & 0 & 0 & \underline{\mathbf{3}} & 3 & \underline{\mathbf{3}} & 1 & 1 & 1 \\ 4 & 2 & 5 & 4 & 2 & 5 & 4 & 2 & 5 \end{pmatrix} \quad \overline{A} = \begin{pmatrix} 7 & 3 & 6 \\ 7 & 3 & 0 \\ 7 & \underline{\mathbf{5}} & 6 \\ 7 & 5 & 0 \\ 1 & 3 & 6 \\ 1 & 3 & 0 \\ 1 & 5 & 6 \\ 1 & 5 & 0 \end{pmatrix}, \ \overline{B} = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 3 & 5 \\ 0 & \underline{\mathbf{2}} & 1 \\ 0 & 2 & 5 \\ 4 & 3 & 1 \\ 4 & 3 & 5 \\ 4 & 2 & 1 \\ 4 & 2 & 5 \end{pmatrix}.$$

**Example 3.** (For Assertions 2 and 4).

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 6 \\ 3 & 1 \\ 1 & 4 \end{pmatrix}, NE(\Gamma) = \emptyset.$$

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & \underline{\mathbf{0}} & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & \underline{\mathbf{0}} & 2 & 2 \\ 4 & 0 & 4 & 0 & 4 & \underline{\mathbf{0}} & 4 & 0 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} 2 & 2 & 2 & 2 & 6 & \underline{\mathbf{6}} & 6 & 6 \\ 3 & 3 & 1 & 1 & 3 & \underline{\mathbf{3}} & 1 & 1 \\ 1 & 4 & 1 & 4 & 1 & \underline{\mathbf{4}} & 1 & 4 \end{pmatrix}.$$

In this game, for the pairs $(i^*, j^*) : (1,2), (2,1), (3,2)$ we have $\min_i \min_j a_{ij} = 0$, and for each row $\max_j b_{ij} = b_{i^*j^*}$, but because each of rows from the matrix $A$ contains the minimum element $a_{12} = 0$, for the $6^{th}$ column the conditions from assertions 2 and 4 do not hold , and $(1,6), (2,6), (3,6) \in NE({}_2\Gamma)$. □

For the generation of the extended matrices $\overline{A}$ and $\overline{B}$ (or the $\widetilde{A}$ and the $\widetilde{B}$, respectively) we can use the next methods.

**The first method** is based on representation of decimal numbers in the base which represent the number of rows or the number of columns in the initial matrices.

For the game ${}_1\Gamma$ we need to represent the numbers $0, 1, \ldots, (m^n - 1)$ in the base $m$ with $n$ components: $N_m = (C_0C_1 \ldots C_{n-1})_m$, where $C_j \in \{0, 1, \ldots, m-1\}$, $j = \overline{0, n-1}$, that is $\left(C_0m^0 + C_1m^1 + \ldots + C_{n-1}m^{n-1}\right) = N_{10}$. Each of these numbers $N_m$ represented in the base $m$ will correspond to one column in the extended matrix.

Then for elements from column $j$ it must replace:

$0 \to a_{1j}, 1 \to a_{2j}, \ldots, i \to a_{(i+1)j}, \ldots, (m-1) \to a_{mj}$ (similarly for the matrix $B$).

For the game ${}_2\Gamma$ it must represent the numbers $0, 1, \ldots, (n^m - 1)$ in the base $n$ with $m$ components: $N_n = (C_0C_1 \ldots C_{m-1})_n$, where $C_i \in \{0, 1, \ldots, n-1\}$, $i = \overline{0, m-1}$, that is $\left(C_0n^0 + C_1n^1 + \ldots + C_{m-1}n^{m-1}\right) = N_{10}$. Each of these numbers $N_n$ represented in the base $n$ will correspond to one row into the extended matrix.

Then for the elements from the row $i$ it must replace:

$0 \to a_{i1}, 1 \to a_{i2}, \ldots, j \to a_{i(j+1)}, \ldots, (n-1) \to a_{in}$ (similarly for the matrix $B$).

**The second method** consists in assigning two numbers to each of the elements from the initial matrices. One of these numbers represents the number of blocks (series) formed by this element, and the second number represents the length of the block (that is, the number of repetitions of this element in the block).

Denote by $nrbl$ the number of blocks for some element $a_{ij}$ ($b_{ij}$) and by $L$ the length of each of blocks (the number of repetitions of this element in the block).

