Lectures on the Differential Geometry of Curves and Surfaces

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To Cristina, with love

Contents

Foreword

This book is based on the lecture notes of several courses on the differential geometry of curves and surfaces that I gave during the last eight years. These courses were addressed to different audience and, as such, the lecture notes have been revised again and again and once almost entirely rewritten. I tried to expose the material in a modern language, without loosing the contact with the intuition. There is a common believe that mathematics is eternal and it never changes, unlike other sciences. I think that nothing could be farther from the truth. Mathematics *does* change continuously, either for the need of more rigor, either for the need of more structure. The mathematics we are doing and teaching today is essentially different (and not only as language!) from the mathematics of the nineteenth century. This variation also applies, of course, to the teaching of mathematics.

There are several reason why we need to renew the textbooks from time to time:

- the mathematics evolves and we need to introduce new results and notions;
- some parts of specific courses become obsolete and should be removed;
- the curriculum as a whole evolves and we have to keep pace with the developments of the neighboring fields;
- some courses move from the undergraduate level to the graduate level or (more often) the other way around and we have to modify the contents and the language accordingly.

These are only some of the reasons I decided to write this book.

The material included is fairly classical. I preferred to discuss in more details the foundations rather than introduce more and more topics. In particular, unlike most of books of this kind, I decided not to discuss at all problems of *global* differential geometry. I believe it is not very useful to discuss such topics, if there is not enough room to give them all the attention they deserve. Also, the Levi-Civita parallelism and the covariant differentiation are not discussed, because I think they would feel a lot better in the course of Riemannian geometry. The notion of differentiable manifold is not introduced but it is only one step away. In fact, I decided to adapt the language of the course in such a way to qualify for a prerequisite for a course in smooth manifolds.

Is not easy to be original when you teach something that have been taught again and again for more than two centuries and, on the other hand, should we be always original at any cost?! As such, it goes without saying that this book owes much to many excellent classical or more modern textbooks. All of them are mentioned in the bibliography. I have to mention, however, the books of Gray, do Carmo and Fedenko.

I have to confess that I am *not* a self-educated person (not entirely, anyway), and my ideas regarding differential geometry have been influenced by my teachers (the late Prof. Marian Ţarină and Prof. Dorin Andrica), as well by the colleagues with whom I had the pleasure to work and to share the ideas, Prof. Pavel Enghiş, Csaba Varga, Cornel Pintea, Daniel Văcăreţu and Liana Ţopan.

This book would have never existed without my students and their enthusiasm (or lack of enthusiasm) determined many changes during the years.

A book is not something that you leave at the office and this one was no exception and it would never have been finished without the infinite patience of my wife who tolerated the (undeterministic) chaos created by books and manuscripts.

Cluj-Napoca, April 2005.

Part I

Curves

CHAPTER 1

Space curves

1.1 Introduction

Intuitively, the curves are just deformations of straight lines. They can be thought of, therefore, as "one-dimensional" objects. We are familiar with some of them already from elementary mathematics, since, obviously, the graphs of functions can be considered as curves, from this point of view. On the other hand, clearly, usually the curves *are not* graphs of functions (well, at least not globally); it is enough to think of a conical section (for instance an ellipse or a circle, in particular). Thus, in general we cannot represent a curve by an equation of the form $y = f(x)$, as it would be the case for a graph. On the other hand, we can represent a conical section also by an implicit equation of the form $f(x, y) = 0$, where, in this particular case, of course, as it is known, f is a second degree polynomial in x and y. Finally, we can represent the coordinates of each point of the curve as functions of one parameter. As we shall see, this is, usually, the most convenient way to represent, locally, an arbitrary curve.

An important issue that has to be dealt with is related to the smoothness of the functions used to describe a curve. Of course, we are interested to apply the tools of differential calculus. We shall assume, therefore, that all the functions are at least once continuous differentiable and, if higher order derivatives are involved, we shall assume, without stating it in clear, that all the derivatives exists and are continuous. We shall use for the functions which are satisfying these conditions the generic term "smooth". Apart from the computational aspects, there are other, deeper reasons for considering functions at least once continuously differentiable. Suppose, for instance, that a curve is described by a set of equations of the form

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases} .$$

Now, it can be shown that if the functions f and g are only continuous, the curve can fill an entire square (or the entire plane). The first example of such an anomalous curve (which, obviously, contradicts the common sense image of a curve as an one-dimensional object) has been constructed by the Italian mathematician Giuseppe Peano, at the end of the XIX-th century. Moreover, this strange phenomenon does not disappear not even if the functions f and g are differentiable, but not *continuously* differentiable. In the figure 1.1 we indicate an iterative process that defines a square-filling (Peano) curve. The curve itself is the limit of the curves obtained by this iterative process. It is possible, actually, to describe analytically this curve (i.e. we can find an expression for each iteration), but, as these "curves" are not the subject of our investigation, we prefer to let the reader satisfy the curiosity by himself. We have to say, however, that the functions we use to describe a curve do not have to be, necessarily, continuous differentiable to avoid aforementioned anomalies. What it is required is a little bit less, namely the functions have to be of *bounded variation*. It is a well-known fact that the continuous differentiable functions do verify this condition and, moreover, as we already emphasized, they also endow us with the necessary computational tools, which are not available for arbitrary functions with bounded variation.

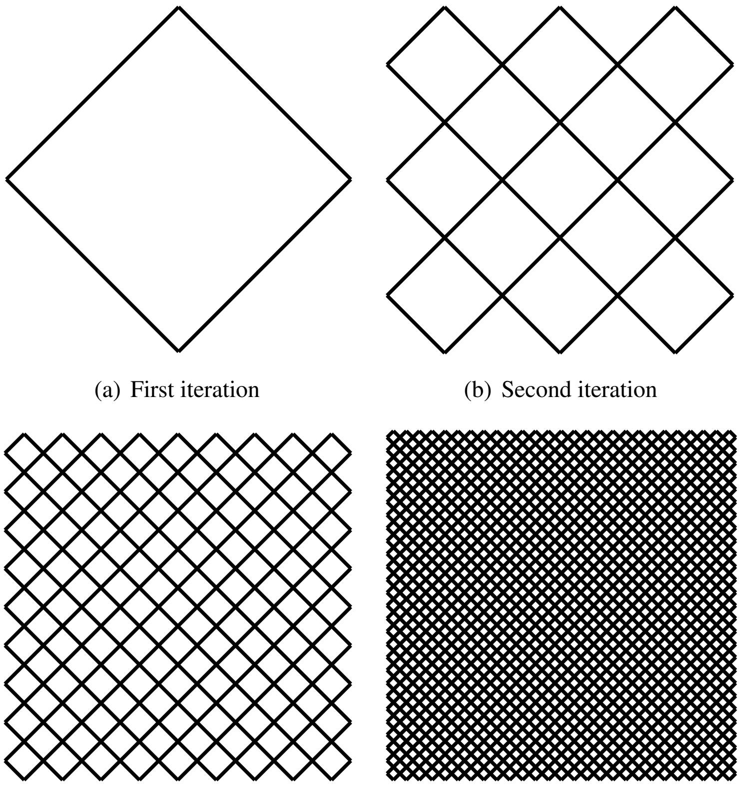
1.2 Parameterized curves (paths)

Let I be an interval on the real axis $\mathbb{R}$. We shall not assume always that the interval is open. Sometimes it is even important that the interval be closed. In particular, the interval can be unbounded and can coincide with the entire real axis.

Definition 1.2.1. A C^k $(k > 0)$ *parameterized curve* (or *path*) in the Euclidean space $\mathbb{R}^3$ is a C^k mapping

$$\mathbf{r} : I \to \mathbb{R}^3 : t \to (x(t), y(t), z(t)). \tag{1.2.1}$$

A parameterized curve is, typically, denoted by $(I, \mathbf{r})$, $(I, \mathbf{r} = \mathbf{r}(t))$ or, when the interval is implicit, just $\mathbf{r} = \mathbf{r}(t)$. We note that, in fact, a path is C^k iff the (real valued) functions x, y, z are C^k. If the interval is not open, we shall assume, first of all, that the functions we are considering are of class C^k in the interior of the interval and all the derivatives, up



(a) First iteration (b) Second iteration

(c) Third iteration (d) Fourth iteration

Figure 1.1: The Peano's curve (first four iterations)

to the k-th order, have a finite lateral (left, or right) limit at the extremities of the interval, if these extremities belong to the interval.

The path is called *compact, half-open* or *open*, if the interval I is, respectively, compact, half-open or open.

If the interval I is bounded from below, from above or from both parts, then the imagine of any extremity of I is called an *endpoint* of the path. If, in particular, the curve is compact and the two endpoints coincide, the path is called *closed*. An alternative denomination used for a closed path is that of a *loop*.

Occasionally (for instance in the theory of the line integral) we might need to consider paths which are of class C^k at all the points of an interval, with the exception of a finite number of points. The following definition will make this more precise.

Definition 1.2.2. We shall say that a compact parameterized curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is *piecewise* C^k if there exists a finite subdivision $(a = a_0, a_1, \ldots, a_n = b)$ of the interval $[a, b]$ such that the restriction of $\mathbf{r}$ to each compact interval $[a_{i-1}, a_i]$ is of class C^k, where $i \in \{1, \ldots, n\}$.

Remark. It is not difficult to show that a parameterized curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is piecewise C^k iff the following conditions are simultaneously fulfilled:

(i) The set

$$S = \left\{t \in [a, b] \mid f^{(k)} \text{ does not exist}\right\}$$

is finite.

(ii) $f^{(k)}$ is continuous on $[a, b] \setminus S$.

(iii) $f^{(k)}$ has a finite left and right lateral limit at each point of S.

Hereafter we shall suppose, all the time, that the order of smoothness k is high enough and we will not mention it any longer (with some exceptions, however), using the generic term of *smooth parameterized curve* (meaning, thus, at least C^1 and, in any particular case, if kth order derivatives are involved, at least C^k).

The image $\mathbf{r}(I) \subset \mathbb{R}^3$ of the interval I through the mapping (1.2.1) is termed the *support* of the path $(I, \mathbf{r})$.

If $\mathbf{r}(t_0) = a$, we shall say that the parameterized curve *passes* through the point a for $t = t_0$. Sometimes, for short, we shall refer to this point as *the point* t_0 of the parameterized curve.

Examples

1. Let $\mathbf{r}_0 \in \mathbb{R}^3$ be an arbitrary point, and $\mathbf{a} \in \mathbb{R}^3$ a vector, $\mathbf{a} \neq 0$, while $I = \mathbb{R}$. The parameterized curve $\mathbb{R} \to \mathbb{R}^3$, $t \to \mathbf{r}_0 + t\mathbf{a}$ is called *a straight line*. Its support is the straight line passing through $\mathbf{r}_0$ (for $t = 0$) and having the direction given by the vector $\mathbf{a}$.

2. $I = \mathbb{R}$, $\mathbf{r}(t) = \mathbf{r}_0 + t^3\mathbf{a}$. The support of this path is the same straight line.

3. $I = \mathbb{R}$, $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $a, b \in \mathbb{R}$. The support of this parameterized curve is called *a circular cylindrical helix* (see figure 1.2).

4. $I = [0, 2\pi]$, $\mathbf{r}(t) = (\cos t, \sin t, 0)$. The support of the path is the unit circle, lying in the xOy-plane, having the center at the origin of the coordinates.

5. $I = [0, 2\pi]$, $\mathbf{r}(t) = (\cos 2t, \sin 2t, 0)$. The support is the same from the previous example.

6. $I = \mathbb{R}$, $\mathbf{r}(t) = (t^2, t^3, 0)$. This curve has a cusp at the point $t = 0$.

Definition 1.2.3. A parameterized curve (1.2.1) is called *regular for* $t = t_0$ if $\mathbf{r}'(t_0) \neq 0$ and *regular* if it is regular for each $t \in I$.

As we shall see a little bit later, the notion of regularity at a point of a parameterized curve is related to the existence of a well defined tangent to the curve in that point.

The curves from the previous example are regular, with the exception of those from the points 2 and 6, which are not regular from $t = 0$.

Remark. The fact that the *same* support can correspond both to a regular and a nonregular parameterized curve suggests that the absence of the regularity at a point doesn't necessarily mean that the corresponding point of the support has some geometric peculiarities. It's only that the regularity *guarantees* the absence of these peculiarities. Indeed, if we look, again, at the curves 2 and 6 from the previous example, we notice immediately that, although they are, both, nonregular for $t = 0$, only for the second curve this analytic singularity implies a geometric singularity (a cusp), while for the first one the support is just a straight line, without any special points.

To each path corresponds a subset of $\mathbb{R}^3$, its support. Nevertheless, as the examples 1 and 2 show, different parameterized curves may have the same support. A parameterized curve can be thought of as a subset of $\mathbb{R}^3$, with a parameterization.[1] The support of a parameterized curve corresponds to our intuitive image of a curve as a one dimensional geometrical object. As we shall see later, the support of a parameterized curve may have self-intersections or cusps, which, for many reasons, are not desirable in applications. The regularity conditions rules out the cusps, but not the self-intersections. To eliminate them, we have to impose some further conditions.

We have seen that different parameterized curves may have the same support. In the end, it is the support, as a set of points, we are interested in. It is, therefore, necessary to identify some relations between the parameterized curves that define the same support. For reasons that will become clear later, for the moment, at least, we are interested only in regular curves. Therefore, for instance, passing from a parameterized representation of the support to another should not change the regularity of the curve.

Definition 1.2.4. Let $(I, \mathbf{r} = \mathbf{r}(t))$, $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ two parameterized curves. A diffeomorphism $\lambda : I \to J : t \to s = \lambda(t)$ such that $\mathbf{r} = \boldsymbol{\rho} \circ \lambda$, i.e. $\mathbf{r}(t) \equiv \boldsymbol{\rho}(\lambda(t))$, is called *a*

[1] Of course, the information is redundant, since the parametric representation defines the support.

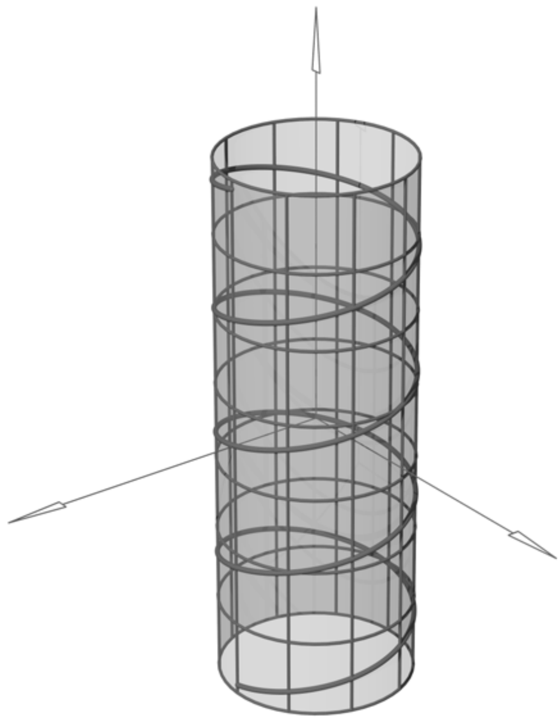


Figure 1.2: The circular helix

parameter change or *a reparameterization*. Two parameterized curves for which there is a parameter change are called *equivalent*, while the points t and $s = \lambda(t)$ are called *correspondent*.

Remarks. 1) The relation just defined is an equivalence relation on the set of all parameterized curves.

2) The reparameterization has a simple kinematical interpretation. If we interpret the parametric equations of the path as being the equations of motion of a particle, then the support is just the trajectory, while the vector $\mathbf{r}'(t)$ is the velocity of the particle. The effect of a reparameterization is the modification of the speed with which the trajectory is traversed. Also, if $\lambda'(t) < 0$, then the trajectory is traversed in the opposite sense. Note that the two velocity vectors of equivalent parameterized curves in correspondent points have the same direction. They may have different lengths (the speed) and sense.

