STUDIA UNIV. "BABEŞ–BOLYAI", INFORMATICA, Volume **XLIV**, Number 2, 1999

COMPLEX VALUE BOUNDARY ELEMENTS METHODS (CVBEM) FOR SOME MIXED BVP

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Abstract. This paper presents a method for setting up a CVBEM for some mixed boundary value problem joined to the Laplace equation, in plane simply conected domains. The considered boundary value problems are met in modelling of different fluid flows as well as in microelectronics, design of electrical machines, magnetohydrodynamics, etc.

The existence and the uniqueness of the classical solutions of the boundary value problems envisaged in this paper have been studied long time ago. Since then such studies have stopped at the analytical stage without emphasizing efficient convergent calculation algorithms, these theoretical results remained rather unusable to applied mathematics . That is why a convergent BEM which is the main purpose of our work, seems to be an useful tool for filling in the above mentioned gap. In the sequel we will foccuss on the Robin problem, an important modified Volterra problem following as an particular case. Some extension as well as some effective numerical approches of certain particular problems will be considered in the future.

1. Let us consider the Robin(mixed, Dirichlet-Neumann) problem for the Laplace operator in a disk Ω (centred at O, of radius R) with the boundary C. With the form of complex functions, the problem leads to the determination of a function $f = u + iv$ holomorphic in Ω , continuous with its derivative in $\Omega \cup C$ such that its real part u satisfies the boundary condition

$$\alpha u + \beta \frac{du}{dn_i}|_C = l, \tag{1}$$

where α , $\beta \in R$, while l is supposed to be a given continuous function on C and $\frac{du}{dn_i}$ is the derivative along the unit inward normal direction $\vec{n}$. Of course for $\beta = 0$ or $\alpha = 0$ we get respectively the Drichlet and Neumann problems. In the last case the compatibility condition requires that $\int\limits_{|z|=R} \frac{l}{\beta} df = 0$. We remark that in the case of a disk, the condition (1) can be replaced by

$$\alpha Ref - \frac{\beta}{R} Re(z\frac{df}{dz})|_{|z|=R} = l,$$

with a Dirichlet condition $ReF(z)|_{|z|=R} = l$ for the holomorphic function

$$F(z) = \alpha f(z) - \frac{\beta}{R} z \frac{df}{dz}$$

Using now a technique already built up by us [3, 4] $f(z)$ is looked under the form of $F(z) = \frac{1}{2\pi i}\int\limits_C \frac{F(t)}{t-z}dt, \text{for} z \in \Omega$. But this Cauchy′s integral for our solution is, in fact, the integral representation required by a BEM which, together with an appropiate boundary integral equation provide the main tools for setting up a BEM.

In our case we will manage to avoid the construction and the solution of any boundary integral equation which, by elliminating of some involved approximations of both contour and itegrals simplifies essentialy our BEM.

Namely, considering a set of nodal points, $z_0,\ z_1,\ \ldots,\ z_n\ (z_0 = z_n)$ on C, disposed contour clockwise and separating the curve C into boundary elements $C_j = (z_{j-1}z_j)$, $j \in \{1, \ldots, n\}$, the envisaged approximation $\tilde{F}(t)$ for the unknow function $F(t)$ is

$$\tilde{F}(t) = \sum_{j=1}^{n} F_j L_j(t) \text{where} F_j = F(z_j),$$

while$L_j(t)$ are the interpolating Lagrange functions, constructed on each arc respectively, i.e.