Thus for the game ${}_2\Gamma$ assign to each element from the row $i$: ($n^{i-1}$) blocks (series) each of them with length ($n^{m-i}$).

So for all elements $a_{ij}$, $b_{ij}$, $i = \overline{1,m}, j = \overline{1,n}$ we can determine the indices of columns $k$ of this element in the extended matrix. Thus for the element from the row $i$ and from the column $j$ and for all $nrbl = \overline{1, n^{i-1}}$, $L = \overline{1, n^{m-i}}$, we calculate the number $k$ by:

$$k = n \cdot n^{m-i} \cdot (nrbl - 1) + (j - 1) \cdot n^{m-i} + L. \tag{1}$$

In such mode we can construct the extended matrices $\widetilde{A}$ and $\widetilde{B}$: $\widetilde{A}[i,k] = A[i,j]$, $\widetilde{B}[i,k] = B[i,j]$.

Similarly, for the game ${}_1\Gamma$ we assign to each element from the column $j$: ($m^{j-1}$) blocks (series) each of them with length ($m^{n-j}$).

Thus for all elements $\forall i = \overline{1,m}, j = \overline{1,n}$, we determine the indices of the rows $k$ of this element in the extended matrix.

In such mode for the element from the row $i$ and from the column $j$ and for all $nrbl = \overline{1, m^{j-1}}$, $L = \overline{1, m^{n-j}}$ we calculate the number $k$ by:

$$k = m \cdot m^{n-j} \cdot (nrbl - 1) + (i - 1) \cdot m^{n-j} + L. \tag{2}$$

In such mode, we can construct the extended matrices $\overline{A}$ and $\overline{B}$ (for each determined $k$): $\overline{A}[k,j] = A[i,j]$, $\overline{B}[k,j] = B[i,j]$.

**Remark.** These two different methods may be used independently. Using it we can construct the extended matrices entirely or partly. If the initial matrices are very big, we can use these methods for partial construction of the extended matrices. Thus the first method may be used when we need to construct only one row (for the informational extended game ${}_1\Gamma$), or only one column (for the game ${}_2\Gamma$), and the second method may be used when we need to determine the position of some element in the extended matrix, i.e. the index of the row (in the game ${}_1\Gamma$) or the index of the column (in the game ${}_2\Gamma$, respectively).

**Example 4.** (The generation of the extended matrices).

$A = \begin{pmatrix} 0 & 3 & 1 \\ 5 & 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}$ $m = 2$, $n = 3$.

For the **first** method:

For the game ${}_1\Gamma$ the matrices are of dimension $\left[2^3 \times 3\right]$. We construct the $5^{th}$ row from the extended matrix $\overline{A}$:

$4_{10} = (100)_2$, next we do the substitution with corresponding elements and we obtain the $5^{th}$ row with elements (5,3,1).

In the same mode we can construct the $8^{th}$ row: $7_{10} = (111)_2$ and we obtain the row (1,5,0) from the extended matrix $\overline{B}$.

For the game ${}_2\Gamma$ the matrices are of dimension $\left[2 \times 3^2\right]$. We construct the $6^{th}$ column:

$5_{10} = (12)_3$, next we do the substitution with corresponding elements and we obtain the $6^{th}$ column: (3,4) from the extended matrix $\widetilde{A}$ and the $6^{th}$ column (3,0) from the matrix $\widetilde{B}$.

In the same mode we can construct the $9^{th}$ column: $8_{10} = (22)_3$ and we obtain the columns (1,4) and (6,0) from the extended matrices ($\widetilde{A}$ and $\widetilde{B}$, respectively).

For the **second** method:

For the same game we determine the positions in the extended matrices for the elements $a_{21} = 5$ and $b_{21} = 1$.

For the game ${}_1\Gamma$ in the first column will contain one series ($2^0$ blocks) which will have $2^2$ elements; the indices of rows are $k = 5, 6, 7, 8$.

For the game ${}_2\Gamma$ in the second row will contain ($3^1$) series (blocks) and each of them will have one element (i. e. $3^0$ elements); the indices of columns are $k = 1, 4, 7$. □

Using these methods we can construct an algorithm for determination of the NE equilibrium. This algorithm does not need the integral construction of the extended matrices, and need only the partial construction of them.

Thus in the case when the dimension of the initial matrices $A$ and $B$ are very big we avoid using a big volume of memory, since the extended matrices will have a bigger dimensions ($[m \times n^m]$ and $[m^n \times n]$, respectively).

