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SURFACES GENERATED BY BLENDING INTERPOLATION

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Dedicated to Professor Emil Muntean on his 60* anniversary

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REZUMAT. - Suprafeţe generate prin interpolare blending. Folosind proprietatea funcţiei interpolatoare blending de a coincide cu funcţia pe care o interpolează pe puncte, segmente sau arce de curbă situate în domeniul de definiţie al funcţiei, sunt generate suprafeţe controlate de valori ale funcţiei şi derivate ale acestora de gradul I sau II

The blending interpolation has many practical applications As it is well know, blending interpolation is the interpolation at an infinite set of points segments, curves, etc Thus, if one gives the contour of an object by such elements (points, segments, curves) using a blending interpolation, we can generate a surface that contains the given contour Hence, we can construct a surface (a blending function interpolant) which mach a given function and certain of its derivatives on the boundary of a plan domain (rectangle, triangle, etc)

Using such a surface fitting technique it was constructed the roof surfaces for large halls (industrial halls, exposition halls, public buildings) [4,5,6,7,8]

Our goal is to construct some new such surfaces using Lagrange's, Hermite's and Birkoff's interpolatory operators

Let $T_h = \{(x,y)\in\mathbb{R}^2 \mid x\geq 0,\ y\geq 0,\ x+h\leq h\}$ be the standard triangle and $f\ T_h \to \mathbf{R}$ a given function

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The operators used are.

1) Lagrange's operators L_1^x, L_1^y and L_1^{xy} defined by

$$\left(L_1^x f\right)(x,y) = \frac{h-x-y}{h-y} f(0,y) + \frac{x}{h-y} f(h-y,y)$$

$$\left(L_1^y f\right)(x,y) = \frac{h-x-y}{h-x} f(x,0) + \frac{y}{h-x} f(x,h-x)$$

$$\left(L_1^{xy} f\right)(x,y) = \frac{x}{x+y} f(x+y,0) + \frac{y}{x+y} f(0,x+y)$$

each of them interpolating the function f on two of the sides of T_h

2) Hermite's operators H_3^x, H_3^y and H_3^{xy} corresponding to the double nodes

$$\begin{aligned}\left(H_3^x f\right)(x,y) = {} & \frac{(h-x-y)^2(h+2x-y)}{(h-y)^3} f(0,y) + \frac{x(h-x-y)^2}{(h-y)^2} f^{(1,0)}(0,y) + \\ & + \frac{x^2(3h-2x-3y)}{(h-y)^3} f(h-y,y) + \frac{x^2(x+y-h)}{(h-y)^2} f^{(1,0)}(h-y,y)\end{aligned}$$

$$\begin{aligned}\left(H_3^y f\right)(x,y) = {} & \frac{(h-x-y)^2(h-x+2y)}{(h-x)^3} f(x,0) + \frac{y(h-x-y)^2}{(h-x)^2} f^{(0,1)}(x,0) + \\ & + \frac{y^2(3h-3x-2y)}{(h-x)^3} f(x,h-x) + \frac{y^2(x+y-h)}{(h-x)^2} f^{(0,1)}(x,h-x)\end{aligned}$$

$$\begin{aligned}\left(H_3^{xy} f\right)(x,y) = {} & \frac{y^2(3x+y)}{(x+y)^3} f(0,x+y) + \frac{xy^2}{(x+y)^2}(f^{(1,0)}-f^{(0,1)})(0,x+y) + \\ & + \frac{x^2(x+3y)}{(x+y)^3} f(x+y,0) - \frac{x^2 y}{(x+y)^2}(f^{(1,0)}-f^{(0,1)})(x+y,0)\end{aligned}$$