Example. The parameterized curves from the examples 1 and 2 are not equivalent, although, as mentioned, they have the same support.

Remark. Sometimes, the equivalence classes determined by the relation defined before between parameterized curves are called *curves*. We shall not follow this line here, because we would like the curves to be objects slightly more general that just supports of parameterized curves. In particular, as we shall see in a moment, usually the curves cannot be represented globally through the same parametric equations. It is enough to think about the circle. One of the most common parametric representation of the circle is

$$\begin{cases} x = \cos\theta, \\ y = \sin\theta \end{cases}.$$

Now, if we let the parameter vary only in the interval $(0, 2\pi)$, then one of the points of the circle is not represented. Of course, we can extend the interval, but then the same point corresponds to more than a value of the parameter, which is, again, not acceptable.

Among all parameterized curves equivalent to a given parameterized curve, there is one which has a special theoretical meaning and which simplifies many proofs in the theory of curve, although, in most cases, it is very difficult to find it analytically and, thus, its practical value is limited.

Definition 1.2.5. We shall say that a parameterized curve is *naturally parameterized* if $\|\mathbf{r}'(s)\| = 1$ for any $s \in I$. Usually, the natural parameter is denoted by s.

Remark. One can see immediately that any smooth, naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$ is *regular*, since, clearly, $\mathbf{r}'(s)$ cannot vanish, as its norm is nowhere vanishing.

It is by no means obvious that for any parameterized curve there is another one, equivalent to it and naturally parameterized. To construct such a curve, we need first another notion.

The *arc length* of a path $(I, \mathbf{r} = \mathbf{r}(t))$ between the points t_1 and t_2 is the number[2]

$$l_{t_1,t_2} = \left| \int_{t_1}^{t_2} \|\mathbf{r}'(t)\| dt \right|.$$

Remark. There is a good motivation for defining in this way the length of an arc. We can consider an arbitrary division $t_1 = a_0 < a_1 < \cdots < a_n = t_2$ of the interval $[t_1, t_2]$ and examine the polygonal line $\gamma_n = \mathbf{r}(a_0)\mathbf{r}(a_1)\cdots\mathbf{r}(a_n)$. The length of this polygonal line is the sum of the lengths of its segments. It can be shown that if the parameterized curve

[2]Although the integrand is positive we did not assume that $t_1 < t_2$, therefore the integral might be negative and the absolute value is necessary, if we want to obtain something positive.

$(I, \mathbf{r})$ is "good enough" (for instance at least once continuously differentiable), then the limit of length of the polygonal line γ_n, when the norm of the division goes to zero, exists and it is equal to the arc length of the path. It is to be mentioned, also, that the definition of the arc length makes sense also for piecewise smooth curves, because in this case the tangent vector has only a finite number of discontinuity points and, therefore, its norm is integrable.

We are going to show that the arc lengths of two equivalent paths between correspondents points are equal, therefore the arc length is, in a way, a characteristic of the support[3].

Indeed, let $\mathbf{r}(t) = \boldsymbol{\rho}(\lambda(t))$, then $\mathbf{r}'(t) = \lambda'(t)\boldsymbol{\rho}'(\lambda(t))$. Therefore,

$$\left|\int_{t_1}^{t_2} \|\mathbf{r}'(t)\| dt\right| = \left|\int_{t_1}^{t_2} \|\boldsymbol{\rho}'(\lambda(t))\| \cdot |\lambda'| dt\right| =$$

$$= \left|\int_{t_1}^{t_2} \|\boldsymbol{\rho}'(\lambda)\| \underbrace{\lambda'(t)dt}_{d\lambda}\right| = \left|\int_{\lambda_1}^{\lambda_2} \|\boldsymbol{\rho}'(\lambda)\| d\lambda\right|.$$

For naturally parameterized curves, $(I, \mathbf{r} = \mathbf{r}(s))$,

$$l_{s_1,s_2} = |s_2 - s_1|.$$

In particular, if $0 \in I$ (which can always be assumed, since the translation is a diffeomorphism), then $l_{0,s} = |s|$, i.e., up to sign, the natural parameter is the arc length.

Proposition 1. *For any regular parameterized curve there is a naturally parameterized curve equivalent to it.*

Proof. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a regular parameterized curve, $t_0 \in I$, and

$$\lambda : I \to \mathbb{R}, \quad t \to \int_{t_0}^{t} \|\mathbf{r}'(\tau)\| d\tau.$$

The function λ is smooth and strictly increasing, since $\lambda'(t) = \|\mathbf{r}'(t)\| > 0$. Therefore, its image will be an open interval J, while the function $\lambda : I \to J$ will be a diffeomorphism. The parameterized curve $(J, \boldsymbol{\rho}(s) = \mathbf{r}(\lambda^{-1}(s))$ is equivalent to $(I, \mathbf{r})$ and it is naturally parameterized, since $\boldsymbol{\rho}'(s) = \mathbf{r}'(\lambda^{-1}(s))(\lambda^{-1})'(s)$, while

$$(\lambda^{-1})'(s) = \frac{1}{\lambda'(\lambda^{-1}(s))} = \frac{1}{\|\mathbf{r}'(\lambda^{-1}(s))\|}$$

[3] In a way, because we may represent the same set of points as support of another parameterized curve, which is not equivalent to the initial one. The new path might have a different arc length between the same points of the support.

and, hence,

$$\|\boldsymbol{\rho}'(s)\| = \|\mathbf{r}'(\lambda^{-1}(s))\| \cdot |(\lambda^{-1})'(s)| = 1.$$

□

Remark. In the proof of the previous proposition we used, in an essential way, the fact that all the points of the curve are regular. On an interval on which the curve has singular points, there is no naturally parameterized curve equivalent to it.

Example. For the circular helix

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt, \end{cases}$$

we get, through a parameter change,

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \|\{-a\sin\tau, a\cos\tau, b\}\| d\tau = \sqrt{a^2+b^2},$$

therefore,

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

Thus, the natural parameterization of the helix is given by the equations

$$\begin{cases} x = a\cos\frac{s}{\sqrt{a^2+b^2}} \\ y = a\sin\frac{s}{\sqrt{a^2+b^2}} \\ z = b\frac{s}{\sqrt{a^2+b^2}}. \end{cases}$$

Exercise 1.2.1. Find a natural parameterization of the curve

$$x = e^t\cos t, \quad ,y = e^t\sin t, \quad z = e^t.$$

Exercise 1.2.2. Show that the parameter along the curve

$$x = \frac{s}{2}\cos\left(\ln\frac{s}{2}\right), \quad y = \frac{s}{2}\sin\left(\ln\frac{s}{2}\right), \quad z = \frac{s}{\sqrt{2}}$$

is a natural parameter.

Remark. It should be noticed that, usually, the natural parameter along a parameterized curve cannot be expressed in finite terms (i.e. using only elementary functions) with respect to the parameter along the curve. This is, actually, impossible even for very simple curves, such that the ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t, \end{cases}$$

with $a \neq b$, for which the arc length can be expressed only in terms of elliptic functions (this is, actually, the origin of their name!). Therefore, although the natural parameter is very important for theoretical consideration and for performing the proofs, as the reader will have more than once the opportunity to see in this book, for concrete examples of parameterized curves we will hardly ever use it.

1.3 The definition of the curve

As we mentioned before, we can imagine, intuitively, a curve as being just a deformation of a straight line, without thinking, necessary, at an analytical representation. We expect the curve to have a well defined tangent at each point. In particular, this condition should rule out both cusps and self intersections.

Definition 1.3.1. A subset $M \subset \mathbb{R}^3$ is called a *regular curve* (or *a 1-dimensional smooth submanifold of* $\mathbb{R}^3$) if, for each point $a \in M$ there is a regular parameterized curve $(I, \mathbf{r})$, whose support, $\mathbf{r}(I)$, is an open neighbourhood in M of the point a (i.e. is a set of the form $M \cap U$, where U is an open neighborhood of a in $\mathbb{R}^3$), while the map $\mathbf{r} : I \to \mathbf{r}(I)$ is a homeomorphism, with respect to the topology of subspace of $\mathbf{r}(I)$. A parameterized curve with these properties is called a *local parameterization* of the curve M around the point a. If for a curve M there is a local parameterization $(I, \mathbf{r})$ which is *global*, i.e. for which $\mathbf{r}(I) = M$, the curve is called *simple*.

Remark. In some books in the definition it is required that the map $\mathbf{r} : I \to \mathbf{r}(I)$ be smooth, which is not completely rigorous, as $\mathbf{r}(I)$ is not an open subset of $\mathbb{R}^3$. What is meant, is, however, exactly the same, i.e. the map $\mathbf{r} : I \to \mathbb{R}^3$ is smooth.

Examples.

1. Any straight line in $\mathbb{R}^3$ is a simple curve, because it has a global parameterization, given by a function of the form $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{r}(t) = \mathbf{a} + \mathbf{b} \cdot t$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, $\mathbf{b} \neq 0$.

2. The circular helix is a simple regular curve, with the global parameterization $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, given by $\mathbf{r}(t) = (a\cos t, b\sin t, bt)$.

3. A circle in $\mathbb{R}^3$ is a curve, but it is not simple, since no open interval can be homeomorphic to the circle, which is a compact subset of $\mathbb{R}^3$.

Thus, a regular curve is just a subset of $\mathbb{R}^3$ obtained "gluing together smoothly" supports of parameterized curves. If we look carefully at the definition of the curve, we see that not any regular parameterized curve can be used as a local parameterization of a curve. For an arbitrary parameterized curve $(I, \mathbf{r})$, the map $\mathbf{r} : I \to \mathbb{R}^3$ is not injective and, thus, it cannot be a local parameterization. Let us, also, mention that even if the function is injective, $\mathbf{r} : I \to \mathbf{r}(I)$ might fail to be a homeomorphism (even if the map is continuous and bijective, the inverse might not be continuous).

If, for instance, we consider the parameterized curve $(I, \mathbf{r})$, with $I = \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$,

$$\mathbf{r}(t) = (\cos t, \sin t, 0),$$

then the support of this parameterized curve is the unit circle in the coordinate plane xOy, centred at the origin. We should not conclude, however, that the circle is a simple curve, since $\mathbf{r}$ is *not* a homeomorphism onto (in fact, it is not even injective, because it is periodic).

Now let us assume that $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(\tau))$ are two local parameterizations of a regular curve M around the same point $a \in M$. Then, as one could expect, the two parameterized curves are equivalent, if we restrict the definition intervals such that they have the same support. More specifically, the following theorem holds:

Theorem 1.3.1. *Let $M \subset \mathbb{R}^3$ be a regular curve and $(I, \mathbf{r} = \mathbf{r}(t))$, $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(\tau))$ – two local parameterizations of M such that $W \equiv \mathbf{r}(I) \cap \boldsymbol{\rho}(J) \neq \emptyset$. Then $(\mathbf{r}^{-1}(W), \mathbf{r}|_{\mathbf{r}^{-1}(W)})$ and $(\boldsymbol{\rho}^{-1}(W), \boldsymbol{\rho}|_{\boldsymbol{\rho}^{-1}(W)})$ are equivalent parameterized curves.*

Proof.

$$(I, \mathbf{r}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$$

and

$$(J, \boldsymbol{\rho}(\tau) = (x(\tau), y(\tau), z(\tau))$$

be two local parameterizations of the curve M. To simplify the notations, we shall assume, from the very beginning, that $\mathbf{r}(I) = \boldsymbol{\rho}(J)$. Clearly, the generality is not reduced by this assumption. We claim that the map $\lambda : I \to J$, $\lambda = \boldsymbol{\rho}^{-1} \circ \mathbf{r}$, is a diffeomorphism, providing, thus, a parameter change between the two parameterized curves.

λ is, clearly, a homeomorphism, as a composition of the homeomorphisms

$$\mathbf{r} : I \to \mathbf{r}(I)$$

and

$$\boldsymbol{\rho}^{-1} : \boldsymbol{\rho}(J) \to J.$$

Moreover $\mathbf{r} = \boldsymbol{\rho} \circ \lambda$. Therefore, all we have to prove is that the maps λ and λ^{-1} are both smooth. One might be tempted, at this point, to write λ as

$$\lambda = \boldsymbol{\rho}^{-1} \circ \mathbf{r}$$

and conclude that λ is smooth, as composition of two smooth functions. While this representation is legitimate, as both $\mathbf{r}$ and $\boldsymbol{\rho}$ are homeomorphism onto, $\boldsymbol{\rho}^{-1}$ *is not* a differentiable function and, for the time being, at least, it doesn't make sense to speak about its differentiability, as its domain of definition is not an open subset of the ambient Euclidean space. We will prove, instead that, locally, $\boldsymbol{\rho}^{-1}$ is the restriction of a differentiable mapping, defined, this time, on an open subset of $\mathbb{R}^3$.

Since the notion of differentiability is a local notion, it is enough to prove that λ is smooth in a neighbourhood of each point of the interval I. This could be achieved, for instance, by representing λ, locally, as a composition of smooth functions. Let $t_0 \in I$, $\tau_0 = \lambda(t_0)$. Due to the regularity of the map $\boldsymbol{\rho}$, we have $\boldsymbol{\rho}'(\tau_0) \neq 0$. We can assume, without restricting the generality, that the first component of this vector is different from zero, i.e. $x'(\tau_0) \neq 0$. From the inverse function theorem, applied to the function x, there is an inverse function $\tau = f(x)$, defined and smooth in an open neighbourhood $V \subset \mathbb{R}$ of the point $x_0 = x(\tau_0)$. Then, in the open neighbourhood $\boldsymbol{\rho}(f(V))$ of the point $(x(\tau_0), y(\tau_0), z(\tau_0))$ from M we will have $\boldsymbol{\rho}^{-1}(x, y, z) = f(x)$, which means that, in fact, we have

$$\boldsymbol{\rho}^{-1}\Big|_{\boldsymbol{\rho}(f(V))} = f \circ pr_1|_{\boldsymbol{\rho}(f(V))},$$

where $pr_1 : \mathbb{R}^3 \to \mathbb{R}$ is the projection of $\mathbb{R}^3$ on the first factor.

With this local expression of $\boldsymbol{\rho}^{-1}$ in hand, we can write λ on the open neighbourhood $\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))$ of t_0 as

$$\lambda|_{\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))} = \boldsymbol{\rho}^{-1}\Big|_{\boldsymbol{\rho}(f(V))} \circ \mathbf{r}|_{\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))} = f \circ pr_1|_{\boldsymbol{\rho}(f(V))} \circ \mathbf{r}|_{\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))}.$$

As the functions f, pr_1 and $\mathbf{r}$ are, all of them, smooth on the indicated domains, it follows that λ is smooth on the open neighbourhood $\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))$ of t_0. As t_0 was arbitrary, λ is smooth on the entire I. The smoothness of λ^{-1} is proven in the same way, changing the roles of $\mathbf{r}$ and $\boldsymbol{\rho}$. □

It follows from the definition that any regular curve is, locally, the support of a parameterized curve. Globally, this is not true, unless the curve is simple. Also, in general, the support of an arbitrary parameterized curve is not a regular curve. Take, for instance, the lemniscate of Bernoulli $(\mathbb{R}, \mathbf{r}(t) = (x(t), y(t), z(t)))$, where

$$\begin{cases} x(t) = \frac{t(1+t^2)}{1+t^4} \\ y(t) = \frac{t(1-t^2)}{1+t^4} \\ z = 0 \end{cases}.$$

$\mathbf{r}$ is continuous, even bijective, but the inverse is not continuous. In fact, the support has a self intersection, because $\lim_{t\to-\infty} \mathbf{r} = \lim_{t\to\infty} \mathbf{r} = \mathbf{r}(0)$ (see the figure 1.3). However, we can always restrict the domain of definition of a parameterized curve such that the support of the restriction is a regular curve.

