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THE RELAXATION OF THE FUNDAMENTAL CONDITIONS OF SCIENTIFIC VISUALIZATION USING EQUIVALENCE CLASSES

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ABSTRACT. The paper addresses the issue of scientific data visualization process validation. Three fundamental conditions for scientific visualization are introduced; one of them – the Precision Condition – is discussed in detail. The theory allows a better formal description of the scientific visualization process.

Index terms – scientific visualization, precision classes, and scientific visualization process validation

1. INTRODUCTION

Scientific Visualization is a computational process that maps scientific data and its attributes into visual objects [1]. Scientific data can be obtained in many different ways, e.g. by running a simulation or by a DAQ process. Usually, **scientific data objects** are finite representations of complex mathematical objects. We note by O the set of such objects, $o \in O$. During the visualization process, initial data objects, o, are processed through different transformation functions $Mat(o) = o'$, into a new set $o' \in O'$. Objects o' are then mapped $Map(o') = g$ into a set of **virtual geometrical objects** $g \in G$, through a set of **graphical primitives**. Objects g usually are n-dimensional (nD), animated (t) and interactive.

Definition 1 *A group of virtual geometrical objects, logically interconnected, is called a* ***logical visualization*** *of that scene.*

Ideal geometrical objects g, nD, animated (t) and interactive are usually represented $Rep(g) = g'$, $g' \in G'$, on real 2D screens.

Definition 2 *The projection of the logical visualization of a scene on a screen is called a* ***physical visualization*** *of that scene.*

The functions $Rep(g) = g'$ implement classical graphical operations such as composition of the scene, volume generation, isosurface generation, simulation of transparency, reflectivity and lighting conditions, $nD \to 2D$ projection, clipping,

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hidden surface removal, shading, animation (t), setting user interactivity (zoom, rotate, translate, pan, etc), etc.

Definition 3. *By* ***interactivity*** *we understand the attributes of visual objects (logical and/or physical) whose setting allows nD $\rightarrow$ 2D projection (zoom, rotate, translate, pan, etc), animation control (t), control of the objects composing the scene and control of the scene as a composite object.*

The scientific visualization process is described by the $Vis(o) = g'$, $Vis(o) = Rep(Map(Mat(o))) = g'$ function. The process is described in figure 1.

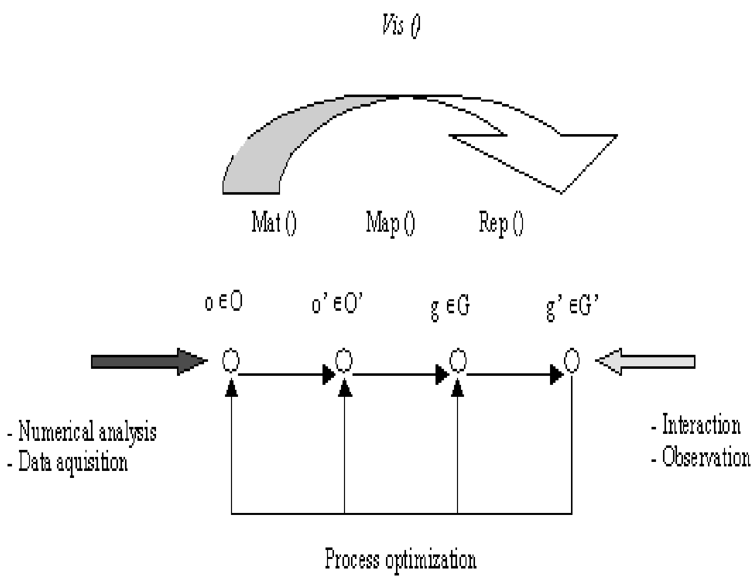

FIGURE 1. Description of the scientific visualisation process

2. Fundamental conditions of scientific visualization

There are many requirements concerning a certain scientific visualization process. We consider three of them to be fundamental. The first one is the **distinctiveness condition**. This condition (although very weak) enables users to distinguish between different data objects based on their display. The condition is necessary as one can imagine many visualization functions that generate images with no use, which reveal none of the data objects characteristics/attributes.

The second condition is the **expressiveness condition**. This condition assures that the attributes of the visual object represent the attributes of the input data set.

The third one is the **precision condition**. This condition insures that the order among data objects is preserved among visual objects.

