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DIVIDED DIFFERENCES AND CONVEX FUNCTIONS OF HIGHER ORDER ON NETWORKS

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Dedicated to Professor Sever Groze on his 65^{th} anniversary

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REZUMAT. - Diferenţe divizate şi funcţii convexe de ordin superior în reţele. În acest articol vom introduce diferenţele divizate pe n puncte, pentru orice funcţie reală, definită pe o submulţime conexă a unei reţele. Este expusă o teoremă de reprezentare a acestor diferenţe divizate generalizate. Ultima secţiune este dedicată funcţiilor convexe de ordin n, definite pe reţele şi studiului unor proprietăţi ale acestui tip de funcţii.

Abstract. For any real function f defined on a connected subset of an oriented network, we are interested to find a way to introduce divided differences on n points. We give a reprezentation theorem for the generalized divided differences and some properties resulting from this theorem.

Next we introduce the concept of convex real function of order n, defined on network. Some properties of these functions will be studied in the last part of this paper.

1. Notations and definitions

First we introduce the concept of network (see [1], [2], [3]). We consider a directed connected graph $G = (W,A)$ without loops. To each vertex $i \in W = \{1,\ldots,n\}$ we associate a point $v_i \in \mathbf{R}^3$. Thus yields a finite subset $V=\{v_1,\ldots,v_n\}$ of $\mathbf{R}^3$, called the vertex set of the

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network. We also associate to each arc $(i,j)\in A$ a rectifiable arc $[v_i,v_j]\subset\mathbb{R}^3$, called arc of the network, which has the orientation from v_i to v_j. Let assume that $[v_j,v_i]$ has the positive length e_{ij} and denote by U the set of all arcs. We define the network N=(V,U) by the union

$$N = \bigcup_{(i,j)\in A} [v_i,v_j].$$

It is obvious that N is a geometric image of G, which follows naturally from an embedding of G in $\mathbb{R}^3$.

Suppose that for each $[v_i,v_j]$ in U there exists a continuous one-one mapping $T_{ij}:[0,1]\to[v_i,v_j]$ with $T_{ij}(0) = v_i$, $T_{ij}(1) = v_j$ and $T_{ij}([0,1]) = [v_i,v_j] \subset \mathbb{R}^3$.

Let Q_{ij} be the inverse of T_{ij}. To each point x from $[v_i,v_j]$ corresponds a unique point $Q_{ij}(x)$ in [0,1].

Any connected and closed subset of an arc $[v_i,v_j]$, bounded by two points x and y of $[v_i,v_j]$ and having the same orientation as $[v_i,v_j]$, is called a closed subarc and is denoted by [x,y]. If one or both of x, y miss we say that the subarc is open in x (or in y) or is open and we denote this by [x,y) or (x,y] or (x,y), respectively.

Using Q_{ij}, it is possible to compute the length of [x,y] as

$$e([x,y]) = |Q_{ij}(x) - Q_{ij}(y)|e_{ij}.$$

Particulary we have $e([v_i,v_j]) = e_{ij}$, $e([v_i,x]) = Q_{ij}(x)e_{ij}$ and $e([x,v_j]) = (1 - Q_{ij}(x))e_{ij}$.

Definition 1.1 A chain L(x,y) linking two points x and y in N is a sequence of arcs and at most two subarcs at extremities. The length of a chain is the lengths sum of all its component arcs and subarcs. If the chain L(x,y) contains only distinct vertices then we call it elementary.

Definition 1.2. A route D(x,y) starting from x and ending in y $(x,y \in N)$ is a chain, which has the same orientation for all the component arcs and subarcs. This also is the route orientation.

Let $L^*(x,y)$ be one of the shortest chains and $D^*(x,y)$ one of the shortest routes between the points x,y in N. We define in N a distance as follows:

$$d(x,y) = e(L^*(x,y)) \text{ for any } x,y \text{ in } N.$$

It is obvious that d is a metric on the oriented network N.

