STUDIA UNIV. BABEŞ–BOLYAI, INFORMATICA, Volume **XLVII**, Number 1, 2002

BL-ALGEBRA STRUCTURE OF RGB MODEL

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Abstract. The aim of this paper is to construct a BL-Algebra structure over RGB Model of colours. First we determin a method to obtain $t-$norms on RGB Model and their associate residuum from $t-$norms on unit interval and their associate residuum. These triangular norms and their associate residuums are used to construct the BL-Algebra structure over RGB model.
Keywords: RGB Model, T-Norms, BL-Algebras, Many-valued logic

Introduction

In a recent paper [6] V. Loia and S. Sessa studied the compression and decompression of gray scale images using fuzzy relations in the Basic Logic over $[0, 1]$.

In this paper we develop an algebraic structure on RGB model of colours [5] that will lead us to consider compression and decompression of colour images. Therefore, we consider that a colour of RGB model is an element of the set

$$RGB = \{(r, g, b) \mid r, g, b \in [0, 2^n - 1]\},$$

where n represents the number of bits on which one colour component is stored in computer's memory.

In RGB set, r, g and b represents the red, green and respectively blue component of the colour. For more details about RGB set see [7].

To construct the BL-Algebra over RGB we have to determin the t-norm and its associate residuum over RGB.

We have to remind now, some notions used later.

A *triangular norm* t on real unit interval (for short $t-norm$) [1], [3] is a binary operation,

$$t : [0, 1]^2 \rightarrow [0, 1],$$

such that t is commutative, associative, non-decreasing in both arguments and $t(0, x) = 0$, $t(x, 1) = x$ for any $x \in [0, 1]$. For brevity, we put $t(x, y) = xty$ for all $x, y \in [0, 1]$.

The *residuum* of the t-norm t [2], [3] is a unique operation $x \rightarrow_t y$ defined as

$$(x \rightarrow_t y) = \max\{z \in [0, 1] \mid xtz \leq y\}$$

2000 *Mathematics Subject Classification.* 03B50, 03G10, 08A70, 08A72.

1998 *CR Categories and Descriptors.* I.3.2 [**Computing Methodologies**]: Computer Graphics – *Graphic Systems*.

such that $xtz \leq y$ if and only if $z \leq (x \rightarrow_t y)$.

The following are the most famous t-norms used in fuzzy logic with the associate residuum:

- Lukasiewicz t-norm

$$xty = \max\{0, x + y - 1\},$$

$$(x \rightarrow_t y) = \min\{1, 1 - x + y\}.$$

- Gödel t-norm

$$xty = \min\{x, y\},$$

$$(x \rightarrow_t y) = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ if } x > y \end{cases}.$$

- Goguen t-norm

$$xty = x \cdot y,$$

$$(x \rightarrow_t y) = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ if } x > y \end{cases}.$$

The following properties hold:

$$x \wedge y = xt(x \rightarrow_t y); \tag{0.1}$$

$$(x \rightarrow_t y) \vee (y \rightarrow_t x) = 1. \tag{0.2}$$

Following [4], a $BL - Algebra$ $L = (L, \wedge, \vee, \star, \rightarrow, 0, 1)$ is an algebra with four binary operations such that for all $x, y, z \in L$:

1. $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
2. $(L, \star, 1)$ is a commutative semigroup having 1 as unit element;
3. $x \wedge y = x \star (x \rightarrow y)$ (divisibility);
4. $z \leq (x \rightarrow y)$ if and only if $x \star z \leq y$ (residuation because of 3., i.e. $(x \rightarrow y) = \vee\{z \in L \mid x \star z \leq y\}$);
5. $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

The properties (0.1), (0.2) guarantee that $L_t = ([0,1], \min, \max, t, \rightarrow_t, 0, 1)$ are important examples of BL-algebras over $[0,1]$, where t is a continuous t-norm with its residuum $\rightarrow_t$.

1. T-Norms defined on RGB set

In this section we will develop a method to obtain from each t-norm defined on unit interval $[0,1]$, a t-norm defined on RGB set.

We start with the construction of the t-norm and its associate residuum on one component of RGB set. Applying this t-norm and its associate residuum on each component we determin the t-norm and its associate residuum on RGB set.

Let $[0,b]$ interval, where $b = 2^n - 1$, be the set of values of one component of RGB set.

Let us consider a one to one onto mapping $f : [0,1] \to [0,b]$, defined as follows:

$$f(x) = b \cdot x,$$

for any $x \in [0,1]$.