The distinctiveness condition. *Different input data (different mathematical objects) are represented by different visual objects.*

This condition can be stated:

$$\begin{aligned} o_1 \neq o_2 \quad &\Rightarrow \quad Vis(o_1) \neq Vis(o_2) \Rightarrow Rep(Map(Mat(o_1))) \neq Rep(Map(Mat(o_2))) \\ &\Rightarrow \quad g1' \neq g2', \text{ for any } o1, o2 \in O, g1', g2' \in G' \end{aligned}$$

The interpretation of this condition is that $Vis()$, $Mat()$, $Map()$ and $Rep()$ functions are injective.

The expressiveness condition. *The visual objects express all and only the characteristics of the input data.*

It results that the visualization function should be one to one.

The two conditions are necessary but not sufficient. Another condition is needed to establish an order relation both among data and visual objects. This condition could be seen as a precision relation.

The precision condition. *For any objects $o_1, o_2 \in O$ such that o_1 is "more precise" than o_2 we have that $Vis(o_1)$ is "more precise" than $Vis(o_2)$, with $Vis(o_1)$, $Vis(o_2) \in G'$.*

The precision condition adds something new. If the visualization function is well defined and the input data objects are strictly ordered, the visual objects can be ordered by their "precision".

The first two conditions introduce criteria of validation and control of the visualization process. The visualization function $Vis()$ fulfilling these criteria results in a scientific visualization. The third condition allows further developments by defining mathematical operations on the given ordering.

3. Equivalence classes

We introduce another approach to describe formally the visualization process. There are examples that prove that the above conditions are too "tight". Because we display the visual objects on real screens (i.e. with finite resolution) it is possible that two or more objects o to be mapped into identical visual objects. Therefore a more relaxed approach to formally describe the visualization process of scientific data is necessary. In order to describe the new approach some mathematical concepts are to be presented.

We have already introduced the basic sets denoted by O, O', G and G'. O represents the set of the so called "data objects". O' is the set of the elements obtained from "data objects" through different transformation functions. G represents the set of virtual geometrical objects, nD. Virtual geometrical objects become real geometrical objects (G') by projection/display (e.g. on 2D screens).

The visualization function can be described as the mapping of the set O into G'.

Definition 4. *Let O and G' be two sets and v be a binary relation. We call v a mapping of O in G' if for each element $o \in O$, there is exactly one element $g' \in G'$ that satisfies $\langle o, g' \rangle \in v$.*

The element g' is called the **image** of the element o through v, and o is called the **inverse image** of g' through v. For the mapping v we introduce the notation $v : o \to g'$ and the functional notation $v(o) = g'$. We can write that $v : O \to G'$ to show that $v = Vis()$ is a mapping of O into G'. O is called the domain of v. If the inverse relation is also a mapping, we will denote it by v^{-1}.

From the set theory we know that a *partition* π of a set O is a subset of $P(O)$ (the power set of O) not containing Φ, satisfying the following property: every $o \in O$ is an element of exactly one $A \in \pi$. The elements of a partition are called *blocks*. If π and π' are partitions of O, we will write $\pi \leq \pi'$ if for every block $B \in \pi$ there exists a block $C \in \pi'$ such that $B \subseteq C$.

We use the fundamental theorem of the equivalence relations in order to underline some important aspects:

Theorem 1. [10] (a) Let π be a partition of O and define a binary relation ϵ_p on O by $o_1 \epsilon o_2$ if and only if o_1 and o_2 are in the same block of the partition π. Then ϵ_π is an equivalence relation on O.

(b) If ϵ is an equivalence relation over a set O, then there exists a partition π_ϵ over O such that $o_1, o_2 \in O$ are elements of the same bloc of π_ϵ if and only if $o_1 \epsilon o_2$.

(c) If $\pi \leq \pi'$, then $\epsilon_\pi \leq \epsilon_{\pi'}$. If $\epsilon \leq \epsilon'$, then $\pi_\epsilon \leq \pi'_\epsilon$.

Theorem 1(a) shows that a binary relation is an equivalence relation if it "conserves" the initial partitioning over the given set. Theorem 1(b) states that a partitioning of a set can be obtained starting from a given equivalence relation ϵ. Theorem 1(c) introduces an order relation.

The following remark has to be stated:

Remark 1. *If more than one element o is mapped into the same visual object g', then the set O can be partitioned into non-empty subsets that include all the o elements mapped into the same visual object.*

Remark 1 introduces the idea of equivalence relations as the main tool in order to obtain a more realistic description of the visualization process. A natural equivalence relation ϵ_v can be defined over O. The relation ϵ_v is called the *equivalence relation induced by v* over the set of objects O and it partitions the set O into subsets of objects sharing the same visualization (see theorem 1). We denote by π_v the induced partitioning over O.