3) Birkhoff's operators B_1^x and B_1^y defined by

$$\left(B_1^x f\right)(x,y) = f(0,y) + (x+y-h) f^{(1,0)}(h-y,y)$$

$$\left(B_1^y f\right)(x,y) = f(x,0) - (x+y-h) f^{(0,1)}(x,h-x)$$

4) Birkhoff's operators B_3^x and B_3^y with

$$\left(B_3^{\lambda}f\right)(x,y) = f(0,y)+\frac{x(x^2-3\lambda x+6h\lambda-3h^2)}{3h(2\lambda-h)}f^{(1,0)}(0,y)+$$
$$+\frac{x^2(2x-3h)}{3(2\lambda-h)}f^{(2,0)}(\lambda,y)+\frac{2x^2(3\lambda-x)}{3h(2\lambda-h)}f^{(1,0)}(h,y)$$

$$\left(B_3^{\gamma}f\right)(x,y) = f(x,0)+\frac{y(y^2-3\gamma y+6h\gamma-3h^2)}{3h(2\gamma-h)}f^{(0,1)}(x,0)+$$
$$+\frac{y^2(2y-3h)}{3(2\gamma-h)}f^{(0,2)}(x,\gamma)+\frac{2y^3(3\gamma-y)}{3h(2\gamma-h)}f^{(0,1)}(x,h)$$

for $\lambda,\gamma\in[0,h]$

1 For the begining we construct a scalar interpolating formula generated by the operators L_1^x, L_1^y and H_3^x, H_3^y and H_3^{xy}, using two levels of interpolation

First, the function f is approximated by the boolean sum of the operators L_1^x and L_1^y

(1) $$\left(L_1^x\oplus L_1^y f\right)(x,y)=\frac{h-x-y}{h-y}f(0,y)+\frac{h-x-y}{h-x}f(x,0)+\frac{y}{h-x}f(x,h-x)-$$
$$-\frac{h-x-y}{h}f(0,0)-\frac{y(h-x-y)}{h(h-y)}f(0,h)$$

In order to obtain a scalar approximant of f, we use in the second level the following approximations

$$f(0,y) \approx \left(H_3^y f\right)(0,y),\ f(x,0) \approx \left(H_3^x f\right)(x,0) \text{ and } f(x,h-x) \approx \left(H_3^{xy} f\right)(x,h-x)$$

Let

(2) $$f=Pf+Rf,$$

with

(3) $$(Pf)(x,y)=\frac{(h-x-y)(h^2+hx+hy-2x^2-2y^2)}{h^3}f(0,0)+\frac{x^2(3h-2x)}{h^3}f(h,0)+$$
$$+\frac{y(2hx+3hy-2x^2-2xy-2y^2)}{h^3}f(0,h)+\frac{x(h-x)(h-x-y)}{h^2}f^{(1,0)}(0,0)+$$
$$+\frac{y(h-y)(h-x-y)}{h^2}f^{(0,1)}(0,0)-\frac{x^2(h-x)}{h^2}f^{(1,0)}(h,0)+\frac{x^2y}{h^2}f^{(0,1)}(h,0)+$$
$$+\frac{xy(h-x)}{h^2}f^{(1,0)}(0,h)-\frac{y[y(h-x-y)+x(h-x)]}{h^2}f^{(0,1)}(0,h)$$

be the obtained interpolation formula

Theorem 1 If there exist $f^{(1,0)}(V_i)$ and $f^{(0,1)}(V_i)$, $i=1,2,3$, where V_i are the vertixes of T_h, then Pf interpolates f and its first partial derivatives at V_i, $i=1,2,3$

Also $Pg=g$ for all $g\in \mathrm{P}_2^2$, i e the exactness degree of P is two

The proof of the theorem is a straighthforward computation

Theorem 2 If $f\in B_{1,2}(0,0)$ [10] then

$$(Rf)(x,y)=\int_0^h \varphi_{30}(x,y,s)f^{(3,0)}(s,0)ds+\int_0^h \varphi_{21}(x,y,s)f^{(2,1)}(s,0)ds+$$
$$+\int_0^h \varphi_{03}(x,y,t)f^{(0,3)}(0,t)dt+\iint_{T_h}\varphi_{12}(x,y,s,t)f^{(1,2)}(s,t)dsdt,$$

where

$$\varphi_{30}(x,y,s)=\frac{(x-s)_+^2}{2}-\frac{x^2(3h-2x)}{h^3}\frac{(h-s)^2}{2}+\frac{x^2(h-x)}{h^2}(h-s)$$
$$\varphi_{21}(x,y,s)=y(x-s)_+ - \frac{x^2y}{h^2}(h-s)$$
$$\varphi_{03}(x,y,t)=\frac{(y-t)_+^2}{2}-\frac{y(2hx+3hy-2x^2-2xy-2y^2)}{h^3}(h-t)^2+$$
$$+\frac{y[y(h-x-y)+x(h-x)]}{h^2}(h-t)$$
$$\varphi_{12}(x,y,s,t)=(x-s)_+^0(y-t)_+$$