$$L_j(t) = \begin{cases} \frac{t-z_{j-1}}{z_j-z_{j-1}} & \text{for } t \in C_j\ . \\ \frac{t-z_{j+1}}{z_j-z_{j+1}} & \text{for } t \in C_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, the approximation $F^*(z)$ of our unknown function $F(z)$ in an interior point $z \in \Omega$ will be defined by

$$F^*(z) = \sum_{i=1}^{n} F_j \tilde{L}_j(z)$$

where,

$$\tilde{L}_j(z) = \frac{1}{2\pi i}\int_C^{\prime} \frac{L_j(t)}{t-z}dt = \frac{1}{2\pi i}(\frac{z-z_{j-1}}{z_j-z_{j-1}} \ln\frac{z-z_j}{z-z_{j-1}} + \frac{z-z_j}{z_j-z_{j+1}} \ln\frac{z-z_{j+1}}{z_j-z_{j+1}})$$

and where one choses the main (principal) determination of the complex logarithm. The continuity of the approximation $F^*(z)$ allows us to write that

$$U_k + iV_k = F_k \approx F^*(z_k) = \sum_{j=1}^{n} F_j \tilde{L}_j(z_k), k \in \{1, \dots, n\}$$

i.e. we are led to the following real linear system of $2n$ equations and $2n$ unknowns

$$U_k = \sum_{i=1}^{n} M_{kj}U_j - \sum_{j=1}^{n} N_{kj}V_jV_k = \sum_{i=1}^{n} M_{kj}V_j - \sum_{j=1}^{n} N_{kj}U_j$$

By solving this system within the data of the Dirichlet boundary problem, we get the looked approximation $\tilde{F}(t)$ of the function $F(t)$ and, implicitely, via Cauchy$'$s formula, the solution of the proposed boundary problem in all the points of the domain D. Concerning the coefficients L_{kj}, for $k \neq j$, they could be directly calculated from the expresion of $\tilde{L}_j(z)$ using the equality

$$\lim_{z \to z_p} (z - z_p) \ln \frac{z_j - z_{j+1}}{z_j - z_{j-1}} = 0$$

in the case $k = j - 1$ or $k = j + 1$. For $k = j$ we get

$$L_{jj} = \frac{1}{2\pi i} \ln \frac{z_j - z_{j+1}}{z_j - z_{j-1}},$$

the same principal determination of the logarithm being considered.
A division $d := (z_0, z_1, \dots, z_n), z_0 = z_n$ of the curve C will be called "acceptable" if for each $t \in C_j, j \in \{1, 2, \dots, n\}$ the following condition is fulfilled

$$max\{|t - z_j|, |t - z_{j-1}|\} < |z_j - z_{j-1}|.$$

By introducing the concept of "acceptable" division

$$d := (z_0, z_1, \dots, z_n)$$

of the norm $||d||$ and using a convergence theorem given by us [3, 4], we state that

$$\lim_{n \to \infty} F^*(z) = F(z)$$

for an acceptable division $||d|| \to 0$. At the same time the solution $f^*(z)$ of the following differential equation

$$\alpha f^*(z) - \frac{\beta}{R} z \frac{df^*(z)}{dz} = F^*$$

i.e.

$$f^*(z) = K - \frac{R}{\beta} \int \frac{F^*(z)}{z} e^{-\frac{R\alpha}{\beta} - 4z} dz] e^{\frac{R\alpha}{\beta} \ln z}$$

where K is an determined constant, which will be fixed by the conditions associated to the problem, and $\ln z$ means the principal (main) determination of the complex logarithm, will converge to the exact solution $f(z)$, due to the continuity of all the functions involved. Summarizing the CVBEM just built up is convergent, i.e.

$$\lim_{n\to\infty} f^*(z) = f(z),$$

for any acceptable division $d = (z_0, \ldots, z_n)$ of norm $||d||$, which $||d|| \to 0$. Obviously the above procedure can be also extended to the case when α and β are real continuous functions. But despite that, the solution of the attached differential equation will be not the same as before, the convergence of the corresponding CVBEM is still valid. We remark that the initial Robin problem can be transformed into a Dirichlet problem, using that $\frac{du}{dn_i}|_C = \frac{dv}{ds}|_C$ obvious extension of Cauchy-Riemann relations (of course the validity of these relations on the boundary is connected with the analycity of $f(z)$ on C, which ensures that the image $F(C)$ is also an analytical curve). Hence

$$\frac{dv}{ds}|_C = -\frac{\alpha}{\beta}u - \frac{l}{\beta},$$

i.e.