2. Divided differences

Our purpose in this section is to extend in a natural way the divided differences on n points of real functions (see [7], [8], [4]) for the functions defined on networks.

In order to define the divided differences of a function f on n points we will consider the notion of metric segment.

Definition 2.1 The metric segment between two different points $x,y \in N$ is the following set:

$$<x,y> = \{z \in N \mid d(x,z) + d(z,y) = d(x,y)\}.$$

Remark. Another way of stating definition 2.1, in geometric language, is to say that the metric segment $<x,y>$ coincides with the union of all the shortest chains between x and y.

So, it is easy to see why the above remark leds us to the following notion.

Definition 2.2 The metric oriented segment between $x,y \in N$ is the set:

$$<x,y>_o = \{z \in N \mid z \in D^*(x,y) \text{ and } d(x,z)+d(z,y)=d(x,y)\},$$

where $D^*(x,y)$ denote a shortest route linking the points $x,y \in N$.

Remarks. 1. Following the definition 2.2 the oriented metric segment is the union of all shortest routes from x to y which have the length d(x,y). Thus we can see that $<x,y>_o$

could be empty.

2. It is obvious that $<x,y>_o \subset <x,y>$.

3. $<x,y>_o \neq <y,x>_o$.

Let $x,y \in N$ be two distinct points such that $<x,y>_o \neq \emptyset$ and consider a shortest route $D^*(x,y) \subset <x,y>_o$. We introduce the function $d_{xy}: D^*(x,y) \times D^*(x,y) \to \mathbb{R}$,

$$d_{xy}(z_1,z_2) = \begin{cases} -d(z_1,z_2), & \text{if from } z_1 \text{ we can reach } z_2 \text{ on } D^*(x,y) \text{ following the route orientation} \\ d(z_1,z_2), & \text{if contrary} \end{cases}$$

Here are some elementary properties of d_{xy}.

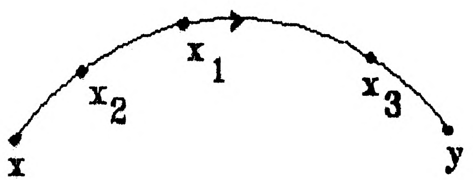
Lemma 2.1 a. $d_{xy}(z_1,z_2) = -d_{xy}(z_2,z_1)$, $\forall z_1,z_2 \in D^*(x,y)$.

b. $\forall x_1,x_2,x_3 \in D^*(x,y)$ are loc $d_{xy}(x_1,x_2)+d_{xy}(x_2,x_3)=d_{xy}(x_1,x_3)$.

Proof.

a. This equality is a direct consequence of d_{xy}'s definition.

b. Let us consider the points x,y,x_1,x_2,x_3 like in the following figure.



Then $d_{xy}(x_1,x_2) + d_{xy}(x_2,x_3) = d(x_1,x_2) - d(x_2,x_3) =$

$= -d(x_1,x_3) = d_{xy}(x_1,x_3)$.

The same reasoning applies to the case of any permutation of points x_1,x_2,x_3.■

Definition 2.3 Let $A\subset N$ and consider the real function $f:A\to\mathbb{R}$. We choose the points $x_1,\dots,x_n\in A$ pairwise distinct and such that there exist $x,y\in A$ and a shortest route $D^*(x,y)\subset\langle x,y\rangle_0$ for which is fullfiled the condition $x_1,\dots,x_n\in D^*(x,y)$. We call divided differecence of function f on points $x_1,\dots,x_n$ the number:

$$[x_1,\dots,x_n;f] = \frac{[x_2,\dots,x_n;f] - [x_1,\dots,x_{n-1};f]}{d_{xy}(x_n,x_1)}$$

where for any $z,t\in D^*(x,y)\cap A, z\neq t$, we set

$$[z,t;f] = \frac{f(t)-f(z)}{d_{xy}(t,z)}.$$

The following divided differences reprezentation formula holds:

Theorem 2.1 Given $f:A\to\mathbb{R}$ $(A\subset N)$ and the points $x_1,\dots,x_n$ under the conditions stated in definion 2.3, then