The following algorithm can be used for determination of Nash equilibria in the informational extended two-matrix games ${}_1\Gamma$ and ${}_2\Gamma$.

**Algorithm.**

Consider the extended game ${}_2\Gamma$.

Using the first method we represent the numbers from 0 to $(n^m - 1)$ in the base $n$. Each of these representations will correspond to one column in the extended matrix $\widetilde{A}$. For each of these representations it must make the substitutions with the corresponding elements from the initial matrix $A$.

For each column $j_0 = \overline{1, n^m}$, obtained in such mode, from the extended matrix $\widetilde{A}$ we will do the next operations.

1. We determine the maximum element from this column of the extended matrix $\widetilde{A}$, and the corresponding element with the same indices from the matrix $\widetilde{B}$; let them $\widetilde{a}_{i_0 j_0}$ and $\widetilde{b}_{i_0 j_0}$.

2. We determine the maximum element from the row $i_0$ in the initial matrix $B$: let it be $b_{i_0 j^*}$.

3. If $\widetilde{b}_{i_0 j_0} = b_{i_0 j^*}$, .then $(i_0, j_0)$ is $NE$ equilibrium for the extended game ${}_2\Gamma$ : $(i_0, j_0) \in NE\left({}_2\Gamma\right)$, and the elements $\widetilde{a}_{i_0 j_0}$ and $\widetilde{b}_{i_0 j_0}$ will be the payoff's values for the first and for the second player respectively.

For the informational extended game ${}_1\Gamma$ we can construct the algorithm in the same mode.

Consider now the extended game ${}_1\Gamma$.

Using the first method we represent the numbers from 0 to $(m^n - 1)$ in the base $m$. For each of these representations it must do the substitutions with the corresponding elements from the initial matrix $B$. Each of these representations will correspond to one row in the extended matrix $\overline{B}$.

For each row $i_0$ $(i_0 = \overline{1, m^n})$ from the matrix $\overline{B}$ (thus obtained) we will do the next operations.

1. We determine the maximum element from this row of the extended matrix $\overline{B}$, and the corresponding element with the same indices from the matrix $\overline{A}$; let them be $\overline{b}_{i_0 j_0}$ and $\overline{a}_{i_0 j_0}$.

2. We determine the maximum element from the column $j_0$ in the initial matrix $A$ : let's consider this element $a_{i^* j_0}$.

3. If $\overline{a}_{i_0 j_0} = a_{i^* j_0}$, then $(i_0, j_0)$ is $NE$ equilibrium for the extended game ${}_1\Gamma$ : $(i_0, j_0) \in NE\,({}_1\Gamma)$, and the elements $\overline{a}_{i_0 j_0}$ and $\overline{b}_{i_0 j_0}$ will be the payoff's values for the first and for the second player respectively.

**Example 5.**

$$A = \begin{pmatrix} 2 & 5 \\ \underline{\mathbf{4}} & 1 \\ 3 & 7 \end{pmatrix}, B = \begin{pmatrix} 5 & 9 \\ \underline{\mathbf{2}} & 1 \\ 6 & 4 \end{pmatrix}.$$

This game has only one Nash equilibrium.

We can determine the Nash equilibria without using the extended matrices.

For the game ${}_2\Gamma$ we need to represent the numbers from 0 to $8 = 2^3$ in the base 2.

For the first column: $0_{10}$ =$(0,0,0)_2$ we do the substitution with corresponding elements (2,4,3), $\max\{2, 4, 3\} = 4 = a_{21}$, and the corresponding element $b_{21}$ is the maximum element from the second row from the matrix $B$, thus follows that: $(2, 1) \in NE({}_2\Gamma)$;

- for the second column : $1_{10}$ =$(0,0,1)_2$ the corresponding elements are (2,4,7), for which $\max\{2, 4, 7\} = 7 = a_{32}$, but the corresponding element $b_{32} \neq \max\{6, 4\}$ from the third row of the matrix $B$, so $(3, 2) \notin NE({}_2\Gamma)$;

- for the third column $2_{10}$=$(0,1,0)_2$ for which $\max\{2, 1, 3\} = 3 = a_{31}$ we have $b_{31} = \max\{6, 4\}$, thus $(3, 3) \in NE({}_2\Gamma)$;

- for the $5^{th}$ column $4_{10}$=$(1,0,0)_2$ we have $\max\{5, 4, 3\} = 5 = a_{12}$ and $b_{12} = \max\{5, 9\}$, so it follows that $(1, 5) \in NE({}_2\Gamma)$;

- for the $7^{th}$ column $6_{10}$ =$(1,1,0)_2$ we have $\max\{5, 1, 3\} = 5 = a_{12}$ and $b_{12} = \max\{5, 9\}$, so $(1, 7) \in NE({}_2\Gamma)$.