The proposed model is based on the concept of equivalence classes.

Definition 5. [9] *The equivalence class of an element $o \in O$, induced by the equivalence relation ϵ, is the subset of those elements from O that are in the relation ϵ with o.*

We denote by $[o]\epsilon$ the equivalence class of $o \in O$, induced by the equivalence relation ϵ. When the equivalence relation is implicit, we use the notation $[o]$.

Further, another theorem is introduced in order for us to be able to formulate the new visualization conditions.

Theorem 2. [10] Any mapping $v : O \to G'$ can de represented as a product of two mappings φ and ϕ, $v = \varphi\phi$, where φ is onto and ϕ is one-to-one; if ϵ is the equivalence relation induced by v, then $\varphi = \varphi_\epsilon : O \to O|\epsilon$ and $\phi : O|\epsilon \to G'$, where $O|\epsilon$ is the set of all equivalence classes induced by ϵ (Figure 2).

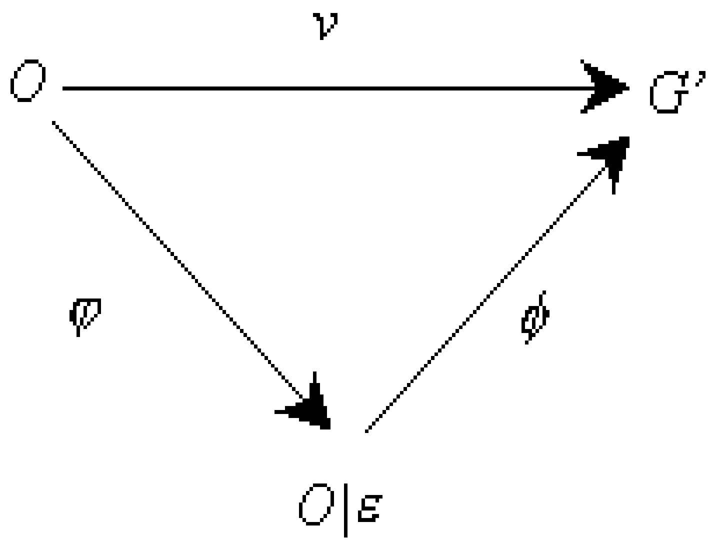

FIGURE 2. Schematic representation for Theorem 2

So, if we have a well-defined equivalence relation over O, then we can consider instead of v a product of two mappings (Figure 3). This approach has the advantage that it reduces the set of objects O to the set of classes $O|\epsilon$. Order relations can be stated over the set of classes.

The presented theory is exemplified below. We consider two data sets having the same format. The equivalence relation ϵ imposes that the attributes of the objects (element by element) have values between:

$$(a_i)_1, (a_i)_2 \in (a_i - \Delta a_i, a_i + \Delta a_i),$$

where $(a_i)_1$ are the attributes of the first object, and $(a_i)_2$ those of the second object. If the resolution of the screen is small enough we observe that, for the same visualization system, the two different data sets will be represented on the screen by the same visual object.

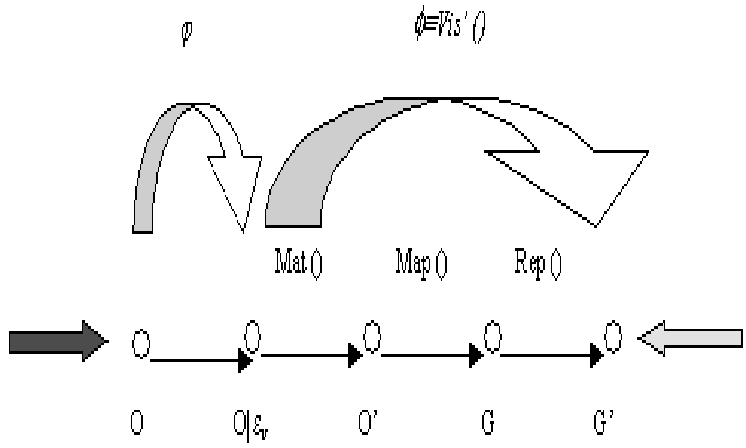

FIGURE 3. Description of the visualization process using equivalence classes

Remark 2. *Assuming that the equivalence relations ϵ_0, ϵ_1, ..., ϵ_{n-1}, defined over the same set exist, we conclude that the partitions π_o, π_1, ..., π_{n-1} also exist.*