(1) $$[x_1,\dots,x_n;f] = \sum_{k=1}^{n}\frac{f(x_k)}{p_k(x_1,\dots,x_n)},\ \text{where}$$

$$p_k(x_1,\dots,x_n) = d_{xy}(x_k,x_1)\dots d_{xy}(x_k,x_{k-1})d_{xy}(x_k,x_{k+1})\dots d_{xy}(x_k,x_n).$$

Proof. We shall prove (1) by induction. For n=3 we have:

$$[x_1,x_2,x_3;f]=\frac{[x_2,x_3;f]-[x_1,x_2;f]}{d_{xy}(x_3,x_1)}=\frac{\dfrac{f(x_3)-f(x_2)}{d_{xy}(x_3,x_2)}-\dfrac{f(x_2)-f(x_1)}{d_{xy}(x_2,x_1)}}{d_{xy}(x_3,x_1)}=$$

$$=\frac{f(x_1)}{d_{xy}(x_2,x_1)d_{xy}(x_3,x_1)}-\frac{f(x_2)}{d_{xy}(x_3,x_1)}\left(\frac{1}{d_{xy}(x_3,x_2)}+\frac{1}{d_{xy}(x_2,x_1)}\right)+$$

$$+\frac{f(x_3)}{d_{xy}(x_3,x_1)d_{xy}(x_3,x_2)}$$

Using lemma 2.1 it follows

$$[x_1,x_2,x_3;f] = \sum_{k=1}^{3}\frac{f(x_k)}{p_k(x_1,x_2,x_3)}.$$

Assume (1) is true and prove the property holds for n+1 points, that is:

$$[x_1,\ldots,x_{n+1};f] = \sum_{k=1}^{n+1} \frac{f(x_k)}{p_k(x_1,\ldots,x_{n+1})}.$$

According to definition 2.3 and using the induction hypothesis we have:

$$[x_1,\ldots,x_{n+1};f] = \frac{1}{d_{xy}(x_{n+1},x_1)}\left[\sum_{k=1}^{n} \frac{f(x_{k+1})}{p_{k+1}(x_2,\ldots,x_{n+1})} - \sum_{k=1}^{n} \frac{f(x_k)}{p_k(x_1,\ldots,x_n)}\right] =$$

$$= \frac{1}{d_{xy}(x_{n+1},x_1)}\left[- \frac{f(x_1)}{d_{xy}(x_1,x_2)\ldots d_{xy}(x_1,x_n)} + \right.$$

$$+ f(x_2)\left(\frac{1}{d_{xy}(x_2,x_3)\ldots d_{xy}(x_2,x_{n+1})} - \right.$$

$$\left. - \frac{1}{d_{xy}(x_2,x_1)d_{xy}(x_2,x_3)\ldots d_{xy}(x_2,x_n)}\right) + \ldots +$$

$$+ f(x_n)\left(\frac{1}{d_{xy}(x_n,x_2)\ldots d_{xy}(x_n,x_{n-1})d_{xy}(x_n,x_{n+1})} - \right.$$

$$\left. - \frac{1}{d_{xy}(x_n,x_1)\ldots d_{xy}(x_n,x_{n-1})}\right) +$$

$$\left. + \frac{f(x_{n+1})}{d_{xy}(x_{n+1},x_1)\ldots d_{xy}(x_{n+1},x_n)}\right].$$