$$v|_C = -\frac{\alpha}{\beta}2\pi R u(0) - \int_C \frac{l}{\beta} ds + k,$$

where k is an arbitrary constant while $u(0)$ is the value of the function u at $z = 0$ (the Gauss theorem). The equivalence of the mixed (Robin) problem with a Dirichlet one can be established even in the case when Ω is an arbitrary, plane, simply connected (bounded) domain, whose boundary C is an analytical curve. Keeping the same requirements on the unknown function $f(z)$, which also ensures even the validity of Cauchy-Riemann relations on C, the Robin condition, with α and β real continuous functions, can be replaced by

$$\alpha u + \beta \frac{dv}{ds}|_C = l$$

At the same time, supposing that $\beta \neq 0$ and using the harmonicity of u, the initial Robin condition leads

$$\int_C \frac{\alpha}{\beta} u ds = \int_C \frac{l}{\beta} ds, \quad \int_C \frac{du}{dn_i} ds = 0,$$

for any harmonical function u. Immediately we can write that $v|_C = K$, where K is an undetermined constant, i.e. the mentioned equivalence has been proved.

2. Let now $\Omega \subset R^2$ be a simply connected domain whose analytical boundary C is divided into C_u and C_v so that $C_u \cap C_v = \emptyset$ and $C_u \cup C_v = C$. We intend to determine the function $u \in C^2(\Omega) \cap C'(\Omega \cup C)$ which satisfies the conditions:

$$\begin{aligned} \triangle u &= 0 \quad \text{in } \Omega \\ u|_{C_u} &= l_1 \quad \text{on } C_u \\ \frac{du}{dn_i}|_{C_v} &= l_2 \quad \text{on } C_v, \end{aligned}$$

l_1 and l_2 being two given real continuous functions, and $\frac{d}{dn_i}$ the derivative in the direction of the inward normal n_i. Of course this problem, which is considered to be a "modififed Volterra" problem [2], could be rewritten under the previous (Robin) form if α and β are a couple of real continuous functions so that $\alpha\beta|_C = 0, \alpha|_{C_u} \neq 0$ and $\beta|_{C_v} \neq 0$, l being now

$$l = \alpha \frac{l_1}{\alpha} + \beta \frac{l_2}{\beta}.$$

It is obvious that this problem is also equivalent with a clssical Volterra problem. Precisely the condition $\frac{du}{dn_i}|_C = l_2$ can be replaced by $v = \int l_2 ds + k$, k being an integration constant whose determination is accomplished within the frame of the proposed problem. But this modified Volterra problem leads, using the same technique as above, to a Dirichlet problem whose data on boundary can have a finite number of singularities of first type. But the existence and the uniqueness of the solution of this Dirichlet problem has been proved too (Fatou), so that, the whole procedure for appropiate CVBEM envisaged before, can be developped again.

References

[1] Caius Iacob, *Introduction mathèmatique à la mécanique des fluides*, Editions Gauthier-Villars et L'Academie Roumaine, 1959

[2] Dorel Homentcovschi, *On the mixed boundary-value problems for harmonic functions in plane domains*, Journal of Applied Mathematics and Physics (ZAMP), **31** (1980), p. 351–366

[3] Titus Petrila, *On certain mathematical problems connected with the use of the complex variable boundary element method to the problems of plane hydrodynamics. Gauss variant of the procedure*, The mathematical heritage of C. F. Gauss, World Scientific,Publishing Company, Singapore, 1991, p. 585–604

[4] Titus Petrila, *An improved CVBEM for plane hydrodynamics*, Revue d'analyse numerique et de la theorie de l'approximation, vol. XVI, fasc. **2** (1987), p. 149–157

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