It is easy to see that its inverse mapping is $f^{-1} : [0,b] \to [0,1]$, defined as follows:

$$f^{-1}(x) = \frac{x}{b},$$

for any $x \in [0,b]$.

Let us consider a t-norm t and a binary operation $T : [0,b]^2 \to [0,b]$, defined for any $x, y \in [0,b]$ as follows:

$$xTy = f\left(f^{-1}(x)\, t f^{-1}(y)\right). \tag{1.1}$$

Lemma 1.1. *The binary operation T defined as above is a t-norm defined on $[0,b]$ interval.*

Proof. It is easy to see that $f^{-1}(x) \in [0,1]$ for any $x \in [0,b]$. It follows that

$$f^{-1}(x)\, t f^{-1}(y) \in [0,1]$$

and fulfills all the properties of a $t-$norms.

Since f is a linear mapping it follows that

$$f\left(f^{-1}(x)\, t f^{-1}(y)\right) \in [0,b]$$

and fullfills all the properties of a $t-$norm.

Therefore T is a $t-$norm on $[0,b]$ interval. ■

In what follows we will develop a similar method to obtain the associate residuum of the $t-$norm defined on $[0,b]$ interval.

The residuum of the $t-$norm T defined on $[0,b]$ interval is the unique operation defined for all $x,\ y \in [0,b]$ as follows:

$$x \to_T y = \max\{z \in [0,b] \mid xTz \le y\}.$$

Lemma 1.2. *Let T be a $t-$norm defined on $[0,b]$ interval. The binary operation defined for all $x,\ y \in [0,b]$ as follows:*

$$x \to_T y = f\left(f^{-1}(x) \to_t f^{-1}(y)\right) \tag{1.2}$$

is the residuum of $t-$norm T, where t represents the $t-$norm defined on unit interval from which we have obtained the $t-$norm T as in (1.1).

Proof. Let $x,\ y \in [0,b]$. Since f and f^{-1} are non-decreasing functions we have:

$$x \rightarrow_T y = \max\left\{z \in [0,b] \mid xTz \leq y\right\}$$

$$= f\left(\max\left\{u \in [0,1] \mid f^{-1}(x)\, tu \leq f^{-1}(y)\right\}\right)$$

$$= f\left(f^{-1}(x) \rightarrow_t f^{-1}(y)\right).$$

This completes the proof. ■

The equations (1.1) and (1.2) are used now to determine the form of the most famous $t-$norms and of their associate residuums.

Example 1.3. Lukasiewicz type $t-$norm defined on $[0,b]$ interval:

$$xTy = f\left(\max\left\{0, f^{-1}(x) + f^{-1}(y) - 1\right\}\right)$$

$$= b \cdot \max\left\{0, \frac{x}{b} + \frac{y}{b} - 1\right\}$$

$$= \max\left\{0, x + y - b\right\};$$

$$x \rightarrow_T y = f\left(\min\left\{1, 1 - f^{-1}(x) + f^{-1}(y)\right\}\right)$$

$$= b \cdot \min\left\{1, 1 - \frac{x}{b} + \frac{y}{b}\right\}$$

$$= \min\left\{b, b - x + y\right\}.$$

Example 1.4. Gödel type $t-$norm defined on $[0,b]$ interval:

$$xTy = f\left(\min\left\{f^{-1}(x), f^{-1}(y)\right\}\right)$$

$$= b \cdot \min\left\{\frac{x}{b}, \frac{y}{b}\right\}$$

$$= \min\left\{x, y\right\};$$

$$x \rightarrow_T y = f\left(f^{-1}(x) \rightarrow_t f^{-1}(y)\right)$$

Since f^{-1} is not a decreasing function it follows that:

(i) if $x \leq y \Rightarrow f^{-1}(x) \leq f^{-1}(y)$. Therefor we get:

$$f^{-1}(x) \rightarrow_t f^{-1}(y) = 1.$$

Using equation (1.2) we obtain:

$$x \rightarrow_T y = f\left(f^{-1}(x) \rightarrow_t f^{-1}(y)\right)$$

$$= f(1) = b.$$

(ii) if $x > y \Rightarrow f^{-1}(x) > f^{-1}(y)$. Therefor we get:

$$f^{-1}(x) \rightarrow_t f^{-1}(y) = f^{-1}(y).$$

Applying equation (1.2) we obtain:

$$x \rightarrow_T y = f\left(f^{-1}(x) \rightarrow_t f^{-1}(y)\right)$$

$$= f\left(f^{-1}(y)\right) = y.$$

¿From (i) and (ii) it follows that:

$$x \rightarrow_T y = \begin{cases} b & \text{if } x \leq y \\ y & \text{if x} > y \end{cases} .$$