Taking in account lemma 2.1 under the transcription

$d_{xy}(x_1,x_k)+d_{xy}(x_k,x_{n+1})=d_{xy}(x_1,x_{n+1}), k=2,\ldots,n$ **and**

$d_{xy}(x_k,x_j) = - d_{xy}(x_j,x_k), k=1,\ldots,n+1, j=1,\ldots,n+1,$

we conclude:

$$[x_1,\ldots,x_{n+1};f] = \frac{1}{d_{xy}(x_{n+1},x_1)}\left[-\frac{f(x_1)}{d_{xy}(x_1,x_2)\ldots d_{xy}(x_1,x_n)}-\right.$$

$$-\frac{f(x_2)d_{xy}(x_1,x_{n+1})}{d_{xy}(x_2,x_1)\ldots d_{xy}(x_2,x_{n+1})}-\ldots-$$

$$\left.-\frac{f(x_n)d_{xy}(x_1,x_{n+1})}{d_{xy}(x_n,x_1)\ldots d_{xy}(x_n,x_{n-1})d_{xy}(x_n,x_{n+1})}+\frac{f(x_{n+1})}{d_{xy}(x_{n+1},x_2)\ldots d_{xy}(x_{n+1},x_n)}\right]=$$

$$=\sum_{k=1}^{n+1}\frac{f(x_k)}{p_k(x_1,\ldots,x_{n+1})}.$$ ■

Theorem 2.1 is an analogue for networks of a similar result concerning the usual divided differences ([7], [8], [4]). A immediate consequence of this characterisation theorem is:

Theorem 2.2 If f, g are real function defined on $A\subset N$, $\alpha\in\mathbb{R}$ and $x_1,\ldots,x_n\in N$ satisfy the conditions stated in definition 2.3, then:

$$[x_1,\ldots,x_n;f+g] = [x_1,\ldots,x_n;f] + [x_1,\ldots,x_n;g]$$

$$[x_1,\ldots,x_n;\alpha f] = \alpha[x_1,\ldots,x_n;f].$$

Proof. The above relations follow directly from (1).■

3. Functions of order n on networks.

The last part of this paper is devoted to define and study the functions of higher order.

Definition 3.1 If $x_1,\ldots,x_n\in N$ are n points ($n>1$) pairwise disctinct, we call this points a metric sequence if the following conditions are fullfiled:

a. There exist $x,y\in N$ and $D^*(x,y)\subset\langle x,y\rangle_0\neq\emptyset$ such that $x_1,\ldots,x_n\in D^*(x,y)$.

b. $d_{xy}(x_1,x_2) < 0$.

c. $\sum_{k=1}^{n-1} d(x_k,x_{k+1}) = d(x_1,x_n)$.

d. $d(x_{k-1},x_k)+d(x_k,x_{k+1})=d(x_{k-1},x_{k+1})$, $k=2,\ldots,n-1$.

Remark. One can easily seen that any subsequence of $\{x_1,\ldots,x_n\}$ is also a metric sequence.

Theorem 3.1 If f is a real valued function defined $A\subset N$ and $x_1,\ldots,x_n\in A$ is a metric sequence, then

$$(2)\quad [x_1,\ldots,x_n;f]=\sum_{k=1}^{n}\frac{(-1)^{n-k}f(x_k)}{d(x_k,x_1)\ldots d(x_k,x_{k-1})d(x_k,x_{k+1})\ldots d(x_k,x_n)}$$

Proof. Since $x_1,\ldots,x_n$ is a metric sequence it follows

$$(3)\qquad d_{xy}(x_k,x_j)=\begin{cases}-d(x_k,x_j),\text{dacă } j>k\\ d(x_k,x_j),\text{dacă } j<k\end{cases}$$

Using (3) in (1) we obtain (2).

Definition 3.2 A real valued function f defined on $A\subset N$ is called convex, notconcave, polynomial, notconvex, concave of order n on A if for any metric sequence $x_1,\ldots,x_{n+2}\in A$ the following inequalities holds respectively:

$$[x_1,\ldots,x_{n+2};f] >, \geq, =, \leq, < 0.$$

All these functions are of order n.

Remark. If we take $n=1$ for notconcavity in definition 3.2 we recover the usual d-convex functions in metric spaces ([10]).

Definition 3.3 A real valued function $f:A\to\mathbb{R}$ is d-convex on A $A\subset N$ if

$$(4)\qquad f(z)\leq\frac{d(z,y)}{d(x,y)}f(x)+\frac{d(x,z)}{d(x,y)}f(y),$$

for any points x,y,z in A such that $z\in\langle x,y\rangle_0$ (which is the same with saying that x,z,y is metric sequence).