The proof follows by Peano's theorem for a triangular domain [2]

The approximation formula (2) is tested on the function $f(x,y)=1/(x^2+y^2+1)$ The graphs of the function f and of the approximation Pf are given in Fig 1 and Fig 2.

Remark Such an interpolation formula can be used to obtain a cubature formula over a triangle

2 Next, it will be used the given interpolatory operators to generate some surfaces on

the domain $D_h=\{(x,y)\in\mathbb{R}^2 \mid |x|+|y|\le h\}$

Such a surface is constructed first on the triangle T_h, ofter that is extended by symmetry with respect to the coordinate axes on all D_h

First examples of such surfaces are obtained from the approximation function Pf (3), for

(A) $f(0,0)=4, f(h,0)=f(0,h)=f^{(1,0)}(0,0)=f^{(0,1)}(0,0)=$
$=f^{(1,0)}(h,0)=f^{(0,1)}(0,h)=0$

and $f^{(1,0)}(0,h)=f^{(0,1)}(h,0)=-0.5$ (Fig 3)

respectively

(B) $f(0,0)=4, f(h,0)=f(0,h)=f^{(1,0)}(h,0)=f^{(0,1)}(0,h)=0$,

$f^{(1,0)}(0,0)=f^{(0,1)}(0,0)=-1$ and

$f^{(1,0)}(0,h)=f^{(0,1)}(h,0)=-0.25$ (Fig 4)

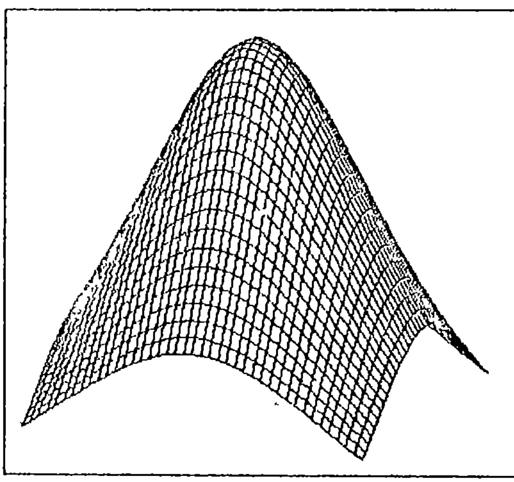

Fig 1

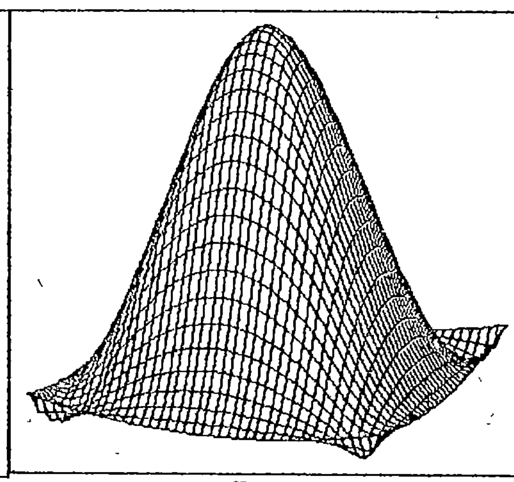

Fig 2

Now one supposes that the function f take the value zero on the border of D, i e $f|_{\partial D}=0$ This is equivalent with the condition $f(x,h-x)=0$ for $x\in[0,h]$ Using this condition