If we will build the extended matrices, we will see that for the informational extended game ${}_2\Gamma$ there are only four Nash equilibria.

$$\begin{pmatrix} 2 & 2 & 2 & 2 & \underline{\mathbf{5}} & 5 & \underline{\mathbf{5}} & 5 \\ \underline{\mathbf{4}} & 4 & 1 & 1 & 4 & 4 & 1 & 1 \\ 3 & 7 & \underline{\mathbf{3}} & 7 & 3 & 7 & 3 & 7 \end{pmatrix}, \qquad \begin{pmatrix} 5 & 5 & 5 & 5 & \underline{\mathbf{9}} & 9 & \underline{\mathbf{9}} & 9 \\ \underline{\mathbf{2}} & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 6 & 4 & \underline{\mathbf{6}} & 4 & 6 & 4 & 6 & 4 \end{pmatrix}.$$

If we need to determine the indices of the columns in the extended matrices in the game ${}_2\Gamma$ for the elements $a_{21}$, $b_{21}$, and we know that $(2,1) \in NE(\Gamma)$, we can use relation (1) from the second method. So in this case indices of columns are $k = 1, 2, 5, 6$, but only one of these columns contains $NE$ equilibrium $(2,1) \in NE({}_2\Gamma)$. □

**Remark.** In the case when the numbers $n^m$ and $m^n$ are very big this algorithm for determination of $NE$ equilibria for the informational extended games and the generation methods of the extended matrices are more complex. But all these operations can be executed operating with the corresponding numbers represented in the base $m$ or $n$ respectively to the informational extension (${}_1\Gamma$ or ${}_2\Gamma$ respectively).

**The operating with numbers represented in the base $n$.**

Consider the informational extended game ${}_2\Gamma$.

For the game ${}_2\Gamma$ the extended matrices will have dimensions $[m \times n^m]$ (by definition).

According to the second method, to each element from the row $i$ two numbers correspond : $nrbl = n^{i-1}$ of blocks, each of them have the length $L = n^{m-i}$.

The relation (1) used in the second method for the game ${}_2\Gamma$ can be written in the next form:

$$k = n^{m-i} \cdot (n \cdot nrbl - n + (j-1)) + L. \tag{3}$$

We will represent all numbers from the relation (3) in the base $n$ with $m$ components:

$n = \left(00\ldots0\underset{2}{1}0\right)_n$;

$n^{i-1} = N_n = \left(0\ldots0\underset{i}{1}0\ldots\underset{1}{0}\right)_n$, $i = \overline{1,m}$;

$n^{m-i} = N_n = \left(0\ldots0\underset{m-i+1}{1}0\ldots\underset{1}{0}\right)_n$, $i = \overline{1,m}$;

the number of blocks is determined by: $nrbl = \overline{1, n^{i-1}}$, so

$nrbl = (00\ldots01)_n, \ldots, \left(0\ldots0\underset{i}{1}0\ldots\underset{1}{0}\right)_n$;

the length of blocks is determined by: $L = \overline{1, n^{m-i}}$, thus

$L = (00\ldots01)_n, \ldots, \left(0\ldots0\underset{m-i+1}{1}0\ldots\underset{1}{0}\right)_n$.

Using the relation (3) all operations can be done, operating with numbers represented in the base $n$.

Thus, using in the relation (3) the numbers represented in the base $n$, we determine $k$.

All arithmetic operations (*,+,-) will be executed in the base $n$.

**Remark.** The operation "*" in the base $n$ for one number with other number in the form $\left(0\ldots0\underset{i+1}{1}0\ldots\underset{1}{0}\right)_n = n^i$ is equivalent to moving to the left with $i$ positions of the components from the first number (so add $i$ zeroes to the right).

**Remark.** The operations (+,-) for two numbers in the base $n$ are done according to the well-known rules characteristic for the base 10.

