ON THE CONVERGENCE OF THE THREE-ORDER METHODS IN FRECHET SPACES

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REZUMAT. - Asupra convergenţei metodelor de ordinul trei în spaţii Fréchet. În lucrare se demonstrează existenţa şi unicitatea existenţei ecuaţiei (1) precum şi convergenţa metodei iterative (2), renunţînd la uniform mărginirea operatorului $\Lambda = [x',x'';P]^{-1}$.

1. It is known that the rapidity of convergence for the sequence of approximates (x_n) of solution of the operatorial equation

$$P(x) = \theta \tag{1}$$

given by an iterative method, can be improved if the first and the second order divided differences, which enter in the algorithm exprimation, are taken on special nodes.

In the case of operatorial equation

$$P(x) \equiv x-F(x) = \theta \tag{2}$$

using the metod

$$x_{n+1} = x_n-\Lambda_n(I-[x_n,u_n,v_n;P]\ \Lambda_n\ P(u_n)\ \ \bar{\Lambda}_n\ )^{-1}\ P(x_n) \tag{3}$$

where

$$\Lambda_n = [x_n,u_n;P]^{-1};\ \ \bar{\Lambda}_n = [u_n,v_n;P]^{-1}$$

and

$$u_n = F(x_n);\ v_n=F(u_n) = (F(x_n))$$

this property is proved in the paper [1]. The following theorem

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are proved:

THEOREM A. *If for $x_0 \in X$, there exist μ_0, B, $M>1$ and N so that the following conditions:*

1) $\| P(x_0) \| < \mu_0$;

2) *For any $x', x'', x''', x^{IV} \in S(x_0, R)$, R remaining to be defined, we have*

a. $\Lambda = [x', x''; P]^{-1}$ exists and $\| \Lambda \| < B$;

b. $\| [x', x''; F] \| < M$;

c. $\| [x', x'', x'''; P] \| < K$;

d. $\| [x', x'', x^{IV}; P] - [x', x'', x'''; P] \| <$
$< N \| x^{IV} - x''' \|$

3) $G_0 h_0 < 1$ *where* $h_0 := B^2 M K \mu_0 < 1/2$ *and*

$$G_0^2 := \frac{M(1+BK\mu_0)\,[1+BK\mu_0(1+M)]}{(1+h_0)^2(1-2h_0)}\left(1+\frac{N}{BK^2}\right)$$

hold, then the equation (2) has the solution $x^* \in S(x_0, R)$, where

$$R = (1+M)\mu_0 + M^2 Q \quad \text{and} \quad Q = \frac{B\mu_0}{1-h_0}\sum_{m=0}^{n}(G_0 h_0)^{3^m-1}$$

solution which is the limit of the sequence genereted by (3), the rapidity of convergence being given by

$$\|x^* - x_m\| \leq (G_0 h_0)^{3^m-1} Q.$$

THEOREM B. *In the conditions of Theorem A, the solution of equation (2) is unique.*

In the following, we will change the condition 2a of Theorem A, removing the uniform bounded of the operator Λ.

2. Let us consider the equation

$$P(x) = x - F(x) = \theta$$

where $P: X \to X$ is a continuous operator considered with its generalized divided difference [2] up to the second order, inclusively, X is a Fréchet space with a quasinorm induced by a distance invariant to translation, i.e $)|\, x\, |(= d(x,\theta)$, $x,\theta \in X$ [3].

To solve the equation (2) we consider the algoritm (3).

Concerning the convergence of the method (3), we prove

THEOREM. *If for $x_0 \in X$ exists $\overline{\mu_0}$, $\bar M > 1$, $\bar K$ and $\bar N$ such that the following conditions:*

1^0 *For any $x', x'', x''', x^{IV} \subset S$, where $S = \{x \mid)|x - x_0|(< R\}$,*

$$R = (1+\bar M)\,\mu_0 + \bar M^2 \bar D\ ; \quad \bar D = \frac{\bar\mu_0}{1-\bar h_0}\sum_{m=0}^{n} (\bar G_0 \bar h_0)^{3^m-1}$$

we have:

a. $\Lambda = [x', x''; P]^{-1}$ *exists;*

b. $)|\, \Lambda[x', x''; F]\, |(< \bar M;$

c. $)|\, \Lambda[x', x'', x'''; P]\, |(< \bar K;$

d. $)|\, \Lambda\, ([x', x'', x^{IV}; P] - [x', x'', x'''; P])\, |(< \bar N\,)|x^{IV} - x'''\, |($