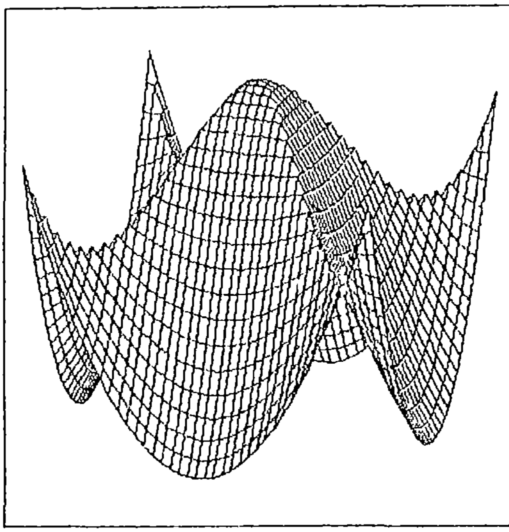

Fig 3

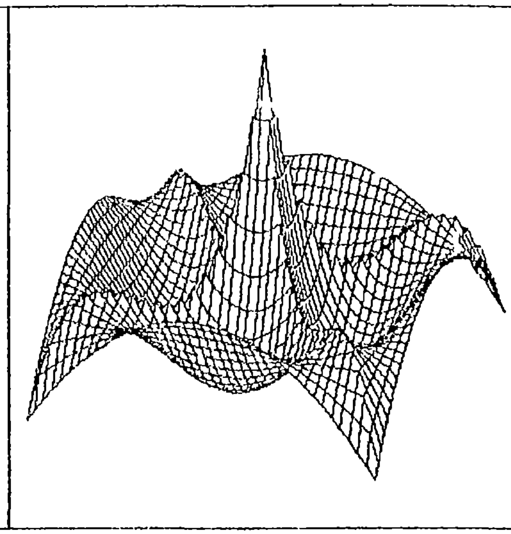

Fig 4

from (1) one obtains

$$L(x,y)=\frac{h-x-y}{h-y}f(0,y)+\frac{h-x-y}{h-x}f(x,0)-\frac{h-x-y}{h}f(0,0)$$

Taking $f(0,h)=\left(H_3^y f\right)(0,y)$ and $f(x,0)=\left(H_3^x f\right)(x,0)$, in the same condition $f(x,h-x)=0$ for all $x\in[0,h]$, one obtains the class of surfaces

$$H(x,y)=\frac{h-x-y}{h^3}\Big[\left(h^2+hx-2x^2-2y^2\right)f(0,0)+hx(h-x)f^{(1,0)}(0,0)+$$
$$+hy(h-y)f^{(0,1)}(0,0)-hx^2f^{(1,0)}(h,0)-hy^2f^{(0,1)}(0,h)\Big],$$

which depends on the data

$$\left(f(0,0),f^{(1,0)}(0,0),f^{(0,1)}(0,0),f^{(1,0)}(h,0),f^{(0,1)}(0,h)\right)$$

For the data (4,-1,-1,-1,-1) one obtains the surface from the Fig 5

Another class of surfaces is given by the boolean sum of the operators G_3^x and G_3^y obtained from H_3^x respectively H_3^y in the conditions $f(x,h-x)=f^{(1,0)}(x,h-x)=f^{(0,1)}(x,h-x)=0$ for all $x\in[0,h]$, i e

$$\left(G_3^x f\right)(x,y)=\frac{(h-x-y)^2(h+2x-y)}{(h-y)^3}f(0,y)+\frac{x(h-x-y)^2}{(h-y)^2}f^{(1,0)}(0,y)$$

$$\left(G_3^y f\right)(x,y)=\frac{(h-x-y)^2(h-x+2y)}{(h-x)^3}f(x,0)+\frac{y(h-x-y)^2}{(h-x)^2}f^{(0,1)}(x,0)$$