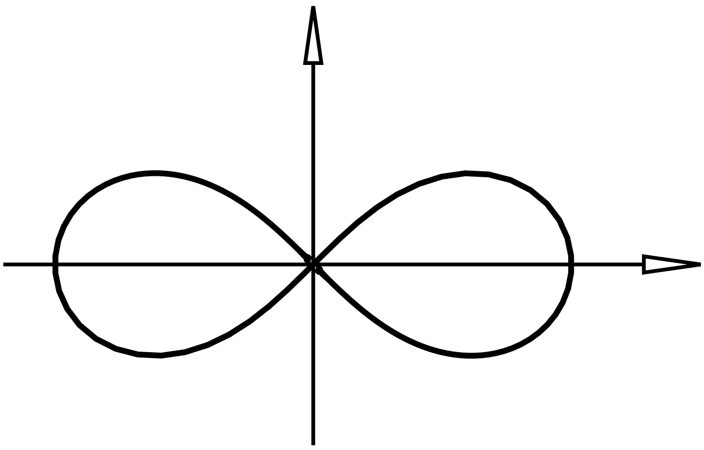


Figure 1.3: The Bernoulli's lemniscate

Theorem 1.3.2. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ a regular parameterized curve. Then each point $t_0 \in I$ has a neighbourhood $W \subset I$ such that $\mathbf{r}(W)$ is a simple regular curve.*

Proof. The regularity of $\mathbf{r}$ at each point means, in particular, that $\mathbf{r}'(t_0) \neq 0$. Without restricting the generality, we may assume that $x'(t_0) \neq 0$. Let us consider the mapping $\psi : I \times \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$\psi(t, u, v) = \mathbf{r}(t) + (0, u, v),$$

where $(u, v) \in \mathbb{R}^2$. ψ is, clearly, smooth and its Jacobi matrix at the point $(t_0, 0, 0)$ is given by

$$J(\psi)(t_0, 0, 0) = \begin{bmatrix} x'(t_0) & 0 & 0 \\ y'(t_0) & 1 & 0 \\ z'(t_0) & 0 & 1 \end{bmatrix}$$

Its determinant is

$$\det J(\psi)(t_0, 0, 0) = x'(t_0),$$

therefore ψ is a local diffeomorphism around the point $(t_0, 0, 0)$. Accordingly, there exist open neighborhoods $U \subset \mathbb{R}^3$ of $(t_0, 0, 0)$ and $V \subset \mathbb{R}^3$ of $\psi(t_0, 0, 0)$ such that $\psi|_V$ is a diffeomorphism from U to V. Let us denote by $\varphi : V \to U$ its inverse (which, of course, is, equally, a diffeomorphism from V to U, this time). If we put

$$W := \{t \in I : (t, 0, 0) \in U\},$$

then, clearly, W is an open neighborhood of t_0 in I such that

$$\varphi(V \cap \mathbf{r}(W)) = \varphi(\psi(W \times \{(0, 0)\}) = W \times \{(0, 0)\}.$$

□

Remark. The previous theorem plays a very important conceptual role. It just tells us that any *local* property of regular parameterized curves is valid, also, for regular curves, if it is invariant under parameter changes, *without* the assumptions of the parameterized curves being homeomorphisms onto. Of course, all the precautions should be taken when we investigate the *global* properties of regular curves.

1.4 Analytical representations of curves

1.4.1 Plane curves

A regular curve $M \subset \mathbb{R}^3$ is called *plane* if it is contained into a plane π. We shall, usually, assume that the plane π coincides with the coordinate plane xOy and we shall use, therefore, only the coordinates x and y to describe such curves.

Parametric representation. We choose an arbitrary local parameterization $(I, \mathbf{r}(t) = (x(t), y(t), z(t)))$ of the curve. Then the support $\mathbf{r}(I)$ of this local parameterization is an open subset of the curve. For a global parameterization of a simple curve, $\mathbf{r}(I)$ is the entire curve. Thus, any point a of the curve has an open neighbourhood which is the support of the parameterized curve

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} . \tag{1.4.1}$$

The equations (1.4.1) are called *the parametric equations* of the curve in the neighbourhood of the point a. Usually, unless the curve is simple, we cannot use the same set of equations to describe the points of an entire curve.

Explicit representation. Let $f : I \to \mathbb{R}$ be a smooth function, defined on an open interval from the real axis. Then its graph

$$C = \{(x, f(x)) \mid x \in I\}, \tag{1.4.2}$$

is a simple curve, which has the global parameterization

$$\begin{cases} x = t \\ y = f(t) \end{cases} . \tag{1.4.3}$$

The equation

$$y = f(x) \tag{1.4.4}$$

is called *the explicit equation* of the curve (1.4.2).

In the literature for the explicit representation of a curve it is, also, used the term *non-parametric form*, which we do not find particularly appealing since, in fact, an explicit representation can be thought of as a particular case of parameter representation (the parameter being the coordinate x).

Implicit representation. Let $F : D \to \mathbb{R}$ be a smooth function, defined on a domain $D \subset \mathbb{R}^2$, and let

$$C = \{(x, y) \in D \mid F(x, y) = 0\} \tag{1.4.5}$$

be the 0-level set of the function F. Generally speaking, C is not a regular curve (we can only say that it is a closed subset of the plane). Nevertheless, if at the point $(x_0, y_0) \in C$ the vector $\operatorname{grad} F = \{\partial_x F, \partial_y F\}$ is non vanishing, for instance $\partial_y F(x_0, y_0) \neq 0$, then, by the implicit functions theorem there exist:

- an open neighborhood U of the point (x_0, y_0) in $\mathbb{R}^2$;

- a smooth function $y = f(x)$, defined on an open neighborhood $I \subset \mathbb{R}$ of the point x_0,

such that
$$C \cap U = \{(x, f(x)|x \in I\}.$$

If grad $F \neq 0$ in all the points of C, then C is a regular curve (although, in general, not a simple one).

Figure 1.4: The bisectors of the coordinate axes

Examples. 1. $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 + y^2 - 1$. Let
$$(x_0, y_0) \in C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}.$$

Then we have
$$grad\, F(x_0, y_0) = \{2x_0, 2y_0\}.$$

Obviously, since $x_0^2 + y_0^2 = 1$, the vector grad F cannot vanish on C and, thus, C is a curve (the unit circle, centred at the origin).

2. $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 - y^2$. C is not a curve in this case (the gradient is vanishing at the origin). In fact, the set C has a self intersection at the origin (C is just the union of the two bissectors of the coordinate axes, see figure 1.4). It might not be obvious why we encounter problems in the neighbourhood of the origin for this "curve". The point is that there is no neighbourhood of the origin (on C), homeomorphic to an open interval on the real axis. An neighbourhood of the origin of C is the intersection of an open neighbourhood of the origin in the plane and the set C. Now, if we restrict the neighbourhood of the origin in the plane, its intersection with the set C will be a cross. If we remove the origin from the cross, the remaining set will have four connected components. On the other hand, suppose there is a homeomorphism f from the cross to an open interval from the real axis. If we remove from the interval the image of the origin through the homeomorphism f, we will get, clearly, only two connected components, or it can be proved that the number of connected components resulted by removing a point is homeomorphisms-invariant.

Remark. It should be clear that the condition of nonsingularity of the gradient of F is only a *sufficient* condition for the equation $F(x, y) = 0$ to represent a curve. If the gradient of F is zero at a point, we cannot claim that the equation represent a curve in the neighborhood of that point, but we cannot claim the opposite, either. Consider, as a trivial example, the equation

$$F(x, y) \equiv (x - y)^2 = 0.$$

Then we have

$$\operatorname{grad} F(x, y) = 2\{x - y, -(x - y)\}$$

and if we denote

$$C = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\},$$

then $\operatorname{grad} F = 0$ at all the points of C. But, clearly, C *is* a curve (it is easy to see that it is the first bissector of the coordinate axes, i.e. a straight line).

1.4.2 Space curves

Parametric representation. As in the case of plane curves, with a local parameterization

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \tag{1.4.6}$$

we can represent either the entire curve, or only a neighbourhood of one of its points.

Explicit representation. If $f, g : I \to \mathbb{R}$ are two smooth functions, defined on an open interval from the real axis, then the set

$$C = \{(x, f(x), g(x)) \in \mathbb{R}^3 \mid x \in I\} \tag{1.4.7}$$

is a simple curve, with a global parameterization given by

$$\begin{cases} x = t \\ y = f(t) \\ z = g(t) \end{cases} . \tag{1.4.8}$$

The equations

$$\begin{cases} y = f(x) \\ z = g(x) \end{cases} \tag{1.4.9}$$

are called *the explicit equations* of the curve. Let us note that, in fact, each equation of the system (1.4.9) is the equation of a cylindrical surface, with the generators parallel to one of the coordinate axis. Therefore, representing explicitly a curve actually means representing it as an intersection of two cylindrical surfaces, with the two families of generators having orthogonal directions.

Implicit representation. Let $F, G : D \to \mathbb{R}$, defined on a domain $D \subset \mathbb{R}^3$. We consider the set

$$C = \{(x, y, z) \in D \mid F(x, y, z) = 0,\ G(x, y, z) = 0\},$$

in other words, the solutions' set for the system

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{1.4.10}$$

In the general case, the set C is not a regular curve. Nevertheless, if at the point $a = (x_0, y_0, z_0) \in C$ the rank of the Jacobi matrix

$$\begin{pmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x G & \partial_y G & \partial_z G \end{pmatrix} \tag{1.4.11}$$

is equal to two, then there is an open neighborhood $U \subset D$ of the point (x_0, y_0, z_0) such that $C \cap U$ — the set of solutions of the system (1.4.10) in U — is a curve. Indeed, suppose, for instance, that

$$\det \begin{pmatrix} \partial_y F(a) & \partial_z F(a) \\ \partial_y G(a) & \partial_z G(a) \end{pmatrix} \neq 0.$$

Then, from the implicit functions theorem, there is an open neighborhood $U \subset D$ such that the set $C \cap U$ can be written as

$$C \cap U = \{(x, f(x), g(x)) | x \in W\},$$

where W is an open neighborhood in $\mathbb{R}$ of the point x_0, while $y = f(x)$, $z = g(x)$ are smooth functions, defined on W. Clearly, $C \cap U$ is a simple curve, while the pair $(W, \mathbf{r}(t) = (t, f(t), g(t))$ is a global parameterization of it.

If the rank of the matrix (1.4.11) is equal to two everywhere, then C is a curve (although, generally, not a simple one).

Example (The Viviani's temple). An important example of space curve given by implicit equations is the so-called *temple of Viviani*[4]. This curve is obtained as the intersection between the sphere with the center at the origin and radius $2a$ and the circular cylinder

[4]Vincenzo Viviani (1622–1703) was a Italian mathematician and architect, who was in contact with Galileo Galilei in the last years of the great scientist, and who liked to recommend himself as "Galileo's last student".

with radius a and axis parallel to the z-axis, situated at a distance a from this axis. In other words, the equations of the Viviani's temple are

$$\begin{cases} x^2 + y^2 + z^2 = 4a^2, \\ (x-a)^2 + y^2 = a^2. \end{cases}$$

It is instructive to make some computations for the case of the Viviani's temple. As we shall see, it is not, globally, a curve. We will have to eliminate a point to get, indeed, a regular curve. In fact, the shape of Viviani's temple is easy to understand. The curve has a shape which is similar to a Bernoulli's lemniscate, lying on the surface of a sphere. So, let

$$\begin{cases} F(x,y,z) = x^2 + y^2 + z^2 - 4a^2, \\ G(x,y,z) = (x-a)^2 + y^2 - a^2. \end{cases}$$

Then the equations of the curve reads

$$\begin{cases} F(x,y,z) = 0, \\ G(x,y,z) = 0. \end{cases}$$

We have, now,

$$\begin{pmatrix} F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 2(x-a) & 2y & 0 \end{pmatrix} = 2 \begin{pmatrix} x & y & z \\ x-a & y & 0 \end{pmatrix}.$$

To get a singular point, the following system of equations has to be fulfilled

$$\begin{cases} y = 0 \\ yz = 0 \\ (x-a)z = 0 \end{cases}.$$

Clearly, the only solution of the system that verifies, also, the equations of the curve is $x = 2a, y = z = 0$. Thus, the Viviani's temple (see figure 1.5) is a regular curve everywhere except the point which has these coordinates. It is not difficult to show that Viviani's temple is, in fact, the support of the parameterized curve

$$\mathbf{r}(t) = (a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}),$$

with $t \in (-2\pi, 2\pi)$. If we compute $\mathbf{r}'(t)$ we get

$$\mathbf{r}'(t) = \{-a\sin t, a\cos t, a\cos\frac{t}{2}\},$$

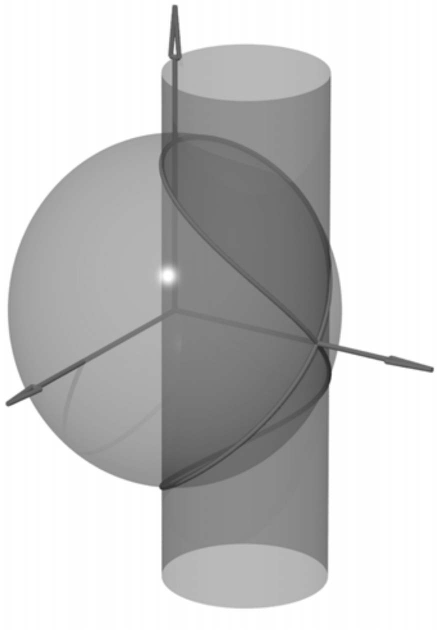


Figure 1.5: Viviani's temple

which means that this parameterized curve is regular. In particular, the existence of this regular parametric representation of the Viviani's temple shows that the point of coordinates $x = 2a, y = z = 0$ is, in fact, a point of self-intersection, rather then a singular point.

1.5 The tangent and the normal plane. The normal at a plane curve

Definition 1.5.1. For a parameterized curve $\mathbf{r} = \mathbf{r}(t)$ the vector $\mathbf{r}'(t_0)$ is called the *tangent vector* or the *velocity vector* of the curve at the point t_0. If the point t_0 is regular, then the straight line passing through $\mathbf{r}(t_0)$ and having the direction of the vector $\mathbf{r}'(t_0)$ is called the *tangent to the curve at the point* $\mathbf{r}(t_0)$ (or at the point t_0).

The vectorial equation of the tangent line reads, thus:

$$R(\tau) = \mathbf{r}(t_0) + \tau\mathbf{r}'(t_0). \tag{1.5.1}$$

Example. The cylindrical helix has the parameterization

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt),$$

therefore, for a point t_0,

$$\mathbf{r}'(t_0) = \{-a\cos t_0, a\sin t_0, b\}.$$

Thus, the equation of the tangent is

$$\begin{aligned} R(\tau) =&(a\cos t_0 - \tau a\sin t_0, a\sin t_0 + \tau a\cos t_0, bt_0 + \tau b) = \\ =&(a(cost_0 - \tau\sin t_0), a(sint_0 + \tau\cos t_0), b(t_0 + \tau)). \end{aligned}$$

Proposition 1.5.1. *The tangent vectors of two equivalent parameterized curves at corresponding points are colinear, while the tangent lines coincide.*

Proof. Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ the two equivalent parameterized curves and $\lambda : I \to J$ the parameter change, i.e. $\mathbf{r} = \boldsymbol{\rho}(\lambda(t))$. Then, according to the chain rule,

$$\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t),$$

with $\lambda'(t) \neq 0$. □

Remarks.