(4) is equivalent with

$$\frac{f(x)}{d(x,y)d(x,z)} - \frac{f(z)}{d(x,z)d(y,z)} + \frac{f(y)}{d(x,y)d(y,z)} \geq 0.$$

Using (2) the above relation becomes:

$$[x,z,y;f] \geq 0, \forall x,z,y \in A$$

where x,z,y is a metric sequence. This corresponds to definition 3.2 in the case of first order notconcave functions.

Definition 3.4 A circuit in a oriented network N is a route D(x,y) with x=y. In what follows C allways denote a circuit.

The main result of this section is:

Theorem 3.2 If C is a elementary circuit, then any real function $f:C \to \mathbb{R}$ of order n is polynomial.

Proof. We shall prove that any notconcave function is polynomial on C. The same reasonning applies to notconvex functions. It is obvious that if the above statement is true then there exist no convex or concave function of order n on C.

If $f:C \to \mathbb{R}$ is notconcave, then for any metric sequence $x_1, \ldots, x_{n+2} \in C$ we have

(5) $$[x_1,\ldots,x_{n+2};f] \geq 0.$$

Consider a metric sequence $x_1, \ldots, x_{n+2} \in C$. It is easy to see that there exists m>n and the points $x_{n+3}, \ldots, x_m$ satifying the conditions:

1. The numbering of this points is made such that for any $k \in \{1,\ldots,m\}$, from x_k we can reach x_{k+1} following the orientation of C. We assume $x_{m+1}=x_1$.

2. The sets $\{x_1,\ldots,x_{n+2}\},\{x_2,\ldots,x_{n+3}\},\ldots,\{x_m,x_1,\ldots,x_{n+1}\}$

are metric sequences.

Using definition 2.3 and (5) we obtain:

$$(6) \qquad [x_i,\dots,x_{i+n+1};f] = \frac{[x_{i+1},\dots,x_{i+n+1};f]-[x_i,\dots,x_{i+n};f]}{d(x_{i+n+1},x_i)} \geq 0,$$

for each $i\in\{1,\dots,m\}$ and under the assumption that

$$x_{i+n+1} = \begin{cases} x_{i+n+1}, \text{ if } i+n+1\leq m \\ x_{i+n+1-m}, \text{ if } i+n+1>m \end{cases}$$

It is obvious that (6) implies:

$$[x_{i+1},\dots,x_{i+n+1};f] \geq [x_i,\dots,x_{i+n};f], \forall i\in\{1,\dots,m\}.$$

Thus we can write the following sequence of inequalities:

$[x_1,\dots,x_{n+1};f] \leq [x_2,\dots,x_{n+2};f] \leq \dots \leq [x_m,x_1,\dots,x_n;f] \leq$

$\leq [x_1,\dots,x_{n+1};f]$. This clearly forces the equality:

(7) $[x_1,\dots,x_{n+1};f] = [x_2,\dots,x_{n+2};f] = \dots = [x_m,x_1,\dots,x_n;f]$.

Using (7) in (6) yields $[x_1,\dots,x_{n+2};f]=0$. Since the metric sequence was arbitrarily choosen we can conclude that f is polinomial of order n on C.■

It is ovious that

Corollary 3.1 A n order real function is polinomial on any union of elementary adjacent circuits.

Corollary 3.2 Any real valued function f, defined on a set A⊂N which contains a elementary circuit C cannot be convex or concave of order n, whatever the natural number n is.

For usual d-convex functions we sate now a stronger result. The proof is adapted from [9], p. 127.

Theorem 3.3 Any real valued and d-convex function, defined on a elementary circuit

C is constant.