We have

$$\begin{aligned}\left(G_3^x \oplus G_3^y f\right)(x,y)=(h-x-y)^2\Bigg[&\frac{h+2x-y}{(h-y)^3}f(0,y)+\frac{x}{(h-y)^2}f^{(1,0)}(0,y)+\\&+\frac{h-x+2y}{(h-x)^3}f(x,0)+\frac{y}{(h-x)^2}f^{(0,1)}(x,0)-\frac{h^2+2hx+2hy+6xy}{h^4}f(0,0)-\\&-\frac{x(h+2y)}{h^3}f^{(1,0)}(0,0)-\frac{y(y+2x)}{h^3}f^{(0,1)}(0,0)-\frac{xy}{h^2}f^{(1,1)}(0,0)\Bigg]\end{aligned}$$

Now, for

$$f(0,y)=\left(B_1^y f\right)(0,y)=f(0,0)+(y-h)f^{(0,1)}(0,h)$$

$$f(x,0)=\left(B_1^x f\right)(x,0)=f(0,0)+(x-h)f^{(1,0)}(h,0)$$

and

$$f^{(1,0)}(0,y)=\left(L_1^y f^{(1,0)}\right)(0,y)=\frac{h-y}{h}f^{(1,0)}(0,0)+\frac{y}{h}f^{(1,0)}(0,h)$$

$$f^{(0,1)}(x,0)=\left(L_1^x f^{(0,1)}\right)(x,0)=\frac{h-x}{h}f^{(0,1)}(0,0)+\frac{x}{h}f^{(0,1)}(h,0)$$

one obtains

$$\begin{aligned}G(x,y)=(h-x-y)^2\Bigg\{&\left[\frac{h+2x-y}{(h-y)^3}+\frac{h-x+2y}{(h-x)^3}-\frac{h^2+2hx+2hy+6xy}{h^4}\right]f(0,0)+\\&+\frac{xy(2y-h)}{h^3(h-y)}f^{(1,0)}(0,0)+\frac{xy(2x-h)}{h^3(h-x)}f^{(0,1)}(0,0)-\frac{xy}{h^2}f^{(1,1)}(0,0)-\\&-\frac{h-x+2y}{(h-x)^2}f^{(1,0)}(h,0)+\frac{xy}{h(h-y)^2}f^{(1,0)}(0,h)-\\&-\frac{h+2x-y}{(h-y)^2}f^{(0,1)}(0,h)+\frac{xy}{h(h-x)^2}f^{(0,1)}(h,0)\Bigg\}\end{aligned}$$

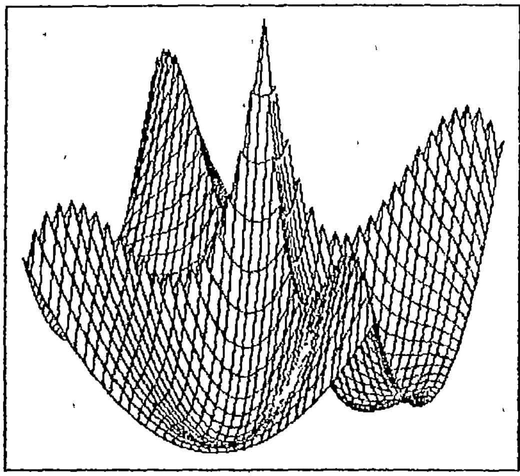

Fig 5.