2^0 $)|\, \Lambda P(x_0)\, |(< \overline{\mu_0};$

3^0 $\bar G_0\, \bar h_0 < 1$ *where* $\bar h_0 := \bar M^2\, \bar K\, \overline{\mu_0} < 1/2$ *and*

$$\bar G_0^2 = \frac{\bar N(1+\bar K\bar\mu_0)\,[1+\bar K\bar\mu_0(1+\bar M)]}{(1+\bar h_0)^2(1-2\bar h_0)}\left(1+\frac{\bar N}{K^2}\right)$$

hold, then the equation (2) has the unique $x^* \subset S(x_0, R)$, which is *the limit of the sequence genereted by* (3), *the rapidity of convergence being given by*

$$\| x^*-x_n \| < (\overline{C}_0 \overline{h}_0)^{3^{n}-1} \cdot \overline{\rho}.$$

Proof. We consider the equation

$$\overline{P}(x) = \theta \qquad (6)$$

where

$$\overline{P}(x) = \Lambda P(x) = \Lambda(x-F(x)), \quad \Lambda=[x',x'';P]^{-1}$$

equation which is equivalent to (2).

Indeed, if x^* is a solution of equation (2), i.e. $P(x^*)=\theta$, due to liniarity of Λ, it results

$$\Lambda P(x^*) = \overline{P}(x^*) = \theta. \qquad (7)$$

Reciprocal, if x^* is a solution of equation (6), i.e.

$$\overline{P}(x^*) = \Lambda P(x^*) = \theta$$

from the existence of operator Λ, it results $\Lambda^{-1}=[x',x'';P]$ which, applied to the left of the equation (6), leads to $P(x^*)=\theta$.

For solving this equation, we have the iterative method

$$\overline{x}_{n+1}=\overline{x}_n-\overline{\Lambda}_n(I-[\overline{x}_n,\overline{u}_n,\overline{v}_n;\overline{P}]\overline{\Lambda}_n\overline{P}(\overline{u}_n)\ \tilde{\overline{\Lambda}}_n)^{-1}\overline{P}(\overline{x}_n) \quad . \qquad (8)$$

Using the induction, one can prove that for $x_0=\overline{x_0}$, $u_0=\overline{u_0}$, $v_0=\overline{v_0}$

the sequence given by (8) is identical with the sequence (3).

For the operator $\bar{P}$, the conditions of Theorem A and B are true. Indeed

1^0 $\|\bar{P}(x_0)\| = \|\Lambda P(x_0)\| < \bar{\mu}_0$;

2^0 For any $x', x'', x''', x^{IV} \in S(x_0,R)$, we have

a) $\bar{\Lambda} = [x',x'';\bar{P}]^{-1} = (\Lambda[x',x'';P]^{-1} = I$, then

$\bar{\Lambda}$ exists and $\|\bar{\Lambda}\| = 1 = \bar{B}$;

b) $\|[x',x'';\bar{P}]\| = \|\Lambda[x',x'';F]\| < \bar{M}$;

c) $\|[x',x'',x''';\bar{P}]\| = \|\Lambda[x',x'',x''';P]\| < \bar{K}$

d) $\|[x',x'',x^{IV};\bar{P}] - [x',x'',x''';\bar{P}]\| =$

$= \|\Lambda([x',x'',x^{IV};P] - [x'',x'',x''';P])\| <$

$< \bar{N}\|x^{IV} - x'''\|$;

3^0 $\bar{C}_0\bar{h}_0 < 1$, where $\bar{h}_0 := \bar{B}^2\bar{M}\bar{K}\bar{\mu}_0 < \frac{1}{2}$ and

$$\bar{C}^2 := \frac{\bar{M}(1+\bar{B}\bar{K}\bar{\mu}_0)[1+\bar{B}\bar{K}\bar{\mu}_0(1+\bar{M})]}{(1+h_0)^2(1-2h_0)}\left(1+\frac{\bar{N}}{BK^2}\right)$$

It results that the hypotesis of Theorem A are satisfied by P, hence the equation (6) has a solution $x^* \in S$, which is the limit of sequence generated by the algoritm (3) or (8), the rapidity of convergente being given by (4).

Because (6) is equivalente to (2), the statement results.

REFERENCES

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