Example 1.5. Goguen type $t-$norm defined on $[0, b]$ interval:

$$xTy = f\left(f^{-1}(x) \cdot f^{-1}(y)\right)$$

$$= b \cdot \left(\frac{x}{b} \cdot \frac{y}{b}\right)$$

$$= \frac{x \cdot y}{b};$$

The residuum is obtained in the same way as it was obtained the residuum of Gödel $t-$norm. It has the following definition:

$$x \rightarrow_T y = \begin{cases} b & \text{if } x \leq y \\ y & \text{if x} > y \end{cases} .$$

Once we have determined the $t-$norm and its associate residuum on $[0, b]$ interval, we can determin the $t-$norm and its associate residuum on RGB set.

The $t-$norm T is defined on $[0, b]$ interval, which represents one component of RGB set. Therefore if we apply the $t-$norm T on each component of RGB set we obtain the $t-$norm defined on it as follows:

Definition 1.6. For all $x, y \in RGB$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we have:

$$xT_{RGB}y = (x_1Ty_1, x_2Ty_2, x_3Ty_3,)$$

where T represents the $t-$norm defined on $[0, b]$ interval as in (1.1).

We are able to determin now the associate residuum of $t-$norm T_{RGB} defined on RGB set. Applying the residuum $\rightarrow_T$ on each component of RGB set we obtain the associate residuum of $t-$norm T_{RGB} defined as follows:

Definition 1.7. For all $x, y \in RGB$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we have:

$$x \rightarrow_{T_{RGB}} y = (x_1 \rightarrow_T y_1, x_2 \rightarrow_T y_2, x_3 \rightarrow_T y_3,)$$

where $\rightarrow_T$ represents the associate residuum of $t-$norm T defined on $[0, b]$ interval as in (1.2).

2. The BL-algebra over RGB set

Once determined the $t-$norm T_{RGB} and its associate residuum $\rightarrow_{T_{RGB}}$ we can go further to develop a BL-algebra over RGB set.

First it is necessary to define the supremum and the infimum of two elements of RGB set. In what follows we will follow the same steps as in the case of $t-$norm T_{RGB}.

We start with the construction of these two binary operations on $[0, b]$ interval, which represents one component of RGB set, using the operations defined on unit interval.

Let $\vee$ be the supremum and $\wedge$ the infimum of two elements defined on unit interval. Then we define the supremum $\vee_{[0,b]}$ and the infimum $\wedge_{[0,b]}$ of any two elements x, y from $[0, b]$ interval as follows:

$$x \vee_{[0,b]} y = f\left(f^{-1}(x) \vee f^{-1}(y)\right); \tag{2.1}$$

$$x \wedge_{[0,b]} y = f\left(f^{-1}(x) \wedge f^{-1}(y)\right); \tag{2.2}$$

where f and f^{-1} are the functions defined in the previous section.

Applying $\vee_{[0,b]}$ and $\wedge_{[0,b]}$ on each component of RGB set, we obtain the supremum and the infimum on RGB set defined as follows:

Definition 2.1. For all $x, y \in RGB$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we have:

$$x \vee_{RGB} y = \left(x_1 \vee_{[0,b]} y_1, x_2 \vee_{[0,b]} y_2, x_3 \vee_{[0,b]} y_3\right); \tag{2.3}$$

$$x \wedge_{RGB} y = \left(x_1 \wedge_{[0,b]} y_1, x_2 \wedge_{[0,b]} y_2, x_3 \wedge_{[0,b]} y_3\right); \tag{2.4}$$

where $\vee_{[0,b]}$ and $\wedge_{[0,b]}$ represents the supremum and the infimum defined on $[0, b]$ interval as in (2.1) and (2.2).

In what follows we will check some properties of supremum and infimum defined on RGB set that will be used later.

Lemma 2.2. *For any $x, y \in RGB$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the following properties hold:*

$$x \wedge_{RGB} y = xT_{RGB}\left(x \rightarrow_{T_{RGB}} y\right); \tag{2.5}$$

(2.6) $$(x \rightarrow_{T_{RGB}} y) \vee_{RGB} (y \rightarrow_{T_{RGB}} x) = 1_{RGB},$$

where $1_{RGB} = (b, b, b)$.

Proof. It is enough to prove that the above properties hold on each component of RGB set.