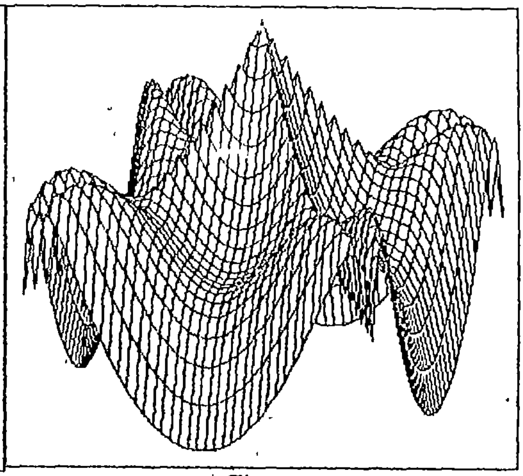

Fig 6

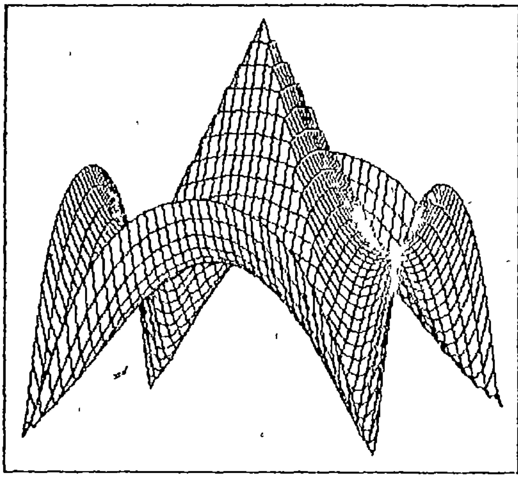

Fig 7.

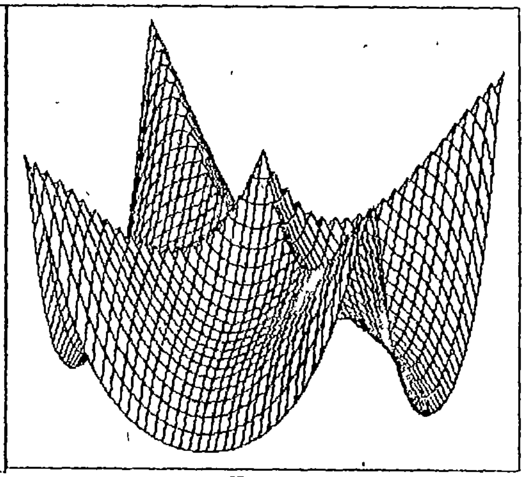

Fig 8

This surfaces depend on the data

$$(f(0,0), f^{(1,0)}(0,0), f^{(0,1)}(0,0), f^{(1,1)}(0,0),$$
$$f^{(1,0)}(h,0), f^{(1,0)}(0,h), f^{(0,1)}(0,h), f^{(0,1)}(h,0))$$

As an example (Fig 6) is given the surface obtained for the data (4,-1,-1,1,0,5,0,5)

The last class of surfaces is generated using the Fejer's type operators F_3^x and F_3^y obtained from H_3^x and H_3^y for

$$f^{(1,0)}(0,y)=f^{(1,0)}(h-y,y)=f^{(0,1)}(x,0)=f^{(0,1)}(x,h-x)=0$$

Taking into account the general condition that $f(x,h-x)=0$ for $x\in[0,h]$, one obtains

$$\left(F_3^x \oplus F_3^y f\right)(x,y)=(h-x-y)^2\left[\frac{h+2x-y}{(h-y)^3}f(0,y)+\right.$$
$$\left.+\frac{h-x+2y}{(h-x)^2}f(x,0)-\frac{(h+2x-y)(h+2y)}{h^3(h-y)}f(0,0)\right]$$

In order to control the inflexion points we take

$$f(0,y)=\left(B_3^y f\right)(0,y)$$
$$f(x,0)=\left(B_3^x f\right)(x,0)$$

One obtains

$$F(x,y)=(h-x-y)^2\left[\frac{h+2x-y}{(h-y)^3}\left(B_3^y f\right)(0,y)+\right.$$
$$\left.+\frac{h-x+2y}{(h-x)^2}\left(B_3^x f\right)(x,0)-\frac{(h+2x-y)(h+2y)}{h^3(h-y)}f(0,0)\right]$$

that depends on

$$(f(0,0),f^{(1,0)}(0,0),f^{(0,1)}(0,0),f^{(1,0)}(0,h),$$
$$f^{(0,1)}(0,h),f^{(2,0)}(\lambda,0),f^{(0,2)}(0,\gamma)),$$

where $\lambda,\gamma\in[0,h]$

Two example are taken here, for the data (4,-1,-1,0,0,0,0) with $\lambda=\gamma=5$ (Fig 7) and (4,-0 75,-0 75,0,0,2,2) with $\lambda=\gamma=15$ (Fig 8)

Finally, we remark that for any of the presented classes of surfaces, for convenable data, can be obtained a large variety of surfaces

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