**Example 6.**

Consider that the game $\Gamma$ have matrices of dimension $[6 \times 6]$, i. e. $m = 6, n = 6$, and we need to determine the index of the column $k$ for the elements $a_{25}$ and $b_{25}$ in the extended matrices for the game ${}_2\Gamma$ (i. e. $i = 2$, $j = 5$), $m-i+1 = 5$; it's known that for the number of blocks (series) holds next $(1 \leqslant nrbl \leqslant n^{i-1} = n)$, so we have $nrbl = (0...01)_6, ..., (0...010)_6$ in the base 6, and $n^{m-i} = (010000)_6$. Consider that $nrbl = 000005$ and $L = (015355)_6$. Using the relation (3), all operations can be done operating with numbers represented in the base 6 :

$$
\begin{array}{rl}
000005 & = nrbl \\
*\underline{000010} & = n \\
000050 & \\
+\underline{000004} & = j-1 \\
000054 & \\
-\underline{000010} & = n \\
000044 & \\
*\underline{010000} & = n^{m-i} \\
440000 & \\
+\underline{015355} & = L \\
455355 & = k
\end{array}
$$

Thus, we just have obtained one of the indices (represented in the base 6: $k = 455355$) of the columns for the elements $a_{25}$, $b_{25}$ in the extended matrices for the game ${}_2\Gamma$.

**Remark.** In this algorithm we can do operations in other order for determination Nash equilibria in the informational extended games ${}_1\Gamma$, ${}_2\Gamma$. Using this modified algorithm, we can determine also the number of Nash equilibria in the games ${}_1\Gamma$, ${}_2\Gamma$, without using of the extended matrices. Thus for the game ${}_1\Gamma$, (${}_2\Gamma$) firstly we determine the maximum payoff for the first (second) player and the corresponding strategy for this maximum element; then we determine the corresponding combinations for that we obtain the maximum payoff and the corresponding strategy for the second (first) player, respectively.

In this way for the game ${}_1\Gamma$, firstly we can determine the maximum elements for the first player, and for corresponding elements we determine if exist some combinations in the matrix of the second player for that we have Nash equilibria.

**The modified algorithm.**

For the game ${}_1\Gamma$, we determine the maximum element in each column from the matrix $A$, i. e. $a_{i_j j} = \max\limits_i \{a_{1j}, a_{2j}, \ldots, a_{mj}\}$, for $\forall j = \overline{1, n}$.

For each element $a_{i_j j}$, $j = \overline{1,n}$ thus obtained, we determine the corresponding elements with the same indices from the matrix $B : b_{i_j j}$, $j = \overline{1,n}$.

For each of these pairs $a_{i_j j}, b_{i_j j}, \left(j = \overline{1,n}\right)$ we determine if these values can be the payoffs for players for some Nash equilibria.

Thus if $\forall k \in X_2 \backslash \{j\}\ \exists b_{ik} : b_{ik} \leqslant b_{i_j j}$, then the pair $a_{i_j j}, b_{i_j j}$ can be the payoffs for players for some Nash equilibria in the game ${}_1\Gamma$; consider this pair $a_{i^* j^*}, b_{i^* j^*}$.

It is possible that for the pair $a_{i^* j^*}, b_{i^* j^*}$ there are many Nash equilibria.

If we wish to determine how many Nash equilibria there are in the game ${}_1\Gamma$ for the pair $a_{i^* j^*}, b_{i^* j^*}$ we determine the number of elements which there are in each column $k \in X_2 \backslash \{j\}$ from the matrix $B$ for that $b_{ik} \leqslant b_{i^* j^*}$. Denote by $n_j$, $j = \overline{1,n}$ the number of elements $b_{ij}$ from the column $j$ for that $b_{ij} \leqslant b_{i^* j^*}$, and for $j^*$ we have $n_{(j^*)} = 1$.

Then the number of Nash equilibria for that the players will have the payoff $a_{i^* j^*}$ and $b_{i^* j^*}$, respectively, can be determined by:

$$N_{j^*} = n_1 \cdot n_2 \cdot \ldots \cdot n_{(j^*-1)} \cdot 1 \cdot n_{(j^*+1)} \cdot \ldots n_n, \tag{4}$$

And the number of all Nash equilibria in the game ${}_1\Gamma$ can be determined by: $N = \sum\limits_j N_j$.

If the pair of elements $a_{i_j j}, b_{i_j j}$ can be the payoffs of the players for some Nash equilibrium in the informational extended game ${}_1\Gamma$, then $j$ will be the strategy for the second player. And because $\overline{X_1} \neq X_1$, we have to determine the strategy for the first player, for which the elements $a_{i^* j^*}, b_{i^* j^*}$ will correspond to one Nash equilibrium.