To prove first property, we use equation (2.2) and we obtain:

$$x_1 \wedge_{[0,b]} y_1 = f\left(f^{-1}(x) \wedge f^{-1}(y)\right),$$

and from equation (0.1) it follows:

$$x_1 \wedge_{[0,b]} y_1 = f\left(f^{-1}(x)\, t\left(f^{-1}(x) \rightarrow_t f^{-1}(y)\right)\right).$$

Using now the equations (1.1) and (1.2) we get:

$$x_1 \wedge_{[0,b]} y_1 = f\left(f^{-1}(x_1)\, t f^{-1}\left(f\left(f^{-1}(x_1) \rightarrow_t f^{-1}(y_1)\right)\right)\right)$$

$$= f\left(f^{-1}(x_1)\, t f^{-1}(x_1 \rightarrow_T y_1)\right)$$

$$= x_1 T (x_1 \rightarrow_T y_1).$$

For the proof of the second property, by (1.2) and (2.1) we have:

$$(x_1 \rightarrow_T y_1) \vee_{[0,b]} (y_1 \rightarrow_T x_1) = f\left(f^{-1}(x_1 \rightarrow_T y_1) \vee f^{-1}(y_1 \rightarrow_T x_1)\right)$$

$$= f\left(f^{-1}\left(f\left(f^{-1}(x_1) \rightarrow_t f^{-1}(y_1)\right)\right) \vee f^{-1}\left(f\left(f^{-1}(y_1) \rightarrow_t f^{-1}(x_1)\right)\right)\right)$$

$$= f\left(\left(f^{-1}(x_1) \rightarrow_t f^{-1}(y_1)\right) \vee \left(f^{-1}(y_1) \rightarrow_t f^{-1}(x_1)\right)\right).$$

Using the equation (0.2) we get:

$$(x_1 \rightarrow_T y_1) \vee_{[0,b]} (y_1 \rightarrow_T x_1) = f(1) = b.$$

This completes the proof. ■

In what follows we introduce the BL-algebra structure over RGB set.

Let $0_{RGB} = (0, 0, 0)$ and $1_{RGB} = (b, b, b)$ be two constants from RGB set, then

Lemma 2.3. *The structure* $(RGB, \wedge_{RGB}, \vee_{RGB}, T_{RGB}, \rightarrow_{T_{RGB}}, 0_{RGB}, 1_{RGB})$ *is a BL-algebra.*

Proof. To prove that $(RGB, \wedge_{RGB}, \vee_{RGB}, T_{RGB}, \rightarrow_{T_{RGB}}, 0_{RGB}, 1_{RGB})$ is a BL-algebra, we have to show that the following properties hold:

(i) $(RGB, \wedge_{RGB}, \vee_{RGB}, 0_{RGB}, 1_{RGB})$ is a bounded distributive lattice;

(ii) $(RGB, T_{RGB}, 1_{RGB})$ is a commutative semigroup having 1_{RGB} as unit element;

(iii) $x \wedge y = x T_{RGB} (x \rightarrow_{T_{RGB}} y)$ (divisibility);

(iv) $z \leq (x \rightarrow_{T_{RGB}} y)$ iff $x T_{RGB} z \leq y$;

(v) $(x \to_{T_{RGB}} y) \vee_{RGB} (y \to_{T_{RGB}} x) = 1_{RGB}$.

The properties (i) and (ii) are obvious. The properties (iii) and (v) are proved in Lemma 2.2. The property (iv) follows from the definition of the associate residuum of the $t-$norm t defined on unit interval and from the mode in which we have obtained the $t-$norm T_{RGB} and its associate residuum $\to_{T_{RGB}}$ from it. ■

Once defined a BL-algebra structure on RGB model we can start to use it in image compression and decompression, but this is the subject of a next paper.

References

[1] Dumitrescu, D., Lazzerini, B., Jain, L.C. (2000) "Fuzzy Sets and their Application to Clustering and Training", Boca Raton, Fl.

[2] Hajek, P. (1998) "Basic Fuzzy Logic and BL-algebras", Soft Computing, Vol. 2, pp. 124-128.

[3] Klement, E.P., Mesiar, R., Pap, E. (2000) "Triangular Norms", Kluwer Academic Publishers, Dordrecht, The Netherlands.

[4] Hajek, P. (1998) "Metamathematics of Fuzzy Logic", Kluwer Academic Publishers, Dordrecht, The Nederlands.

[5] Ionescu, F. (2000) "Grafica in realitatea virtuala", Editura Tehnica, Bucuresti.

[6] Loia, V., Sessa, S. (2002) "Compression and Decompression of Fuzzy Relations in the Basic Logic over [0,1], (submitted).

[7] Noje, D., Bede, B. (2001)"MV-Algebra structure of RGB Model", Studia Informatica, Cluj-Napoca.

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