1. Clearly, $\mathbf{r}'$ and $\boldsymbol{\rho}'$ have the same sense when $\lambda' > 0$ (the parameter change does not modify the sense in which the support of the parameterized curve is traversed) and they have opposite sense when $\lambda' < 0$.
2. Since the parameter change modifies the tangent vector, it doesn't make sense to define the tangent vector at a point of a regular curve, through a local parameterization. Nevertheless, as we have seen, only the orientation and the length of the tangent vector can vary, but not the direction. Thus, it makes sense to speak about the tangent line at a point of a regular curve, defined through *any* local parameterization of the curve around that point.

We can use a more "geometric" way to define the tangent to a parameterized curve. Let $\mathbf{r}(t_0 + \Delta t)$ be a point of the curve close to the point $\mathbf{r}(t_0)$. Then, according to the Taylor's formula,

$$\mathbf{r}(t_0 + \Delta t) = \mathbf{r}(t_0) + \Delta t \cdot \mathbf{r}'(t_0) + \Delta t \cdot \boldsymbol{\epsilon}, \tag{1.5.2}$$

with $\lim\limits_{\Delta t \to 0} \boldsymbol{\epsilon} = 0$. We consider an arbitrary straight line π, passing through $\mathbf{r}(t_0)$ and having the direction given by the unit vector $\mathbf{m}$. Let

$$d(\Delta t) \stackrel{\text{def}}{=} d((\mathbf{r}(t_0 + \Delta t), \pi).$$

Theorem 1.5.1. *The straight line π is the tangent line to the parameterized curve* $\mathbf{r} = \mathbf{r}(t)$ *at the point t_0 iff*

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|} = 0. \tag{1.5.3}$$

Proof. From the Taylor's formula (1.5.2), we have

$$\Delta \mathbf{r} \equiv \mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0) = \Delta t \cdot \mathbf{r}'(t_0) + \Delta t \cdot \boldsymbol{\epsilon}.$$

The distance $d(\Delta t)$ is equal to

$$\|\Delta \mathbf{r} \times \mathbf{m}\| = |\Delta t| \|\mathbf{r}'(t_0) \times \mathbf{m} + \boldsymbol{\epsilon} \times \mathbf{m}\|.$$

Thus,

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|} = \lim_{\Delta t \to 0} \|\mathbf{r}'(t_0) \times \mathbf{m} + \underbrace{\boldsymbol{\epsilon} \times \mathbf{m}}_{\to 0}\| = \|\mathbf{r}'(t_0) \times \mathbf{m}\|.$$

Now, if the straight line π is the tangent line at t_0, then the vectors $\mathbf{r}'(t_0)$ and $\mathbf{m}$ are colinear, therefore $\mathbf{r}'(t_0) \times \mathbf{m} = 0$.

Conversely, if the condition (1.5.3) is fulfilled, then $\|\mathbf{r}'(t_0) \times \mathbf{m}\| = 0$, therefore either $\mathbf{r}'(t_0) = 0$ (which cannot happen, since the parameterized curve is regular), either the vectors $\mathbf{r}'(t_0)$ and $\mathbf{m}$ are colinear, i.e. π is the tangent line at t_0. □

Remark. The condition (1.5.3) is expressed by saying that the tangent line and the curve have *a first order contact* (or tangency contact). Another way of interpreting this formula is that the tangent line is the limit position of the straight line determined by the chosen point and a neighbouring point on the curve, when the neighbouring point is approaching indefinitely to the given one.

Hereafter, if not specified otherwise, all the parameterized curves considered will be regular.

Definition 1.5.2. Let $\mathbf{r} = \mathbf{r}(t)$ be a parameterized curve and $t_0 \in I$. The *normal plane* at the point $\mathbf{r}(t_0)$ of the curve $\mathbf{r} = \mathbf{r}(t)$ is the plane which passes through $\mathbf{r}(t_0)$ and it is perpendicular to the tangent line to the curve at $\mathbf{r}(t_0)$.

If $\mathbf{r} = \mathbf{r}(t)$ is a *plane* parameterized curve (i.e. its support is contained into a plane, which we will assume to be identical to the coordinate plane xOy), then the *normal* to the curve at the point $\mathbf{r}(t_0)$ will be the straight line through $\mathbf{r}(t_0)$, which is perpendicular to the tangent line to the curve at the point $\mathbf{r}(t_0)$.

Remark. Because it makes sense to define the tangent line at a point of a regular curve, by using an arbitrary local parameterization around that point, the same is true for the normal plane (or the normal line, in the case of plane curves).

The vectorial equation of the normal plane (line) follows immediately from the definition:

$$(R - \mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0. \tag{1.5.4}$$

1.5.1 The equations of the tangent line and normal plane (line) for different representations of curves

Parametric representation

If we start from the vectorial equation (1.5.1) of the tangent line and project it on the coordinate axes, we obtain the parametric equations of the tangent line, i.e., for space curves,

$$\begin{cases} X(\tau) = x(t_0) + \tau x'(t_0), \\ Y(\tau) = y(t_0) + \tau y'(t_0), \\ Z(\tau) = z(t_0) + \tau z'(t_0), \end{cases} \tag{1.5.5}$$

and, for plane curves,

$$\begin{cases} X(\tau) = x(t_0) + \tau x'(t_0), \\ Y(\tau) = y(t_0) + \tau y'(t_0). \end{cases} \tag{1.5.6}$$

If we eliminate the parameter τ, we get the canonical equations:

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}, \tag{1.5.7}$$

for space curves, respectively

$$\frac{X - x}{x'} = \frac{Y - y}{y'}, \tag{1.5.8}$$

for plane curves.

As for the equation of the normal plane (line), we can obtain it from (1.5.4), expressing it as

$$\{X - x, Y - y, Z - z\} \cdot \{x', y', z'\} = 0,$$

for space curves and

$$\{X - x, Y - y\} \cdot \{x', y'\} = 0,$$

for plane curves. Expanding the scalar products, we get:

$$(X - x)x' + (Y - y)y' + (Z - z)z' = 0, \tag{1.5.9}$$

for the equation of the normal plane to a space curve and, for the normal line to a plane curve,

$$(X - x)x' + (Y - y)y' = 0. \tag{1.5.10}$$

Explicit representation

If we have a space curve given by the equations

$$\begin{cases} y = f(x) \\ z = g(x) \end{cases},$$

then we can construct a parameterization

$$\begin{cases} x = t \\ y = f(t) \\ z = g(t). \end{cases}$$

For the derivatives we obtain immediately the expressions

$$\begin{cases} x' = 1 \\ y' = f' \\ z = g', \end{cases}$$

which, when substituted into the equations (1.5.7), give

$$X - x = \frac{Y - f(x)}{f'(x)} = \frac{Z - g(x)}{g'(x)}, \tag{1.5.11}$$

while for the equation of the normal plane, after substituting the derivatives into the equation (1.5.9), we obtain

$$X - x + (Y - f(x))f'(x) + (Z - g(x))g'(x) = 0. \tag{1.5.12}$$

For a plane curve given explicitly

$$y = f(x),$$

we have the parametric representation

$$\begin{cases} x = t \\ y = f(t), \end{cases}$$

and, thus, the equation of the tangent line is

$$X - x = \frac{Y - f(x)}{f'(x)} \tag{1.5.13}$$

or, in a more familiar form,

$$Y - f(x) = f'(x)(X - x), \tag{1.5.14}$$

while for the normal we get

$$X - x + (Y - f(x))f'(x) = 0 \tag{1.5.15}$$

ore

$$Y - f(x) = -\frac{1}{f'(x)}(X - x). \tag{1.5.16}$$

Implicit representation

Let us consider a curve given by the implicit equations

$$\begin{cases} F(x, y) = 0 \\ G(x, y) = 0. \end{cases} \tag{1.5.17}$$

Let us suppose that, at a point (x_0, y_0, z_0)

$$\det \begin{pmatrix} F'_y & F'_z \\ G'_y & G'_z \end{pmatrix} \neq 0$$

Then, as we saw before, around this point, the curve can be represented as

$$\begin{cases} y = f(x) \\ z = g(x), \end{cases} \tag{1.5.18}$$

i.e. the system (1.5.17) can be written as

$$\begin{cases} F(x, f(x), g(x)) = 0 \\ G(x, f(x), g(x)) = 0. \end{cases}$$

By computing the derivatives with respect to x of F and G, we get the system

$$\begin{cases} F'_x + f'(x)F'_y + g'(x)F'_z = 0 \\ G'_x + f'(x)G'_y + g'(x)G'_z = 0, \end{cases}$$

therefore

$$\begin{cases} f'F'_y + g'F'_z = -F'_x \\ f'G'_y + g'G'_z = -G'_x. \end{cases}$$

From this system, one can obtain f' and g', through the Cramer method:

$$\Delta = \begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix} \overset{not}{=} \frac{D(F,G)}{D(y,z)} \overset{hyp}{\neq} 0,$$
$$\Delta_{f'} = \begin{vmatrix} -F'_x & F'_z \\ -G'_x & G'_z \end{vmatrix} = \begin{vmatrix} F'_z & F'_x \\ G'_z & G'_x \end{vmatrix} \overset{not}{=} \frac{D(F,G)}{D(z,x)}$$
$$\Delta_{g'} = \begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix} = \begin{vmatrix} F'_x & F'_y \\ G'_x & G'_y \end{vmatrix} \overset{not}{=} \frac{D(F,G)}{D(x,y)},$$

therefore

$$\begin{cases} f' = \dfrac{\frac{D(F,G)}{D(z,x)}}{\frac{D(F,G)}{D(y,z)}} \\ g' = \dfrac{\frac{D(F,G)}{D(x,y)}}{\frac{D(F,G)}{D(y,z)}} \end{cases} \tag{1.5.19}$$

As we saw before, for the curve (1.5.18) the equations of the tangent are

$$X - x_0 = \frac{Y - f(x_0)}{f'(x_0)} = \frac{Z - g(x_0)}{g'(x_0)}$$

or, using (1.5.19),

$$X - x_0 = \frac{Y - f(x_0)}{\frac{\frac{D(F,G)}{D(z,x)}}{\frac{D(F,G)}{D(y,z)}}} = \frac{Z - g(x_0)}{\frac{\frac{D(F,G)}{D(x,y)}}{\frac{D(F,G)}{D(y,z)}}},$$

from where, having in mind that $f(x_0) = y_0$ and $g(x_0) = z_0$, we have

$$\frac{X - x_0}{\frac{D(F,G)}{D(y,z)}} = \frac{Y - y_0}{\frac{D(F,G)}{D(z,x)}} = \frac{Z - z_0}{\frac{D(F,G)}{D(x,y)}}.$$

For a plane curve

$$F(x, y) = 0,$$

if at the point (x_0, y_0) there is fulfilled the condition $F'_y \neq 0$, then, from the implicit functions theorem, locally, $y = f(x)$, hence the equation of the curve can be written as

$$F(x, f(x)) = 0.$$

By differentiating this relation with respect to x, one obtains

$$F'_x + f'F'_y = 0 \Longrightarrow f' = -\frac{F'_x}{F'_y}.$$

Thus, from the equation of the tangent:

$$Y - y_0 = f'(x_0)(X - x_0),$$

we deduce

$$Y - y_0 = -\frac{F'_x}{F'_y}(X - x_0)$$

or

$$(X - x_0)F'_x + (Y - y_0)F'_y = 0,$$

while for the normal we obtain the equation

$$(X - x_0)F'_y - (Y - y_0)F'_x = 0.$$

1.6 The osculating plane

Definition 1.6.1. A parameterized curve $\mathbf{r} = \mathbf{r}(t)$ is called *biregular* (or *in general position*) at the point t_0 if the vectors $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$ are not colinear, i.e.

$$\mathbf{r}'(t_0) \times \mathbf{r}''(t_0) \neq 0.$$

The parameterized curve is called *biregular* if it is biregular at each point[5].

Remark. It is not difficult to check that the notion of a biregular point is independent of the parameterization: if a point is biregular for a given parameterized curve, then its corresponding point through any parameter change is, also, a biregular point.

Definition 1.6.2. Let $(I, \mathbf{r})$ be a parameterized curve and $t_0 \in I$ – a biregular point. The *osculating plane* of the curve at $\mathbf{r}(t_0)$ is the plane through $\mathbf{r}(t_0)$, parallel to the vectors $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$, i.e. the equation of the plane is

$$(R - \mathbf{r}(t_0), \mathbf{r}'(t_0), \mathbf{r}''(t_0)) = 0, \tag{1.6.1}$$

or, expanding the mixed product,

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0. \tag{1.6.2}$$

[5] A biregular curve is also called in some books a *complete* curve. We find this term a little bit misleading, since this terms has, usually, other meaning in the global theory of curves (and, especially, surfaces).

Theorem 1.6.1. *The osculating planes to two equivalent parameterized curves at corresponding biregular points coincides.*

Proof. Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be two equivalent parameterized curves and $\lambda : I \to J$ – the parameter change. Then

$$\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)),\\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t),\\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot (\lambda'(t))^2 + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t).
\end{aligned}$$

Since $\lambda'(t) \neq 0$, from these relations it follows that the system of vectors $\{\mathbf{r}'(t), \mathbf{r}''(t)\}$ and $\{\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t))\}$ are equivalent, i.e. they generate the same vector subspace $\mathbb{R}^3$, hence the osculating planes to the two parameterized curves at the corresponding points t_0 and $\lambda(t_0)$ have the same directing subspace, therefore, they are parallel. As they have a common point (since $\mathbf{r}(t_0) = \boldsymbol{\rho}(\lambda(t_0))$), they have to coincide. □

Remark. From the previous theorem it follows that the notion of an osculating plane makes sense also for regular curves.

As in the case of the tangent line, there is a more geometric way of defining the osculating plane which is, at the same time, more general, since it can be applied also for the case of the points which are not biregular.

Let $\mathbf{r}(t_0)$ şi $\mathbf{r}(t_0 + \Delta t)$ two neighboring points on a parameterized curve, with $\mathbf{r}(t_0)$ biregular. We consider a plane α, of normal versor $\mathbf{e}$, passing through $\mathbf{r}(t_0)$. and denote $d(\Delta t) = d(\mathbf{r}(t_0 + \Delta t), \alpha)$.

Theorem 1.6.2. *α is the osculating plane to the parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the biregular point $\mathbf{r}(t_0)$ iff*

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|^2} = 0,$$

i.e. the curve and the plane have a second order contact.

Proof. From the Taylor formula we have

$$\mathbf{r}(t_0 + \Delta t) = \mathbf{r}(t_0) + \Delta t \cdot \mathbf{r}'(t_0) + \frac{1}{2}(\Delta t)^2 \cdot \mathbf{r}''(t_0) + (\Delta t)^2 \cdot \boldsymbol{\epsilon},$$

with $\lim\limits_{\Delta t \to 0} \boldsymbol{\epsilon} = 0$.

On the other hand,

$$\begin{aligned}
d(\Delta t) &= |\mathbf{e} \cdot (\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0))| =\\
&= |(\mathbf{e} \cdot \mathbf{r}'(t_0)) \cdot \Delta t + \frac{1}{2}(\mathbf{e} \cdot \mathbf{r}''(t_0)) \cdot (\Delta t)^2 + (\mathbf{e} \cdot \boldsymbol{\epsilon}) \cdot (\Delta t)^2|.
\end{aligned}$$

Thus,

$$\lim_{\Delta t\to 0}\frac{d(\Delta t)}{|\Delta t|^2}=\lim_{\Delta t\to 0}\left|\frac{\mathbf{e}\cdot\mathbf{r}'(t_0)}{\Delta t}+\frac{1}{2}\cdot(\mathbf{e}\cdot\mathbf{r}''(t_0))+\underbrace{\mathbf{e}\cdot\boldsymbol{\epsilon}}_{\to 0}\right|=$$

$$=\lim_{\Delta t\to 0}\left|\frac{\mathbf{e}\cdot\mathbf{r}'(t_0)}{\Delta t}+\frac{1}{2}\cdot(\mathbf{e}\cdot\mathbf{r}''(t_0))\right|.$$

If $\lim\limits_{\Delta t\to 0}\frac{d(\Delta t)}{|\Delta t|^2}=0$, then $\mathbf{e}\cdot\mathbf{r}'(t_0)=0$ and $\mathbf{e}\cdot\mathbf{r}''(t_0)=0$, i.e. $\mathbf{e}\parallel\mathbf{r}'(t_0)\times\mathbf{r}''(t_0)$, meaning that α is the osculating plane.