Proof. We consider the d-convex function $f:C\rightarrow\mathbb{R}$, where $C\subset N$ is a elementary circuit. We want to prove that $f(x)=f(y)$ holds for any pair $x,y\in C$. It is easy to see that there exist the points $z_1, \ldots, z_n\in C$ $(n\geq 5)$ satifying the conditions:

1. $z_i\in<z_{i-1},z_{i+1}>_o$, $i=2,\ldots,n$, where $z_{n+1}=z_1$,

2. $z_1=x$ and $\exists k\in\{2,\ldots,n\}$ such that $z_k=y$.

Assume f reach the maximum value on the set $\{z_1,\ldots,z_n\}$ in z_p. Since f is d-convex we have:

$$f(z_p) \leq \frac{d(z_{p+1},z_p)}{d(z_{p-1},z_{p+1})}f(z_{p-1}) + \frac{d(z_{p-1},z_p)}{d(z_{p-1},z_{p+1})}f(z_{p+1}) \leq$$

$$\leq \frac{d(z_{p+1},z_p)}{d(z_{p-1},z_{p+1})}f(z_p) + \frac{d(z_{p-1},z_p)}{d(z_{p-1},z_{p+1})}f(z_p) = f(z_p),$$

which leds us to $f(z_p)=f(z_{p-1})=f(z_{p+1})$. Repeated application of this reasoning enable us to write $f(z_1)=\ldots=f(z_n)$, and thus $f(x)=f(y)$.

Since x and y was arbitrarily chosen we conclude that f is constant on C.■

Definition 3.5 We call the elementary circuits $C_1,C_2\subset N$ adjacent if $C_1\cap C_2\neq\varnothing$.

Corollary 3.3 Any d-convex function defined on the connected set $A\subset N$ is constant on the union of all adjacent circuits in A.

Proof. This corollary is the direct consequence of theorem 3.3 and definition 3.5.■

We mention now some elementary properties of functions of order n.

Theorem 3.4

1. Given the real number $\alpha>0$ and two real functions convex (notconcave, polynomial,

notconvex, concave) of order n f,g defined on A⊂N, then f+g and αf are also convex (notconcave, polynomial, notconvex ,concave) of order n.

2. The limit of a punctualy convergent sequence of convex (or notconcave) functions of order n is notconcave of order n.

3. The limit of a punctualy convergent sequence of concave (or notconvex) functions of order n is notconvex of order n.

4. The limit of a punctualy convergent sequence of polynomial functions of order n is also polynomial of order n.

Proof.

1. This follows from theorem 2.2.

2. Let us consider a metric sequence $x_1,\dots,x_{n+2}\in N$ and $f_i:N\to\mathbf{R}$, convex (or notconcave) of order n, for each $i\in\mathbf{N}$. If $f:N\to\mathbf{R}$, $f(z)=\lim\limits_{i\to\infty} f_i(z)$ then

$$[x_1,\dots,x_{n+2};f]=$$

$$=\sum_{k=1}^{n+2}\frac{(-1)^{n+2-k}f(x_k)}{d(x_k,x_1)\dots d(x_k,x_{k-1})d(x_k,x_{k+1})\dots d(x_k,x_{n+2})}=$$

$$=\sum_{k=1}^{n+2}\frac{(-1)^{n+2-k}\lim\limits_{i\to\infty} f_i(x_k)}{d(x_k,x_1)\dots d(x_k,x_{k-1})d(x_k,x_{k+1})\dots d(x_k,x_{n+2})}=$$

$$=\lim_{i\to\infty}\sum_{k=1}^{n+2}\frac{(-1)^{n+2-k}\, f_i(x_k)}{d(x_k,x_1)\dots d(x_k,x_{k-1})d(x_k,x_{k+1})\dots d(x_k,x_{n+2})}=$$

$$=\lim_{i\to\infty}[x_1,\dots,x_{n+2};f_i]\geq 0.$$

For 3. and 4. one can use a proof similar with that made for 2. ■

The technique we use here to introduce divided differences and function of higher order allows us to make other natural extensions to networks of some types of generalized convex function, for example cvasiconvex function of order n ([5], [6]).

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