STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXVI, 3, 1991

ON INDEPENDENT SETS OF GRAPHS

V. CIOBAN, V. PREJMEREAN, S. MOTOGNA*

Received: March 3, 1991
AMC Subject Classification: 05C90,68R10

Rezumat. - Asupra seturilor independente de grafe. Lucrarea trece în revistă unii algoritmi de determinare a mulţimilor independente (mulţimi interior stabile) referitoare la un graf. În prima parte se prezintă cîţiva algoritmi care au la bază expresii şi/sau ecuaţii booleene precum şi un algoritm recursiv şi anume algoritmul dat de Taulbee şi Bednarek. În final autorii dau un algoritm recursiv inspirat din acest ultim algoritm.

1. **Definition, properties.** Let $G = (V,T)$ be an undirected graph where:

- V is the set of vertices and $|V| = n$;
- $T : V \to V$ is the aplication which defines the graph.

DEFINITION : Let $S \subset V$. S is an independent set (IS) iff $\forall\, v \in S,\ T_v \cap S = \emptyset$.

(where we denote $T(v)$ by T_v).

In other words the vertices of S don't have any edges between each other.

Observations:

1^0 We may define $G = (V,E)$ where E is the set of edges, $E \subset V \times V$, an edge is $[x,y]$, $x,y \in V$ and $[x,x] \notin E$.

2^0 Let S be an IS. S is called maximal if S is maximal by sets inclusion.

3^0 We denote by $\mathbf{S}$ the set of all maximal IS of G.

We remember that:

a) $\alpha(G)$ is the number of internal stability:

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* *University of Cluj-Napoca, Department of Computer Science, 3400 Cluj-Napoca, Romania*

$$\alpha(G) = \max_{S \in \mathcal{S}} |S|$$

b) $\gamma(G)$ is the chromatic number of G, $\gamma(G)$ is the smallest number of IS, disjoints, which cover G.

4^0 Moon and Moser have proved that:

$$\alpha(G) \le \begin{cases} 3^{\frac{n}{3}} & ,\text{if } n=3k \\ 4*3^{\frac{n-1}{3}-1} & ,\text{if } n=3k+1 \\ 2*3^{\frac{n-2}{3}} & ,\text{if } n=3k+2 \end{cases}$$

2. **Algorithms for determining *IS*.** In many problems it is important to find the *IS* family.

There are some algebraic or combinatorial algorithms to find *IS*.

2.1. Maghout and Weissman'S algorithm based on boolean expression.

2.2. Malgrange algorithm's which finds every squared matrix containing only 0 (zero) of the adjacent matrix where:

$$A = (a_{i,j}) \ ; \ i=\overline{1,n} \ ; \ j=\overline{1,n} \text{ with}$$

$$a_{ij} = \begin{cases} 1 & ,\text{if } [v_i, v_j] \in E \\ 0 & ,\text{otherwise} \end{cases}$$

2.3. The Rudeanu's method, using the boolean equations which caracterize the *IS* family.

Let $G = (V,E)$, where $V = \{v_1,\ldots,v_n\}$.

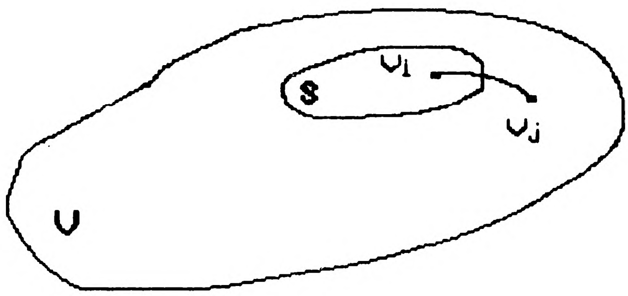
If $S \subset V$ is an *IS* then we associate to each $v_i \in V$ an boolean variable b_i define by :

$$b_i=\begin{cases}1 & ,\text{if } v_i\in S\\ 0 & ,\text{if } v_i\notin S\end{cases}$$

We have the following result:

If $a_{ij} = 1$ then it results $a_{ij}.b_i.b_j = 0$ (1)

(because $v_i \in S$ and $v_j \notin S$, see the diagram).