Theorem 1(c) and remark 2 introduce an order relation between the equivalence classes, relation that can be regarded as "precision" relation. For the above example, we consider another equivalence relation ϵ' imposing that the attributes of the objects (element by element) have values between:

$$(a_i{}')_1, (a_i')_2 \in (a_i' - \Delta a_i', a_i' + \Delta a_i'),$$

where $(a_i')_1$ are the attributes of the first object, and $(a_i')_2$ those of the second object, with $\Delta a_i \leq \Delta a_i'$. In this case $\pi \leq \pi'$, where π and π' represent the partitions corresponding to the equivalence relations ϵ and ϵ'. From theorem 1.3 it results that $\epsilon \leq \epsilon'$.

4. THE PRECISION RELATION OVER THE SCIENTIFIC VISUALIZATION PROCESS

An order relation is necessary over the visualization process. We have introduced the "precision relation" as a fundamental condition of the scientific visualization. Now, the equivalence classes allow a further development of the idea. We are especially interested in the O (or $O|\epsilon$), G and G' sets.

Definition 6. *A class of objects, defined by the equivalence relation ϵ (see theorem 1), is "more precise" than another one, defined by the equivalence relation ϵ', if $\epsilon \leq \epsilon'$.*

So, $[o_1]\epsilon \leq [o_2]\epsilon'$ ($\leq$ describes the precision relation) if $\epsilon \leq \epsilon'$.

The set of virtual geometrical objects is denoted by G. A virtual geometrical object g can be regarded as a composition of graphical primitives. We denote by P the set of all types of graphical primitives. Let us denote by $SUM(N,P)$ the sum $\sum_{i=1}^{n} p_i$. Then the virtual geometrical object g can be described as:

$$g = SUM(N,P), \text{ where } p_i \in P, \text{ for a finite N.}$$

Definition 7. *1. A virtual geometrical object $g_1 = SUM(N_1,P)$ is said to be "strictly more precise" than another virtual geometrical object $g_2 = SUM(N_2,P)$ if $N_1 > N_2$.*

2. If $N_1 = N_2$, then a virtual geometrical object $g_1 = Map(Mat(o_1))$, $o_1 \leq [o_1]\epsilon$ is said to be "more precise" than another object $g_2 = Map(Mat(o_2))$, $o_2 \leq [o_2]\epsilon'$ if the class $[o1]\epsilon$ is "more precise" than the class $[o2]\epsilon'$.

Remarks. 1. An object can be represented using several ways (Figure 4). The representation considered "the most (strictly) precise" is the one that uses the highest number of graphical primitives. We call this kind of precision **representation precision**.

2. If the representation uses the same number of graphical primitives then the set G conserves the precision relation over O. The precision induced over G is called **order precision**.

If different numbers of graphical primitives are used, then the representation precision is considered as order relation.

Definition 8. *A visual geometrical object $g_1 \in G$ is said to be "(strictly) more precise" than another visual object $g_2 \in G$ if $g_1 = Rep^{-1}(g_1)$ is "(strictly) more precise" than $g_2 = Rep^{-1}(g_2)$.*

5. The relaxation of fundamental conditions of the scientific visualization

The fundamental conditions of the scientific visualization can be restated:

The distinctiveness condition. *Different equivalence classes are mapped into different visual objects.*

Formally: $[o_1] \neq [o_2] \Rightarrow \phi([o_1]) \neq \phi([o_2]) \Rightarrow g'_1 \neq g'_2$
$[o_1], [o_2] \in O|\epsilon, g'_1, g'_2 \in G'$.

The expressiveness condition. *The visual objects express all the characteristics of input equivalence classes, and only those characteristics.*

Formally: $\forall g' \in G', \exists [o] \in O|\epsilon$ such that $\phi([o]) = g'$.

The distinctiveness condition and the expressiveness condition impose the mapping ϕ to be one-to-one.