The converse is obvious. □

Remarks. (i) The previous theorem justifies the name of the osculating plane. Actually, the name (coined by Johann Bernoulli), comes from the Latin verb *osculare*, which means *to kiss* end emphasizes the fact that, among all the planes that are passing through a given point of a curve, the osculating plane has the higher order ("the closer") contact.

(ii) Defining the osculating plane through the contact, one can define the notion of an osculating plane also for the points which are not biregular, but in that case any plane passing through the tangent is an osculating plane, in the sense that it has a second order contact with the curve. Saying that the osculating plane at a biregular point is the only plane which has at that point a second order contact with the curve is the same with saying that the osculating plane is the limit position of a plane determined by the considered point and two neighbouring points when this ones are approaching indefinitely to the given one. Also, on can define the osculating plane at biregular point of a parameterized curve as being the limit position of a plane passing through the tangent at the given point and a neighboring point from the curve, when this point is approaching indefinitely the given one.

A natural question that one may ask is what happens with the osculating plane in the particular case of a plane parameterized curve. The answer is given by the following proposition, whose prove is left to the reader:

Proposition 1.6.1. *If a biregular parameterized curve is plane, i.e. its support is contained into a plane π, then the osculating plane to this curve at each point coincides to the plane of the curve. Conversely, if a given biregular parameterized curve has the same osculating plane at each point, then the curve is plane and its support is contained into the osculating plane.*

1.7 The curvature of a curve

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a regular parameterized curve. Let $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be a naturally parameterized curve, equivalent to it. Then $\|\boldsymbol{\rho}'(s)\| = 1$, while the vector $\boldsymbol{\rho}''(s)$ is orthogonal to $\boldsymbol{\rho}'(s)$[6].

One can show that $\boldsymbol{\rho}''(s)$ does not depend on the choice of the naturally parameterized curve equivalent to the given curve $\mathbf{r} = \mathbf{r}(t)$. Indeed, if $(J_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(\tilde{s}))$ is another equivalent naturally parameterized curve with the parameter change $\tilde{s} = \lambda(s)$, then, from the condition

$$\|\boldsymbol{\rho}'(s)\| = \|\boldsymbol{\rho}_1'(\lambda(s))\| = 1$$

we get $|\lambda'(s)| = 1$ for any $s \in J$. Thus, $\lambda' = \pm 1$ and, therefore, $\tilde{s} = \pm s + s_0$, where s_0 is a constant. It follows that

$$\boldsymbol{\rho}''(s) = \boldsymbol{\rho}''{}_1(\tilde{s})\underbrace{(\lambda'(s))^2}_{=1} + \boldsymbol{\rho}'{}_1 \cdot \underbrace{\lambda''(s)}_{=0} = \boldsymbol{\rho}''{}_1(\tilde{s}).$$

Definition 1.7.1. The vector $\mathbf{k} = \boldsymbol{\rho}''(s(t))$ is called the *curvature vector* of the parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the point t, while its norm, $k(t) = \|\boldsymbol{\rho}''(s(t))\|$ – the *curvature* of the parameterized curve at the point t.

We shall express now the curvature vector $\mathbf{k}(t)$ as a function of $\mathbf{r}(t)$ and its derivatives. We choose as natural parameter the arc length of the curve. Then we have

$$\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(s(t)) \Rightarrow \\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(s(t)) \cdot s'(t) \\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(s(t)) \cdot (s'(t))^2 + \boldsymbol{\rho}'(s(t)) \cdot s''(t),
\end{aligned}$$

where $s'(t) = \|\mathbf{r}'(t)\|$, $s''(t) = \|\mathbf{r}'(t)\|' = \dfrac{d}{dt}\left(\sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)}\right) = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$. Thus, we have

$$\mathbf{k}(t) = \boldsymbol{\rho}''(s(t)) = \frac{\mathbf{r}''}{\|\mathbf{r}'\|^2} - \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|^4} \cdot \mathbf{r}'. \tag{1.7.1}$$

Now, since the vectors $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}''$ are orthogonal, while $\boldsymbol{\rho}'$ has unit length, we have

$$k(t) = \|\mathbf{k}(t)\| = \|\boldsymbol{\rho}''\| = \|\boldsymbol{\rho}' \times \boldsymbol{\rho}''\|.$$

[6]Indeed, since $\boldsymbol{\rho}'$ is a unit vector, we have $\boldsymbol{\rho}'^2 = 1$. Differentiating this relation, we obtain that $\boldsymbol{\rho}' \cdot \boldsymbol{\rho}'' = 0$, which expresses the fact that the two vectors are orthogonal.

Substituting $\boldsymbol{\rho}' = \dfrac{\mathbf{r}'}{\|\boldsymbol{\rho}'\|}$, and $\boldsymbol{\rho}''$ by the formula (1.7.1) we obtain

$$k(t) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}. \tag{1.7.2}$$

Remarks. 1. From the formula (1.7.2) it follows that a parameterized curve $\mathbf{r} = \mathbf{r}(t)$ is biregular at a point t_0 iff $k(t_0) \neq 0$.

2. Since for equivalent parameterized curve the naturally parameterized curve to each of them are equivalent between them, the notion of curvature makes sense also for regular curves.

Examples. 1. For the straight line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}$ the curvature vector (and, therefore, also the curvature) is identically zero.

2. For the circle $S^1_R = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = R^2,\ z = 0\}$ we choose the parameterization

$$\begin{cases} x = R\cos t \\ y = R\sin t \qquad 0 < t < 2\pi. \\ z = 0 \end{cases}$$

Then

$$\mathbf{r}'(t) = \{-R\sin t, R\cos t, 0\}, \quad \mathbf{r}''(t) = \{-R\cos t, R\sin t, 0\}$$

and, thus, $\|\mathbf{r}'(t)\| = R$, $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$. Therefore, for the curvature vector we get

$$\mathbf{k}(t) = \left\{-\frac{1}{R}\cos t, -\frac{1}{R}\sin t, 0\right\} = -\frac{1}{R}\{x, y, z\}, \quad k(t) = \frac{1}{R}.$$

Remarks. 1. The computation we just made explains why the inverse of the curvature is called the *curvature radius* of the curve.

2. We saw that the curvature of a straight line is identically zero. The converse is also true, in some sense, as the following proposition shows.

Proposition 1.7.1. *If the curvature of a regular parameterized curve is identically zero, then the support of the curve lies on a straight line.*

Proof. Suppose, for the very beginning, that we are dealing with a naturally parameterized curve $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$. From the hypothesis, $\boldsymbol{\rho}''(s) = 0$, therefore $\boldsymbol{\rho}'(s) = \mathbf{a} = \text{const}$, $\boldsymbol{\rho}(s) = \boldsymbol{\rho}_0 + s\mathbf{a}$, i.e. the support $\boldsymbol{\rho}(I)$ lies on a straight line. As two equivalent parameterized curves have the same support, the result still holds also for non-naturally parameterized curves. $\square$

Remark. Be careful, the fact that a parameterized curve has zero curvature simply means that the *support* of the curve lies on a straight line, but it doesn't necessarily means that the parameterized curve is (the restriction of) an affine map from $\mathbb{R}$ to $\mathbb{R}^3$, nor that it is equivalent to such a particular parameterized curve. We can consider, as we did before, the parameterized curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t^3$, where $\mathbf{a}$ is a constant, nonvanishing, vector from $\mathbb{R}^3$. Then we have, immediately, $\mathbf{r}'(t) = 2\mathbf{a}t^2$ and $\mathbf{r}''(t) = 3\mathbf{a}t$, which means that the velocity and the acceleration of the curve are parallel, hence the curve has zero curvature but, as we also saw earlier, this parameterized curve is not equivalent to an affine parameterized curve.

1.7.1 The geometrical meaning of curvature

Let us consider a naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$. We denote by $\Delta\varphi(s)$ the measure of the angle between the versors $\mathbf{r}(s)$ and $\mathbf{r}(s + \Delta s)$. Then

$$\|\mathbf{r}(s + \Delta s) - \mathbf{r}(s)\| = 2\left|\sin \frac{\Delta\varphi(s)}{2}\right|.$$

Therefore,

$$k(s) = \|\mathbf{r}''(s)\| = \left\|\lim_{\Delta s\to 0} \frac{\mathbf{r}(s + \Delta s) - \mathbf{r}(s)}{\Delta s}\right\| = \lim_{\Delta s\to 0} \frac{2\left|\sin \frac{\Delta\varphi(s)}{2}\right|}{|\Delta s|} =$$
$$= \lim_{\Delta s\to 0}\left|\frac{\Delta\varphi(s)}{\Delta s}\right| \cdot \frac{\left|\sin \frac{\Delta\varphi(s)}{2}\right|}{\left|\frac{\Delta\varphi(s)}{2}\right|} = \lim_{\Delta s\to 0}\left|\frac{\Delta\varphi(s)}{\Delta s}\right| = \left|\frac{d\varphi}{ds}\right|.$$

Thus, if we have in mind that $\Delta\varphi(s)$ is the measure of the angle between the tangents to the curve at s and $s + \Delta s$, the last formula gives us:

Proposition 1.7.2. *The curvature of a curve is the speed of rotation of the tangent line to the curve, when the tangency point is moving along the curve with unit speed.*

1.8 The Frenet frame (the moving frame) of a parameterized curve

At each point of the support of a biregular parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ one can construct a frame of the space $\mathbb{R}^3$. The idea is that, if we want to investigate the local properties of a parameterized curve around a given point of the curve, that it might be easier to do that if we don't use the standard coordinate system of $\mathbb{R}^3$, but a coordinate system with the origin at the given point of the curve, while the coordinate axes have some connection with the local properties of the curve. Such a coordinate system was constructed, independently, at the middle of the XIX-th century, by the French mathematicians Frenet and Serret.

Definition 1.8.1. The *Frenet frame* (or the *moving frame*) of a biregular parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ at the point $t_0 \in I$ is an orthonormal frame of the space $\mathbb{R}^3$, with the origin at the point $\mathbf{r}(t_0)$, the coordinate versors being the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\nu}(t_0), \boldsymbol{\beta}(t_0)\}$, where:

- $\boldsymbol{\tau}(t_0)$ is the versor of the tangent to the curve at t_0, i.e.

$$\boldsymbol{\tau}(t_0) = \frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|}.$$

  $\boldsymbol{\tau}(t_0)$ is also called the *unit tangent* at the point t_0;

- $\boldsymbol{\nu}(t_0) = \mathbf{k}(t_0)/k(t_0)$ is the versor of the curvature vector:

$$\boldsymbol{\nu}(t_0) = \frac{\mathbf{k}(t_0)}{k(t_0)}$$

  and it is called the *unit principal normal* at the point t_0.

- $\boldsymbol{\beta}(t_0) = \boldsymbol{\tau}(t_0) \times \boldsymbol{\nu}(t_0)$ it is called the *unit binormal* at the point t_0.

- The axis of direction $\boldsymbol{\tau}(t_0)$ is, obviously, the tangent to the curve at t_0.

- The axis of direction $\boldsymbol{\nu}(t_0)$ is called the *principal normal*. In fact, this straight line is contained into the normal plane (since it is perpendicular on the tangent), but it is also contained into the osculating plane. Thus, *the principal normal is the normal contained into the osculating plane*.

- The axis of direction $\boldsymbol{\beta}(t_0)$ is called the *binormal*. *The binormal is the normal perpendicular on the osculating plane*.

- The plane determined by the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\nu}(t_0\}$ is the osculating plane (O in the figure 1.8).

- The plane determined by the vectors $\{\boldsymbol{\nu}(t_0), \boldsymbol{\beta}(t_0)\}$ is the normal plane (N in the figure 1.8).

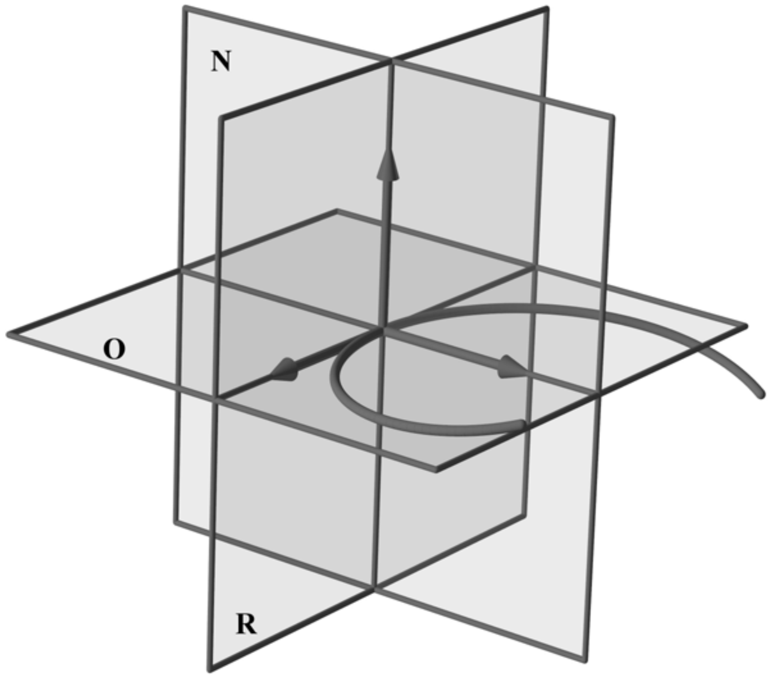


Figure 1.6: The Frenet frame of a parameterized curve

- The plane determined by the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\beta}(t_0)\}$ is called the *rectifying plane* , for reasons that will become clear later (R in the figure 1.8).