So from (1) it results that $\bigvee_{a_{ij}=1} b_i \cdot b_j=0$ iff $\bigwedge_{a_{ij}=1} \overline{b_i \cdot b_j}=1$ iff $\bigwedge_{a_{ij}=1} (\overline{b_i}\vee\overline{b_j})=1$ iff $\bigvee \overline{b_{k_1}}\cdot\overline{b_{k_2}}\ldots\overline{b_{k_p}}=1$

So, for each factor $\overline{b_{k_1}}\cdot\overline{b_{k_2}}\ldots\overline{b_{k_p}} = 1$ we have an *IS* :

$$S = \{x_{k_{p+1}}, x_{k_{p+2}}, \ldots, x_{k_n}\}.$$

2.4. Bednarek and Taulbee's recursiv algorithm

Let $G = (V,E)$ be an undirected graph:

- $\forall\ k = 1,\ldots,n$ we denoted by $V_k = \{v_1,\ldots,v_k\}$;
- for each subgraph with $V_k = \{v_1,\ldots,v_k\}$; we denoted by L_k the maximal *IS* family;
- we also denote $Y_k = \{y \in V_k\ /\ [x_k,y] \notin E\}$.

The steps of the algorithm are:

S1. Let $Y_1 = \{v_1\}$, $L_1 = \{v_1\}$, $k=1$.

S2. One finds the next family: $I_k=\{S/S=M\cap Y_{k+1},\ M\in L_k\}$.

S3. One finds $I'_k = \{I\ /\ I \subset I_k,\ I$ maximal with respect to sets inclusion$\}$.

S4. One finds L^*_{k+1} family, for each $M \in L_k$:

a) if $M \subset Y_{k+1} \rightarrow M \cup \{v_{k+1}\} \in L^*_{k+1}$

b) if $M \not\subset Y_{k+1} \rightarrow M \in L^*_{k+1}$ and $\{v_{k+1}\} \cup (M \cap Y_{k+1}) \in L^*_{k+1}$ iff $M \cap Y_{k+1} \in I'_k$

The L^*_{k+1} family contains only these sets of S4.

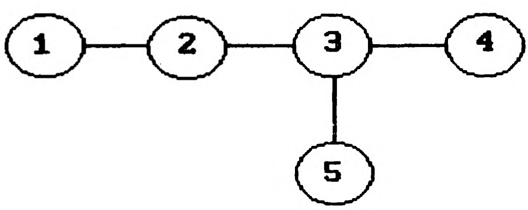
S5. One finds the maximal family L_{k+1} from L^*_{k+1} wich respect to sets inclusion.

S6. Repeat S2,S3,S4,S5 for $k=2,\ldots,n-1$. Finally we have L_n which contains the maximal *IS* of *G*.

Example :

Let $G = (V,E)$ be an undirected graph with

$V = \{1,2,3,4,5\}$, $E = \{[1,2],[2,3],[3,4],[3,5]\}$



In the next table we have:

k	Y_{k+1}	I_k	I'_k	L^*_{k+1}	L_{k+1}
1	{2}	∅	∅	{1},{2}	{1},{2}

2	{1,3}	∅,{1}	{1}	{1,3},{2}, {3}	{2}, {1,3}
3	{1,2,4}	∅,{1},{2}	{1},{2}	{1,3},{1,4}, {2,4},{4}	{1,3}, {1,4}, {2,4}
4	{1,2,4,5}	∅,{1},{1,4}	{1,4}	{1,3}, {1,4,5}, {2,4,5},{5}	{1,3}, {1,4,5} {2,4,5}

$\rightarrow L = \{\{1,3\}, \{1,4,5\}, \{2,4,5\}\}$

2.5. In what follows we suggest the next algorithm:

The notations used:

Let $G = \{V,E\}$ be an undirected graphs and:

$V_k = \{v_1,\ldots,v_k\}$, $|V| = n, 1 \le k \le n$;

L_k = the sets family of *IS* associated with V_k, $1 \le k \le n$.

The steps of the algorithm are:

S1. $L_1 = \{v_1\}$, $k=2$.

S2. One finds L_k :

a) if $M \in L_{k-1} \rightarrow M \in L_k$.

b) if $M \in L_{k-1}$ and $\forall\, y \in M$ with $[y,x_k] \notin E \rightarrow M \cup \{x_k\} \in L_k$.

c) $\{v_k\} \in L_k$.

Repeat S2 for $k=2,3,\ldots,n$.

S3. Reducing L_n with respect to sets inclusion:

$\forall\, M,N \in L_n$ and $M \subset N \rightarrow L_n = L_n \setminus M$.

For the previous graph we have:

L_1 : {1}.

L_2 : {1}, {2}.

L_3 : {1}, {2}, {1,3}, {3}.

L_4 : {1}, {2}, {1,3}, {3}, {1,4}, {2,4}, {4}.

L_5 : {1}, {2}, {1,3}, {3}, {1,4}, {2,4}, {4}, {1,5}, {2,5}
{1,4,5}, {2,4,5}, {4,5}, {5}.

Applying S3 we obtain:

L : { {1,3}, {1,4,5}, {2,4,5} }.

The algorithm is very simple and it works only with a single sets family.

R E F E R E N C E S

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