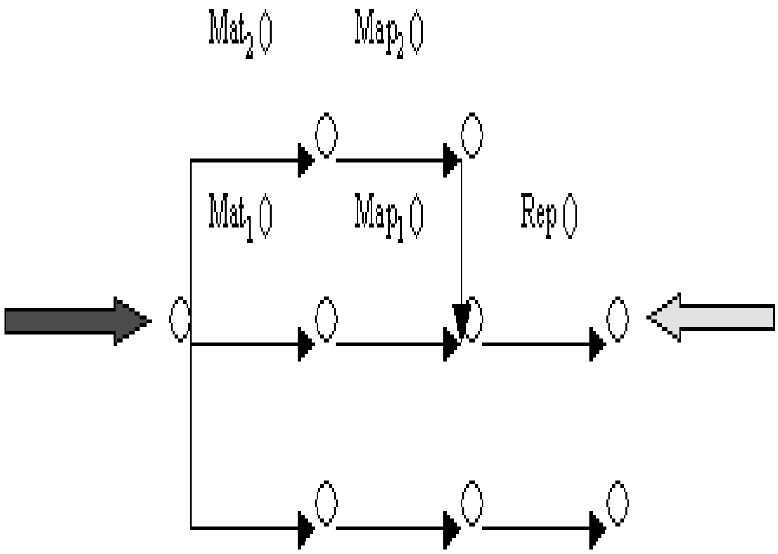

FIGURE 4. Example of visualization pipelines

The precision condition becomes the precision theorem. The equivalence class approach reduces the number of fundamental conditions and in the same time allows the introduction of a well-defined order relation.

Precision Theorem. 1. Let $[o]\epsilon \in O|\epsilon$ be a class of objects and let the ideal geometrical objects $g_1, g_2 \in G$, where g'_1 represents the physical visualization of the the $[o]$ class using N_1 graphical primitives, and g'_2 represents the physical visualization of the $[o]$ class object using N_2 graphical primitives.

i.: If $N_1 > N_2$ then g'_1 is "strictly more precise" than g'_2.
ii.: If $N_1 = N_2$ then g'_1 is represented with the same precision as g'_2.
iii.: If $N_1 < N_2$ then g'_2 is "strictly more precise" than g'_1.

2. Let $[o1]\epsilon, [o2]\epsilon \in O|\epsilon$ be two classes of objects and let the ideal geometrical objects $g'_1, g'_2 \in G$, where g'_1 represents the physical visualization of the the $[o_1]\epsilon$ class using N_1 graphical primitives, and g'_2 represents the physical visualization of the $[o_2]\epsilon$ class object using N_2 graphical primitives. We consider that the class $[o_1]\epsilon$ is "more precise" than $[o_2]\epsilon$.

i.: If $N_1 > N_2$ then g'_1 is "strictly more precise" than g'_2.
ii.: If $N_1 = N_2$ then g'_1 is "more precise" than g'_2.
iii.: If $N_1 < N_2$ then g'_2 is "strictly more precise" than g'_1.

Proof. 1. i. For the objects $g_1 = SUM(N_1, P)$ and $g_2 = SUM(N_2, P)$ we have $N_1 > N_2$. From definition 7.1 it results that g_1 is "strictly more precise" than g_2. From definition 8 we conclude that g_1' is "strictly more precise" than g_2'.

ii. If $N_1 = N_2$ then $g_1 = g_2$ and $g_1' = g_2'$.

iii. The same proof as for i.

2. ii. We assume that a class of objects induced by an equivalence relation ϵ, $[o_1]\epsilon$, is "more precise" than another one, $[o_2]\epsilon$, induced by the equivalence relation ϵ. We have then $[o_1]\epsilon \leq [o_2]\epsilon$.

From the definition of the precision relation for object classes from O, we conclude that $\epsilon \leq \epsilon$ as the result of the relation $[o_1]\epsilon \leq [o_2]\epsilon$. From theorem 1(c) we further conclude that $\pi \leq \pi$, where π and π are partitions of the set O. If π and π are partitions of O, we write $\pi \leq \pi$ if for every block $B \in \pi$ there exists a block $C \in \pi$ such that $B \subseteq C$.

Then, the objects that belong to the equivalence class $[o_1]\epsilon$ are more exact than those from the class $[o_2]\epsilon$ and as a result their representations are more accurate. From definition 7.2 it results that g_1 is "more precise" than g_2.

So, from definition 8, if g_1 is "more precise" than g_2 then $g_1 = Rep(g_1)$ is "more precise" than $g_2 = Rep(g_2)$.

For i. and ii. we use the definition 7.1.

6. Conclusions

This article proves that a more "relaxed" approach of the mathematical description of the process is necessary. The finite screen resolution and the finite accuracy of the system introduce visualization "error": different data sets have sometimes the same display/visualization, i.e. are mapped into the same visual object. The introduced data models allow the definition of different operations between data sets and the definition of a precision relation.

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