For a naturally parameterized curve $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$, the expressions of the vectors of the Frenet frame are quite simple:

$$\begin{cases} \boldsymbol{\tau}(s) & = \boldsymbol{\rho}'(s) \\ \boldsymbol{\nu}(s) & = \dfrac{\boldsymbol{\rho}''(s)}{\|\boldsymbol{\rho}''(s)\|} \\ \boldsymbol{\beta}(s) & \equiv \boldsymbol{\tau}(s) \times \boldsymbol{\nu}(s) = \dfrac{\boldsymbol{\rho}'(s) \times \boldsymbol{\rho}''(s)}{\|\boldsymbol{\rho}''(s)\|} \end{cases}. \tag{1.8.1}$$

For an arbitrary biregular parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ the situation is a little bit more complicated. Thus, obviously, from the definition, at an arbitrary point $t \in I$,

$$\boldsymbol{\tau}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}. \tag{1.8.2}$$

Then, having in mind that

$$\mathbf{k}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} - \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^4}\cdot\mathbf{r}'(t)$$

and

$$k(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3},$$

we get

$$\boldsymbol{\nu}(t) \equiv \frac{\mathbf{k}(t)}{k(t)} = \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}\cdot\mathbf{r}''(t) - \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|\cdot\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}\cdot\mathbf{r}'(t), \tag{1.8.3}$$

while

$$\boldsymbol{\beta}(t) \equiv \boldsymbol{\tau}(t)\times\boldsymbol{\nu}(t) = \frac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}. \tag{1.8.4}$$

Remark. The above computations show that, in practice, for an arbitrary parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ it is easier to compute directly $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ and then compute $\boldsymbol{\nu}$ by the formula

$$\boldsymbol{\nu} = \boldsymbol{\beta}\times\boldsymbol{\tau}.$$

1.8.1 The behaviour of the Frenet frame at a parameter change

A notion defined for parameterized curves makes sense for regular curves iff it is invariant at a parameter change, in other words if it doesn't change when we replace a parameterized curve by another one, equivalent to it. The Frenet frame is "almost" invariant, i.e. we have

Theorem 1.8.1. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$ be two equivalent parameterized curves with the parameter change $\lambda : I \to J$, $u = \lambda(t)$. Then, at the corresponding points t and $u = \lambda(t)$, their Frenet frame coincide if $\lambda'(t) > 0$. If $\lambda'(t) < 0$, then the origins and the unit principal normals coincide, while the other two pairs of versors have the same direction, but opposite senses.*

Proof. Since $\mathbf{r}(t) = \boldsymbol{\rho}(\lambda(t))$, the origins of the Frenet frames coincide in any case. As seen before, the curvature vectors of two equivalent curves coincide and, thus, the same is true for the unit principal normals. From $\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t))\cdot\lambda'(t)$ it follows the coincidence of the Frenet frames for $\lambda'(t) > 0$. If $\lambda'(t) < 0$, the tangent vectors $\boldsymbol{\rho}'(u)$ and $\mathbf{r}'(t)$ have opposite senses, and the same is true for their versors. From $\boldsymbol{\beta} = \boldsymbol{\tau}\times\boldsymbol{\nu}$ it follows that, in this case, the unit binormals also have opposite senses. □

1.9 Oriented curves. The Frenet frame of an oriented curve

As we saw before, the Frenet frame of a parameterized curve is not invariant at a parameter change (well, at least not at *any* parameter change). Therefore, in order to be able to use this apparatus also for regular curve, we need to make it invariant. The idea is to modify a little bit the definition of the regular curve, imposing some further condition on the local parameterization, to make sure that the parameter changes will not modify the Frenet frames.

Definition 1.9.1. Two parameterized curves are $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ are called *positively equivalent* if there is a parameter change $\lambda : I \to J$, $u = \lambda(t)$, with $\lambda'(t) > 0$, $\forall t \in I$.

Definition 1.9.2. An *orientation* of a regular curve $C \subset \mathbb{R}^3$ is a family of local parameterizations $\{(I_\alpha, \mathbf{r}_\alpha = \mathbf{r}_\alpha(t))\}_{\alpha\in A}$ such that

a) $C = \bigcup\limits_{\alpha\in A} \mathbf{r}_\alpha(I_\alpha)$,

b) For any connected component $C^b_{\alpha\beta}$ of the intersection $C_{\alpha\beta} = \mathbf{r}_\alpha(I_\alpha) \cap \mathbf{r}_\beta(I_\beta)$ with $\alpha, \beta \in A$ the parameterized curves $(I^b_\alpha, \mathbf{r}^b_\alpha)$ and $(I^b_\beta, \mathbf{r}^b_\beta)$ with $I^b_\alpha = \mathbf{r}^{-1}_\alpha(C^b_{\alpha\beta})$, $\mathbf{r}^b_\alpha = \mathbf{r}_\alpha|_{I^b_\alpha}$, $I^b_\beta = \mathbf{r}^{-1}_\beta(C^0_{\alpha\beta})$, $\mathbf{r}^b_\beta = \mathbf{r}_\beta|_{I^b_\beta}$ are positively equivalent.

Example. For the unit circle S^1 the following parameterizations:

$$(I_1 = (0, 2\pi), \mathbf{r}_1(t) = (\cos t, \sin t, 0))$$

and

$$(I_2 = (-\pi, \pi), \mathbf{r}_2(t) = (\cos t, \sin t, 0))$$

give an orientation of S^1. $C_{12} = \mathbf{r}_1(I_1) \cap \mathbf{r}_2(I_2)$ has two connected components (the upper and the lower half circles).

Starting with the upper component, C^1_{12}, we have

$$I^1_1 = \mathbf{r}^{-1}_1(C^1_{12}) = (0, \pi),$$
$$I^1_2 = \mathbf{r}^{-1}_2(C^1_{12}) = (0, \pi)$$

and the parameter change is the identity, $\lambda : (0, \pi) \to (0, \pi)$, $\lambda(t) = t$, $\forall t \in (0, \pi)$, therefore the two parameterized curves are, clearly, positively equivalent.

As for the lower connected component, C_{12}^2, we get

$$I_1^2 = \mathbf{r}_1^{-1}(C_{12}^2) = (\pi, 2\pi),$$
$$I_2^2 = \mathbf{r}_2^{-1}(C_{12}^2) = (-\pi, 0)$$

and the parameter change is $\lambda : I_1^2 \to I_2^2$, $\lambda(t) = t - 2\pi$, therefore, since $\lambda'(t) = 1 > 0$, also this time the two local parameterizations are positively equivalent.

Definition 1.9.3. A regular curve $C \subset \mathbb{R}^3$, with an orientation, is called an *oriented regular curve*.

Example. If C is a simple regular curve, it can be turned into an oriented curve by using an orientation given by any *global* parameterization $(I, \mathbf{r})$.

Remark. If C is a connected regular curve, it has only two distinct orientations, corresponding to the two possible senses of moving along the curve.

Definition 1.9.4. A local parameterization $(I, \mathbf{r})$ of an oriented regular curve C is called *compatible with the orientation* defined by the family $\{(I_\alpha, \mathbf{r}_\alpha)\}_{\alpha\in A}$ if on the intersections $\mathbf{r}(I) \cap \mathbf{r}_\alpha(I_\alpha)$ the parameterized curves $(I, \mathbf{r})$ and $(I_\alpha, \mathbf{r}_\alpha)$ are positively equivalent.

$$I_1^1 = \mathbf{r}_1^{-1}(C_{12}^1) = (0, \pi),$$
$$I_2^1 = \mathbf{r}_2^{-1}(C_{12}^1) = (0, \pi)$$

Remark. For an oriented regular curve C, with the orientation given by the family of local parameterizations $\{(I_\alpha, \mathbf{r}_\alpha)\}_{\alpha\in A}$, one can define, by using the vectors $\mathbf{r}'_\alpha(t)$, a sense on each tangent line, since, passing to another local parameterization $\mathbf{r}_\beta(t)$, the vectors $\mathbf{r}'_\alpha$ and $\mathbf{r}'_\beta$ have the same direction and the same sense (only their norms may differ).

The orientation itself can be given through the choice of a sense on each tangent line. Thus, if the direction of the tangent vector at $x \in C$ is given by the vector $\mathbf{a}(x)$, then wave to impose the continuity of the map $C \to \mathbb{R}^3$, $x \to \mathbf{a}(x)$. For this definition of the orientation, a local parameterization $(I, \mathbf{r})$ is compatible with the orientation if, for each point $x \in C$, $x = \mathbf{r}(t)$, the vectors $\mathbf{a}(x)$ and $\mathbf{r}'(t)$ have the same sense.

Definition 1.9.5. The Frenet frame of an oriented biregular curve C at a point $x \in C$ is the Frenet frame of a biregular parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at t_0, where $\mathbf{r} = \mathbf{r}(t)$ is a local parameterization of the curve C, compatible with the orientation, such that $\mathbf{r}(t_0) = x$.

Remark. Clearly, this definition does not depend on the choice of the local parameterization, compatible with the orientation of the curve.

1.10 The Frenet formulae. The torsion

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a biregular parameterized curve. Then the vectors $\boldsymbol{\tau}(t)$, $\boldsymbol{\nu}(t)$, $\boldsymbol{\beta}(t)$ are, in fact, smooth vector functions with respect to the parameter t. We want to find their derivatives with respect to t, more precisely, the decomposition of these derivatives with respect to the vectors $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$. These derivatives show, in fact, the way the vectors of the Frenet frame vary along the curve. From the definition, we have,

$$\boldsymbol{\tau} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{\mathbf{r}'}{\sqrt{\mathbf{r}'^2}}.$$

Therefore,

$$\begin{aligned}\boldsymbol{\tau}' &= \frac{\mathbf{r}'' \cdot \|\mathbf{r}'\| - \mathbf{r}' \cdot \frac{\mathbf{r}'\cdot\mathbf{r}''}{\|\mathbf{r}'\|}}{\|\mathbf{r}'\|^2} = \frac{\mathbf{r}'' \cdot \|\mathbf{r}'\|^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')}{\|\mathbf{r}'\|^3} = \\ &= \|\mathbf{r}'\| \cdot \underbrace{\frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}}_{k} \left[\underbrace{\frac{\|\mathbf{r}'\|}{\|\mathbf{r}' \times \mathbf{r}''\|} \cdot \mathbf{r}'' - \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\| \cdot \|\mathbf{r}' \times \mathbf{r}''\|} \cdot \mathbf{r}'}_{\nu} \right].\end{aligned}$$

Thus,

$$\boldsymbol{\tau}' = \|\mathbf{r}'\| k \cdot \boldsymbol{\nu}. \tag{1.10.1}$$

Further,

$$\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu} \;\Rightarrow\; \boldsymbol{\beta}' = \boldsymbol{\tau}' \times + \boldsymbol{\tau} \times \boldsymbol{\nu}' = k(\underbrace{\boldsymbol{\nu} \times \boldsymbol{\nu}}_{=0}) + \boldsymbol{\tau} \times \boldsymbol{\nu}' \;\Rightarrow \boldsymbol{\beta}' = \boldsymbol{\tau} \times \boldsymbol{\nu}',$$

hence $\boldsymbol{\beta}' \perp \boldsymbol{\tau}$. On the other hand,

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1 \;\Rightarrow\; \boldsymbol{\beta}' \cdot \boldsymbol{\beta} = 0 \;\Rightarrow\; \boldsymbol{\beta}' \perp \boldsymbol{\beta}.$$

Thus, the vector $\boldsymbol{\beta}'$ is colinear to the vector $\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$ and we can write

$$\boldsymbol{\beta}' = -\|\mathbf{r}'\| \cdot \chi \boldsymbol{\nu},$$

where χ is a proportionality factor, the meaning of which will be made clear later.

We differentiate now the equality

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}.$$

We have

$$\boldsymbol{\nu}' = \boldsymbol{\beta}' \times \boldsymbol{\tau} + \beta \times \boldsymbol{\tau}' = -\|\mathbf{r}'\| \cdot \chi(\boldsymbol{\nu} \times \boldsymbol{\tau}) + \|\mathbf{r}'\| \cdot k(\boldsymbol{\beta} \times \boldsymbol{\nu}),$$

therefore,

$$\boldsymbol{\nu}' = \|\mathbf{r}'\|(-k\boldsymbol{\tau} + \chi\boldsymbol{\beta}). \tag{1.10.2}$$

We obtained, thus, the equations

$$\begin{cases} \boldsymbol{\tau}'(t) = \|\mathbf{r}'\|k(t)\boldsymbol{\nu}(t) \\ \boldsymbol{\nu}'(t) = \|\mathbf{r}'\|(-k(t)\boldsymbol{\tau}(t) + \chi(t)\boldsymbol{\beta}(t)) \\ \boldsymbol{\beta}'(t) = -\|\mathbf{r}'\|\chi(t)\boldsymbol{\nu}(t). \end{cases} \tag{1.10.3}$$

These equations are called the *Frenet formulae* for the parameterized curve $\mathbf{r} = \mathbf{r}(t)$. If we are dealing win a naturally parameterized curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$, then the Frenet equations are a little bit simpler:

$$\begin{cases} \boldsymbol{\tau}'(t) = k(t)\boldsymbol{\nu}(t) \\ \boldsymbol{\nu}'(t) = -k(t)\boldsymbol{\tau}(t) + \chi(t)\boldsymbol{\beta}(t) \\ \boldsymbol{\beta}'(t) = -\chi(t)\boldsymbol{\nu}(t). \end{cases} \tag{1.10.3$'$}$$

Definition 1.10.1. The quantity $\chi(t)$ is called the *torsion* (or *second curvature*) of the biregular parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the point t.

We shall compute first the torsion for a naturally parameterized curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$. For such a curve, the versors of the Frenet frame are given by the expressions:

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{\rho}' \\ \boldsymbol{\nu} = \frac{1}{k}\boldsymbol{\rho}'' \\ \boldsymbol{\beta} = \frac{1}{k}\boldsymbol{\rho}' \times \boldsymbol{\rho}''. \end{cases}$$

From the third Frenet formula, we have:

$$\boldsymbol{\beta}' \cdot \boldsymbol{\nu} = -\chi(s) \cdot \underbrace{(\boldsymbol{\nu} \times \boldsymbol{\nu})}_{=1} = -\chi(t).$$

But, on the other hand, from the definition,

$$\boldsymbol{\beta}' = \left(\frac{1}{k}\right)' + \frac{1}{k}\underbrace{\boldsymbol{\rho}'' \times \boldsymbol{\rho}''}_{=0} + \frac{1}{k}\boldsymbol{\rho}' \times \boldsymbol{\rho}''',$$

hence

$$\chi = -\boldsymbol{\beta}' \cdot \boldsymbol{\nu} = -\underbrace{\left(\frac{1}{k}\right)' (\boldsymbol{\rho}' \times \boldsymbol{\rho}'') \cdot \frac{1}{k}\boldsymbol{\rho}''}_{=0} - \frac{1}{k}(\boldsymbol{\rho}' \times \boldsymbol{\rho}''') \cdot \frac{1}{k}\boldsymbol{\rho}'',$$

therefore,

$$\chi = \frac{1}{k^2}\,(\boldsymbol{\rho}', \boldsymbol{\rho}'', \boldsymbol{\rho}''')\,. \tag{1.10.4}$$

Now, the following theorem gives us the way to compute the torsion for an arbitrary (biregular) parameterized curve:

Theorem. *If* $(I, \mathbf{r} = \mathbf{r}(t))$ *and* $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ *are two positively equivalent parameterized curves, with the parameter change* $\lambda : I \to J$, $\lambda' > 0$, *then they have the same torsion at the corresponding points* t *and* $u = \lambda(t)$.