In this way we determine the elements $b_{i_1 1}, b_{i_2 2}, \ldots, b_{i_j j}, \ldots, b_{i_n n}$, for that $b_{i_k k} \leqslant b_{i_j j}$, $\forall k \in X_2 \backslash \{j\}$.

Then using the indices of the rows of these elements, we can determine the strategy for the first player by:

$$i' = (i_1 - 1)\, m^{n-1} + (i_2 - 1)\, m^{n-2} + \ldots + (i_j - 1)\, m^{n-j} + \ldots + (i_n - 1)\, m^0 + 1. \tag{5}$$

So, the pair $i', j$ is Nash equilibrium for the informational extended game ${}_1\Gamma$ : $(i', j) \in NE\left({}_1\Gamma\right)$.

Similarly, for the game ${}_2\Gamma$ , we can determine the strategy for the second player by:

$$j' = (j_1 - 1)\, n^{m-1} + (j_2 - 1)\, n^{m-2} + \ldots + (j_i - 1)\, n^{m-i} + \ldots + (j_m - 1)\, n^0 + 1, \tag{6}$$

where the indices $j_i$ $\left(i = \overline{1,m}\right)$ are determined by the indices of columns of the elements $b_{ij_i} = \max\limits_j \{b_{i1}, b_{i2}, \ldots, b_{in}\}$, $\forall i = \overline{1,m}$.

**Example 7.**

$$A = \begin{pmatrix} \underline{\mathbf{9}} & 2 & 6 & 0 \\ 2 & \underline{\mathbf{7}} & 7 & 2 \\ 5 & 4 & \underline{\mathbf{9}} & \underline{\mathbf{5}} \\ 3 & 5 & 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 5 & 3 & \underline{\mathbf{9}} \\ \underline{\mathbf{8}} & 2 & 5 & 7 \\ \underline{\mathbf{7}} & 5 & 4 & 1 \\ 2 & 3 & 1 & \underline{\mathbf{4}} \end{pmatrix}.$$

For this game $NE(\Gamma) = \emptyset$. For the informational extended games ${}_1\Gamma$, ${}_2\Gamma$ the extended matrices will have the dimension $[256 \times 4]$ and $[4 \times 256]$, respectively.

For the game ${}_1\Gamma$ we determine the maximum elements in each column from the matrix $A$, and for the corresponding elements we determine if there are some combinations in the matrix $B$ such that the pair $(a_{i^*j^*}, b_{i^*j^*})$ will be the payoffs for the players.

So, the pair $(a_{11}, b_{11}) = (9, 3)$ will be the payoffs for the players, and the strategy for the second player will be $j^* = 1$.

We determine the combination of elements for which we have NE in the game ${}_1\Gamma : (b_{11}, b_{22}, b_{13}, b_{34}) = (3, 2, 3, 1)$, for that

$i' = (i_1 - 1)\, 4^3 + (i_2 - 1)\, 4^2 + (i_3 - 1)\, 4^1 + (i_4 - 1)\, 4^0 + 1 = 0 + 1 \cdot 4^2 + 0 + 2 \cdot 4^0 + 1 = 19$, so $(19, 1) \in NE\left({}_1\Gamma\right)$.

For the pair $(a_{11}, b_{11}) = (9, 3)$ we have $\{(19, 1), (31, 1), (51, 1), (63, 1)\} \in NE\left({}_1\Gamma\right)$.

Similarly, for the pair $(a_{33}, b_{33}) = (9, 4)$ we obtain
$\{(27, 3), (28, 3), (59, 3), (60, 3), (219, 3), (220, 3), (251, 3), (252, 3)\} \in NE\left({}_1\Gamma\right)$;

for the pair $(a_{22}, b_{22}) = (7, 2)$ we obtain $\{(223, 2)\} \in NE\left({}_1\Gamma\right)$.

Thus, in the game ${}_1\Gamma$ there are 13 Nash equilibria.

Similarly, for the game ${}_2\Gamma$ we can determine the set of Nash equilibria.

In this case for the pair $(a_{31}, b_{31}) = (5, 7)$ we obtain the follow Nash equilibria: $(3, 65)$, $(3, 66)$, $(3, 67)$, $(3, 68)$, $(3, 113)$, $(3, 114)$, $(3, 115)$, $(3, 116)$, $(3, 193)$, $(3, 194)$, $(3, 195)$, $(3, 196)$, $(3, 241)$, $(3, 242)$, $(3, 243)$, $(3, 244)$ in the game ${}_2\Gamma$.

Thus, in the game ${}_2\Gamma$ there are 16 Nash equilibria.

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