Proof. Let $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$, respectively $\{\boldsymbol{\tau_1}, \boldsymbol{\nu_1}, \boldsymbol{\beta_1}\}$ be the Frenet frames of the two parameterized curves at the corresponding points t and $u\lambda(t)$, respectively, and χ and χ_1 their – torsions at these points. Then

$$\boldsymbol{\beta_1}(\lambda(t)) = \boldsymbol{\beta}(t)$$
$$\boldsymbol{\nu_1}(\lambda(t)) = \boldsymbol{\nu}(t),$$

$$\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \;\Rightarrow\; \mathbf{r}'(t) = \frac{d}{du}(\boldsymbol{\beta_1}(u))\lambda'(t).$$

From the last Frenet equation for the curve $\mathbf{r}$, we get

$$\boldsymbol{\beta}'(t) \cdot \boldsymbol{\nu}(t) = -\|\mathbf{r}'(t)\| \cdot \chi(t),$$

i.e.

$$\begin{aligned}\chi(t) &= -\frac{1}{\|\mathbf{r}'(t)\|} \cdot \boldsymbol{\beta}'(t) \cdot \boldsymbol{\nu}(t) = -\frac{1}{\|\boldsymbol{\rho}'(\lambda(t))\| \cdot \lambda'(t)} \cdot \boldsymbol{\beta_1}'(\lambda(t)) \cdot \lambda'(t) \cdot \boldsymbol{\nu_1}(\lambda(t)) = \\ &= -\frac{1}{\|\boldsymbol{\rho}'(\lambda(t))\|} \cdot (-\|\boldsymbol{\rho}'(\lambda(t))\| \cdot \chi_1(\lambda(t))) = \chi_1(\lambda(t)),\end{aligned}$$

where we used once more the last Frenet equation, but this time for the curve $\boldsymbol{\rho}$, as well as the fact that the vector $\boldsymbol{\nu_1}(\lambda(t))$ has unit length. □

Let now $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ be a naturally parameterized curve, positively equivalent to the parameterized curve $\mathbf{r} = \mathbf{r}(t)$, where $s = \lambda(t)$ is the parameter change. Then $\mathbf{r}$ and its derivatives up to the third order can be expressed as functions of $\boldsymbol{\rho}$ and its derivatives as:

$$\begin{aligned}\mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)) \\ \mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \\ \mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t) \\ \mathbf{r}'''(t) &= \boldsymbol{\rho}'''(\lambda(t)) \cdot \lambda'^3(t) + 3\boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'(t) \cdot \lambda''(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'''(t),\end{aligned}$$

therefore the mixed product of the first three derivatives of $\mathbf{r}$ reads

$$\begin{aligned}(\mathbf{r}'(t),\mathbf{r}''(t),\mathbf{r}'''(t)) = \big(&\boldsymbol{\rho}'(\lambda(t))\cdot\lambda'(t), \boldsymbol{\rho}''(\lambda(t))\cdot\lambda'^2(t)+\boldsymbol{\rho}'(\lambda(t))\cdot\lambda''(t),\\ &\boldsymbol{\rho}'''(\lambda(t))\cdot\lambda'^3(t)+3\boldsymbol{\rho}''(\lambda(t))\cdot\lambda'(t)\cdot\lambda''(t)+\boldsymbol{\rho}'(\lambda(t))\cdot\lambda'''(t)\big) =\\ &=\lambda'^6(t)\,(\boldsymbol{\rho}'(\lambda(t)),\boldsymbol{\rho}''(\lambda(t)),\boldsymbol{\rho}'''(\lambda(t)))\,,\end{aligned}$$

all the other mixed products from the right hand side vanishing, because two of the factors are colinear. Hence

$$(\boldsymbol{\rho}'(\lambda(t)),\boldsymbol{\rho}''(\lambda(t)),\boldsymbol{\rho}'''(\lambda(t))) = \frac{1}{\lambda'^6(t)}\,(\mathbf{r}'(t),\mathbf{r}''(t),\mathbf{r}'''(t))\,.$$

But, since $\boldsymbol{\rho}$ is naturally parameterized and the two curves are positively equivalent, we have $\lambda' = \|\mathbf{r}'\|$, therefore the previous formula becomes

$$(\boldsymbol{\rho}',\boldsymbol{\rho}'',\boldsymbol{\rho}''') = \frac{(\mathbf{r}',\mathbf{r}'',\mathbf{r}''')}{\|\mathbf{r}'\|^6}.$$

Moreover (see (1.7.2)), the curvature can be expressed with respect to the derivatives of $\mathbf{r}$ through the formula

$$k = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3},$$

hence, from the expression of the torsion (1.10.4) and the previous relation, we get

$$\chi(t) = \frac{(\mathbf{r}',\mathbf{r}'',\mathbf{r}''')}{\|\mathbf{r}'\times\mathbf{r}''\|^2}. \tag{1.10.5}$$

Exercise 1.10.1. Let $\boldsymbol{\omega} = \chi\boldsymbol{\tau} + k\boldsymbol{\beta}$. Show that Frenet's formulae can be written as

$$\begin{cases}\boldsymbol{\tau}' = \boldsymbol{\omega}\times\boldsymbol{\tau}\\ \boldsymbol{\nu}' = \boldsymbol{\omega}\times\boldsymbol{\nu}\\ \boldsymbol{\beta}' = \boldsymbol{\omega}\times\boldsymbol{\beta}\end{cases} \tag{1.10.6}$$

The vector $\boldsymbol{\omega}$ is called the *Darboux vector*.

1.10.1 The geometrical meaning of the torsion

The torsion is, in a way, an analogue of the curvature (this is the reason why in the old-fashioned books the torsion is called the *second curvature*). What we mean is that the torsion can be also interpreted as being the speed of rotation of a straight line, this time the binormal. In other words, we have

Proposition 1.10.1. *If* $(I, \mathbf{r} = \mathbf{r}(s))$ *is a naturally parameterized curve and* $\Delta\alpha$ *is the angle of the osculating planes of the curve at* $\mathbf{r}(s)$ *and* $\mathbf{r}(s + \Delta s)$ *(in other words, the angle of the binormals of the curve at that points), then we have*

$$\chi(s) = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s}$$

Notice that, this time, unlike the case of the curvature, the torsion is the *algebraic value* of the limit, not the absolute value. We have to say, however, that the curvature of space curve is *defined* to be positive because one could find no geometrical meaning of a signed curvature. As we shall see in the sequel, for plane curve we can define a *signed curvature*, whose absolute value is the curvature and which will help us to get some more information about the curve.

As we said before, the torsion is analogue to the curvature. Thus, the curvature is a measure of the deviation of a curve from a straight line. On the other hand, the torsion is a measure of the deviation of the curve from a plane curve. More precisely, we have

Theorem 1.10.1. *The support of a biregular parameterized curve lies in a plane iff the torsion of the curve vanishes identically.*

Proof. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a biregular parameterized curve such that $\mathbf{r}(I) \subset \pi$, where π is a plane. Then, obviously, the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are parallel to this plane, which, as we know, is the osculating plane of the curve. Therefore, $\boldsymbol{\beta}(t) = \text{const}$, hence we have

$$0 = \boldsymbol{\beta}'(t) = -\underbrace{\|\mathbf{r}'\|}_{\neq 0} \cdot \chi(t) \cdot \underbrace{\boldsymbol{\nu}(t)}_{\neq 0} \Rightarrow \chi(t) \equiv 0.$$

Conversely, if $\chi(t) \equiv 0$, then the unit binormal, $\boldsymbol{\beta}(t)$, is always equal to a constant vector, $\boldsymbol{\beta}_0$. But $\boldsymbol{\beta}(t) = \mathbf{r}'(t) \times \mathbf{r}''(t)$, therefore the vector $\mathbf{r}'(t)$ is always perpendicular to the constant vector $\boldsymbol{\beta}_0$. Thus, we have the series of implications

$$\mathbf{r}'(t) \cdot \boldsymbol{\beta}_0 = (\mathbf{r} \cdot \boldsymbol{\beta}_0)' = 0 \Rightarrow \mathbf{r} \cdot \boldsymbol{\beta}_0 = \mathit{const} = \mathbf{r}_0 \cdot \boldsymbol{\beta}_0 \Rightarrow (\mathbf{r} - \mathbf{r}_0) \times \boldsymbol{\beta}_0 = 0,$$

i.e. the support of $\mathbf{r}(I)$ the curve is contained in a plane perpendicular to the constant vector $\boldsymbol{\beta}_0$, and passing through the point $\mathbf{r}_0$. □

1.10.2 Some further applications of the Frenet formulae

We saw that the curvature of a parameterized curve vanishes identically if and only if the support of the curve lies on a straight line. On the other hand, we know that the curvature of a circle is constant and equal to the inverse of the radius of the circle. We could expect that the converse is also through, in other words that if the curvature of a parameterized curve is constant, then its support lies on a circle. Unfortunately, this claim is not true. It is enough to think about the circular cylindrical helix, which has constant curvature (and, also, constant torsion). We have, however, the weaker result:

Proposition 1.10.2. *If* $(I, \mathbf{r} = \mathbf{r}(s))$ *is a naturally parameterized curve with a curvature* k *equal to a positive constant* k_0*, while the torsion vanishes identically, then the support of the curve lies on a circle of radius* $1/k_0$.

Proof. Since the torsion is identically zero, the curve is plane. Let us consider the parameterized curve $(I, \mathbf{r}_1 = \mathbf{r}_1(s))$, where

$$\mathbf{r}_1 = \mathbf{r} + \frac{1}{k_0}\boldsymbol{\nu}. \tag{*}$$

We differentiate with respect to s and we get, using the second Frenet formula for $\mathbf{r}$,

$$\mathbf{r}_1' = \mathbf{r}' + \frac{1}{k_0}\boldsymbol{\nu}' = \boldsymbol{\tau} + \frac{1}{k_0}(-k_0\boldsymbol{\tau}) = \boldsymbol{\tau} - \boldsymbol{\tau} = 0.$$

Thus, the curve $\mathbf{r}_1$ reduces to a point, say $\mathbf{r}_1(s) \equiv \mathrm{c} = \text{const}$. But, from (*), we get

$$\|\mathbf{r} - c\| = \left\|\frac{1}{k_0}\boldsymbol{\nu} \cdot \boldsymbol{\nu}\right\| = \frac{1}{k_0},$$

which means that any point of the support of the curve $\mathbf{r}$ is at a (constant) distance $1/k_0$ from the fixed point c, i.e. the support $\mathbf{r}(I)$ lies on the circle of radius $1/k_0$, centred at c. □

Another interesting situation is when the support of the curve lies not in a plane, but on a sphere. In this case we have

Proposition 1.10.3. *If a naturally parameterized curve* $(I, \mathbf{r} = \mathbf{r}(s))$ *has the support on a sphere centred at the origin and radius equal to a, then its curvature is subject to*

$$k \geq \frac{1}{a}.$$

Proof. The distance from a point of the curve to the origin is equal to $\|\mathbf{r}\|$, i.e. we have $\mathbf{r}^2 = a^2$. Differentiating, we get $\mathbf{r} \cdot \mathbf{r}' = 0$ or $\mathbf{r} \cdot \boldsymbol{\tau}$. Differentiating once more, we obtain

$$\mathbf{r}' \cdot \boldsymbol{\tau} + \mathbf{r} \cdot \boldsymbol{\tau}' = 0,$$

or

$$1 + \mathbf{r} \cdot \boldsymbol{\tau}' = 0 \quad \Longleftrightarrow \quad 1 + k\mathbf{r} \cdot \boldsymbol{\nu} = 0 \quad \Longrightarrow k\mathbf{r} \cdot \boldsymbol{\nu} = -1.$$

From the properties of the scalar product, we have

$$|\mathbf{r} \cdot \boldsymbol{\nu}| \le ||\mathbf{r}|| \cdot ||\boldsymbol{\nu}|| = ||\mathbf{r}|| = a,$$

therefore,

$$k = |k| = \frac{1}{|\mathbf{r} \cdot \boldsymbol{\nu}|} \ge \frac{1}{||\mathbf{r}|| \cdot ||\boldsymbol{\nu}||} = \frac{1}{a}.$$

□

In the figure **??** we give an example of a curve which lies on a sphere, it is a so-called *spherical helix*, because it is both a general helix (see the next section) and a spherical curve.

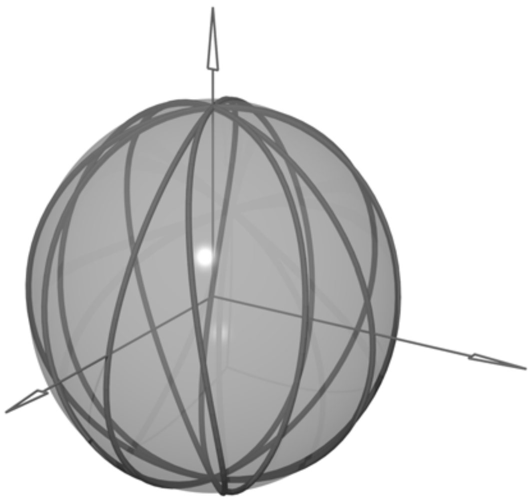


Figure 1.7: A spherical helix

1.10.3 General helices. Lancret's theorem

Definition 1.10.2. A parameterized curve curve $(I, \mathbf{r})$ is called a *general helix* if its tangents make a constant angle with a fixed direction in space.

The following theorem was formulated in 1802 by the French mathematician Paul Lancret, but the first known proof belongs to another celebrated French mathematician, known especially for his contributions to mechanics, A. de Saint Venant (1845).

Theorem 1.10.2 (Lancret, 1802)**.** *A space curve with the curvature $k > 0$ is a general helix if and only if the ratio between its torsion and its curvature is constant.*

Proof. Let us assume, to begin with, that the curve is parameterized by the arc length.

To prove the first implication, let us suppose that $\mathbf{r}$ is a general helix and let $\mathbf{c}$ be the versor of the fixed direction:

$$\boldsymbol{\tau} \cdot \mathbf{c} = \cos \alpha_0 = const.$$

By differentiating, we get

$$\boldsymbol{\tau}' \cdot \mathbf{c} = 0,$$

hence

$$k \cdot \mathbf{c} \cdot \boldsymbol{\nu} = 0.$$

As, by hypothesis, $k > 0$, it follows that

$$\mathbf{c} \cdot \boldsymbol{\nu} = 0,$$

i.e., at each point of the curve, $\mathbf{c} \perp \boldsymbol{\nu}$. This means that $\mathbf{c}$ is in the rectifying plane and therefore

$$\boldsymbol{\beta} \cdot \mathbf{c} = \sin \alpha_0.$$

By differentiating the relation $\boldsymbol{\nu} \cdot \mathbf{c} = 0$ we get, having in mind that $\mathbf{c}$ is constant and using the second formula of Frenet:

$$(-k\boldsymbol{\tau} + \chi\boldsymbol{\beta}) \cdot \mathbf{c} = 0$$

which leads to

$$-k \cdot \cos \alpha_0 + \chi \cdot \sin \alpha_0 = 0,$$

i.e.

$$\frac{\chi}{k} = \cot \alpha_0 = const.$$

Conversely, let us assume that

$$\frac{\chi(s)}{k(s)} = c_0 = const,$$

or

$$c_0 \cdot k - \chi = 0.$$

On the other hand, from the first and the third of the Frenet's equations, we get,

$$(c_0 \cdot k - \chi)\boldsymbol{\nu} = c_0\boldsymbol{\tau}' + \boldsymbol{\beta}' = 0.$$

We integrate once and we obtain

$$c_0\boldsymbol{\tau} + \boldsymbol{\beta} = \mathbf{c}^*,$$

where $\mathbf{c}^* \neq 0$ is a constant vector.

We define

$$\mathbf{c} := \frac{\mathbf{c}^*}{\|\mathbf{c}^*\|} = \frac{c_0\boldsymbol{\tau} + \boldsymbol{\beta}}{\|c_0\boldsymbol{\tau} + \boldsymbol{\beta}\|} = \frac{c_0\boldsymbol{\tau} + \boldsymbol{\beta}}{\sqrt{1 + c_0^2}},$$

whence

$$\mathbf{c} \cdot \boldsymbol{\tau} = \frac{c_0}{\sqrt{1 + c_0^2}} = const \leq 1.$$

Hence the vectors $\mathbf{c}$ şi $\boldsymbol{\tau}$ make a constant angle and the curve is a general helix. □

We end this paragraph by noting, as a historical curiosity, that, although in most of the books this theorem is credited to Lancret, in fact he (and, with him, Saint-Venant), only gave one implication, the first one: on a general helix, the ratio between the torsion and the curvature is constant. The other implication was given and proved later by Joseph Bertrand (see, for instance, the book of Eisenhart [16]).

1.10.4 Bertrand curves

An interesting problem in the theory of curves is whether it is possible for several curves to share the same family of tangents, principal normal or binormals. For the tangents, the answer is easily seen to be negative: the family of tangents uniquely determines the curve. For the principal normals, the problem, raised by the same Saint-Venant, was answered by Joseph Bertrand, who discovered that, for an arbitrary curve, the answer is negative, however, there are special curves for which there might be, also, other curves with the same family of principal normals. These curves are called *Bertrand curves*. Usually, for a Bertrand curve, there is only one curve having the same principal normals. We will say that the two curves are *Bertrand mates*, or that they are *associated, or conjugated Bertrand curves*. It turns out that if a Bertrand curve has more than one

Bertrand mate, then it has an infinity and the curve (and all of its mates) is a circular cylindrical helix.

We shall prove, in what follows, a series of interesting results related to Bertrand curves, without respecting the historical order. Let $\mathbf{r}$ and $\mathbf{r}^*$ be two Bertrand mates. We assume that the first curve is naturally parameterized. Then the second curve is, in fact, also dependent on the arc length s of the first and we assume that $\mathbf{r}^*(s)$ is the point on the Bertrand mate having the same principal normal as the first one at $\mathbf{s}$. The two points are called *corresponding*. We have the following result:

Theorem 1.10.3 (Schell)**.** *The angle of the tangents of two associated Bertrand curves at corresponding points is constant.*

Proof. Clearly, if $\boldsymbol{\nu}(s)$ is the versor of the principal normal of the first curve, then we have

$$\mathbf{r}^*(s) = \mathbf{r}(s) + a(s)\boldsymbol{\nu}(s). \tag{1.10.7}$$

As the two curves have the same principal normals, the second curve ought to have the versor of the principal normal

$$\boldsymbol{\nu}^*(s) = \pm\boldsymbol{\nu}(s), \tag{1.10.8}$$

We differentiate the relation (1.10.7) with respect to s and we get

$$\frac{d\mathbf{r}^*}{ds} = \frac{d\mathbf{r}}{ds} + a\frac{d\boldsymbol{\nu}}{ds} + \frac{da}{ds}\mathbf{r} \tag{1.10.9}$$

or, using the second Frenet formula,

$$\frac{d\mathbf{r}^*}{ds} = (1 - ak)\,\boldsymbol{\tau} + \frac{da}{ds}\boldsymbol{\nu} + a\chi\boldsymbol{\beta}. \tag{1.10.10}$$

The vector $\frac{d\mathbf{r}^*}{ds}$ is tangent to the second curve, therefore it is perpendicular both on $\boldsymbol{\nu}^*$ and on $\boldsymbol{\nu}$. We deduce, by multiplying both sides of (1.10.10) by $\boldsymbol{\nu}$, that $\frac{da}{ds} = 0$, i.e. a is a constant. As such, the relation (1.10.10) turns into

$$\frac{d\mathbf{r}^*}{ds} = (1 - ak)\,\boldsymbol{\tau} + a\chi\boldsymbol{\beta}. \tag{1.10.11}$$

We denote by s^* the arc length of the second curve. Then

$$\boldsymbol{\tau}^* = \frac{d\mathbf{r}^*}{ds^*} = \frac{dr^*}{ds}\frac{ds}{ds^*} \tag{1.10.12}$$

or, using (1.10.11)

$$\boldsymbol{\tau}^* = (1 - ak)\,\frac{ds}{ds^*}\boldsymbol{\tau} + a\chi\frac{ds}{ds^*}\boldsymbol{\beta}. \tag{1.10.13}$$

Let ω be the angle of the tangents of the two curvs at the corresponding points. Then this angle is given by

$$\cos\omega = \boldsymbol{\tau}\boldsymbol{\tau}^*, \tag{1.10.14}$$

where, of course, $\boldsymbol{\tau}^*$ is the versor of the second curve at the point $\mathbf{r}^*(s)$. In terms of ω, $\boldsymbol{\tau}^*$ and the versor of the binormal of the second curve,, $\boldsymbol{\beta}^*$, can be written as

$$\boldsymbol{\tau}^* = \cos\omega\boldsymbol{\tau} + \sin\omega\boldsymbol{\beta}, \tag{1.10.15}$$

$$\boldsymbol{\beta}^* = \varepsilon\left(-\sin\omega\boldsymbol{\tau} + \cos\omega\boldsymbol{\beta}\right), \tag{1.10.16}$$

where $\varepsilon = \pm 1$.

We differentiate the relation (1.10.15) with respect to s and we get:

$$\frac{d\boldsymbol{\tau}^*}{ds} = -\sin\omega\frac{d\omega}{ds}\boldsymbol{\tau} + k\cos\omega\boldsymbol{\nu} + \cos\omega\frac{d\omega}{ds}\boldsymbol{\beta} - \chi\sin\omega\boldsymbol{\nu}. \tag{1.10.17}$$

If we multiply scalarly both sides of the relation (1.10.17) by $\boldsymbol{\tau}$ and then by $\boldsymbol{\beta}$ and use the fact that, on the ground of the first Frenet formula, $\frac{d\boldsymbol{\tau}^*}{ds}$ is colinear to $\boldsymbol{\nu}^*$, and, thsu, also to $\boldsymbol{\nu}$, we obtain the relations:

$$\sin\omega\frac{d\omega}{ds} = 0, \quad \cos\omega\frac{d\omega}{ds} = 0, \tag{1.10.18}$$

i.e. $\frac{d\omega}{ds} = 0$, hence ω is constant. □

The following theorem was proved by Joseph Bertrand and it is considered to be the central result of the entire theory of Bertrand curves.

Theorem 1.10.4 (Bertrand)**.** *A curve* $\mathbf{r}$ *is a Bertrand curve if and only if its torsion and curvature verify a relation of the form*

$$a \cdot k + b \cdot \chi = 1, \tag{1.10.19}$$

with constant a and b.

Proof. Let us assume that $\mathbf{r}$ is a Bertrand curve. Comparing the relations (1.10.13) and (1.10.15), we obtain that

$$\cos\omega = (1 - ak)\frac{ds}{ds^*}, \tag{1.10.20}$$

$$\sin\omega = a\chi\frac{ds}{ds^*}. \tag{1.10.21}$$

Dividing side by side these two equalities, we obtain

$$\operatorname{ctg}\omega = \frac{1 - ak}{a\chi} \tag{1.10.22}$$

or

$$1 - ak = a\chi \operatorname{ctg}\omega. \tag{1.10.23}$$

If we denote

$$b = a \operatorname{ctg}\omega, \tag{1.10.24}$$

we obtain the relation (1.10.19).

Let us suppose now, conversely, that the relation (1.10.19) holds. Let $\mathbf{r}^*$ be the curve given by (1.10.7), where a is the constant from the equation (1.10.19). Differentiating the relation (1.10.7), we get, as we saw earlier, the relation (1.10.11). As the basis $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$ is orthonormal we conclude from (1.10.11) that

$$\left(\frac{ds^*}{ds}\right)^2 = (1 - ak)^2 + a^2\chi^2 \tag{1.10.25}$$

or, using (1.10.25),

$$\left(\frac{ds^*}{ds}\right)^2 = (a^2 + b^2)\chi^2, \tag{1.10.26}$$

whence

$$\chi\frac{ds}{ds^*} = \text{const.} \tag{1.10.27}$$

In the same way follows that

$$(1 - ak)\frac{ds}{ds^*} = \text{const.} \tag{1.10.28}$$

Taking into account these two relations, we differntiate the relation (1.10.13) and we obtain, by using the first and the third Frenet formulae for the curve $\mathbf{r}$:

$$\frac{d\boldsymbol{\tau}^*}{ds} = \left[k(1 - ak) - a\chi^2\right]\frac{ds}{ds^*}\boldsymbol{\nu} \tag{1.10.29}$$

or, using the first Frenet formula for the curve $\mathbf{r}^*$:

$$k^*\boldsymbol{\nu}^* = \left[k(1 - ak) - a\chi^2\right]\left(\frac{ds}{ds^*}\right)^2\boldsymbol{\nu}, \tag{1.10.30}$$

whence it follows that $\boldsymbol{\nu}^* = \pm\boldsymbol{\nu}$, i.e., indeed, $\mathbf{r}$ is a Bertrand curve. □

Corollary 1.10.1. *The circular cylindrical helices, the curves of constant torsion (in particular the plane curves), and the curves of constant curvature are Bertrand curves.*

Remark. For a plane curve, the principal normals are, in fact, the normals, therefore, as one can see easily, any plane curve has an infinity of Bertrand mates, all of them congruent (they are, all of them, parallel to the given one).

Corollary 1.10.2. *If a Bertrand curve has more than one mate, then it has an infinity and it is a circular cylindrical helix.*

Proof. As we saw from the proof of the theorem of Bertrand, the constant a from the relation (1.10.19) is the one that identifies the Bertrand mate of our curve. Thus, if a curve $\mathbf{r}$ has more then one Bertrand mate, this means that there are (at least) two distinct pairs of real numbers (a, b), (a_1, b_1) such that

$$\begin{aligned} a \cdot k + b \cdot \chi &= 1, \\ a_1 \cdot k + b_1 \cdot \chi &= 1. \end{aligned}$$

Subtracting side by side the two relations, we obtain

$$(a - a_1) \cdot k = (b_1 - b) \cdot \chi.$$

At least one of the coefficients $(a - a_1)$ and $(b_1 - b)$ is different from zero, therefore the ratio between the curvature and torsion is a constant. This means, via the Lancret's theorem, that the curve is a general helix. However, as, say, $\chi = c \cdot k$, with c – constant, from (1.10.19) follows that

$$(a + b \cdot c) \cdot k = 1,$$

which means that k (and, hence, also χ) is a constant, therefore the curve is a *circular* cylindrical helix. Clearly, for the case of a cylindrical helix, when both the curvature and the torsion are constants, we can get an infinity of pairs of real numbers that verify (1.10.19), so our curve has an infinity of Bertrand mates. They lie on circular cylinders which have the same axis of symmetry with the cylinder associated to the given curve. □

1.11 The local behaviour of a parameterized curve around a biregular point

Let $(I, \mathbf{r})$ be a naturally parameterized curve. We shall assume that $0 \in I$, as an interior point and $M_0 \equiv \mathbf{r}(0)$ is a biregular point of the curve. We shall make use of the following notations: $\boldsymbol{\tau}_0 = \boldsymbol{\tau}(0)$, $\boldsymbol{\nu}_0 = \boldsymbol{\nu}(0)$, $\boldsymbol{\beta}_0 = \boldsymbol{\beta}(0)$.

We expand $\mathbf{r}$ in a Taylor series around the origin. Up to the third order, the Taylor expansion is

$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + \frac{1}{6}s^3\mathbf{r}'''(0) + o(s^3). \tag{1.11.1}$$

We want to express the derivatives of $\mathbf{r}$ as functions of the Frenet vectors $\boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0$. We have, obviously, since $\mathbf{r}$ is naturally parameterized,

$$\mathbf{r}'(0) = \boldsymbol{\tau}_0. \tag{1.11.2}$$

On the other hand, from the definition of the curvature vector, we have

$$\mathbf{r}''(0) = \mathbf{k}(0) = k(0) \cdot \boldsymbol{\nu}_0. \tag{1.11.3}$$

Moreover, if we differentiate the relation

$$\mathbf{r}''(s) = \mathbf{k}(s) = k(s) \cdot \boldsymbol{\nu} \tag{1.11.4}$$

we get

$$\begin{aligned} \mathbf{r}'''(s) &= k'(s)\boldsymbol{\nu} + k(s)\boldsymbol{\nu}' = k'(s)\boldsymbol{\nu} + k(s)(-k(s)\boldsymbol{\tau} + \chi(s)\boldsymbol{\beta}) = \\ &= -k^2(s)\boldsymbol{\tau} + k'(s)\boldsymbol{\nu} + k(s)\chi(s)\boldsymbol{\beta}, \end{aligned} \tag{1.11.5}$$

where we have made use of the second Frenet equation.

Substituting $s = 0$ in (1.11.5), we get

$$\mathbf{r}'''(0) = -k^2(0)\boldsymbol{\tau}_0 + k'(0)\boldsymbol{\nu}_0 + k(0)\chi(0)\boldsymbol{\beta}_0. \tag{1.11.6}$$

Thus, the relation (1.11.1) becomes

$$\mathbf{r}(s) - \mathbf{r}(0) = s\boldsymbol{\tau}_0 + \frac{1}{2}s^2k(0)\boldsymbol{\nu}_0 + \frac{1}{6}s^3\left(-k^2(0)\boldsymbol{\tau}_0 + k'(0)\boldsymbol{\nu}_0 + k(0)\chi(0)\boldsymbol{\beta}_0\right) + o(s^3) \tag{1.11.7}$$

or

$$\begin{aligned} \mathbf{r}(s) - \mathbf{r}(0) = &\left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right)\boldsymbol{\tau}_0 + \left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right)\boldsymbol{\nu}_0 + \\ &+ \left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right)\boldsymbol{\beta}_0. \end{aligned} \tag{1.11.8}$$

We consider now a coordinate frame with the origin at M_0 and having as axes the Frenet axes. The position vector of a point of the curve with respect to this frame is exactly the vector $\mathbf{r}(s) - \mathbf{r}(0) \equiv \overrightarrow{M_0 M}$, where $M = \mathbf{r}(s)$. Therefore, projecting (1.11.8) on the axes, we get

$$\begin{cases} x(s) & = s - \frac{1}{6}k^2(0)s^3 + o(s^3) \\ y(s) & = \frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3) \\ z(s) & = \frac{1}{6}k(0)\chi(0)s^3 + o(s^3) \end{cases} . \tag{1.11.9}$$

It is not at all difficult to see that locally, close enough to 0, we have the following relations between coordinates, giving, in fact, the equations of the projections of the curve on the coordinate planes of the Frenet frame:

$$\begin{cases} y = \frac{1}{2}k(0)x^2, \\ z = \frac{1}{6}k(0)\chi(0)x^3, \\ z^2 = \frac{2}{9}\frac{\chi^2(0)}{k(0)}y^3. \end{cases} \tag{1.11.10}$$

Thus, the projection of the curve on the xOy-plane (the osculating plane) is a parabola, the projection on the xOz-plane (the rectifying plane) is a cubic, while the projection on the yOz-plane (the normal plane) is a semicubic parabola.

1.12 The contact between a space curve and a plane

We consider a naturally parameterized space curve $(I, \mathbf{r})$. We assume, as we did before, that 0 is an interior point of the interval I and that the point $M_0 = \mathbf{r}(0)$ of the curve is biregular. We consider a plane Π passing through the point M_0. We choose as coordinate frame the Frenet frame of the curve $\mathbf{r}$ at the point M_0, $\mathcal{R}_0 = \{M_0; \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$. With respect to this coordinate frame, he cartesian equation of the plane Π will be of the form

$$F(x, y, z) \equiv ax + by + cz = 0. \tag{1.12.1}$$

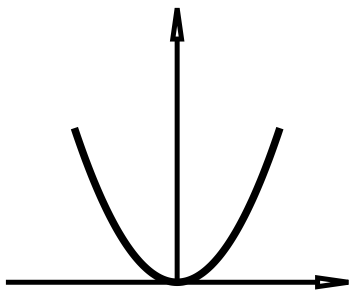
Now, if $x = x(s), y = y(s), z = z(s)$ are the local equations of the curve with respect to the Frenet frame (see (1.11.9)), the intersection condition between the the plane and the curve is $F(x(s), y(s), z(s)) = 0$, i.e.

$$a\left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right) + b\left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right) + c\left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right) = 0 \tag{1.12.2}$$

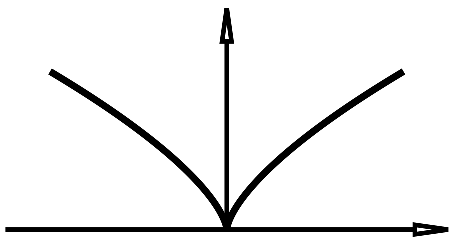
or

$$as + \frac{1}{2}bk(0)s^2 + \frac{1}{6}\left(-ak^2(0) + bk'(0) + ck(0)\chi(0)\right)s^3 + o(s^3) = 0. \tag{1.12.3}$$

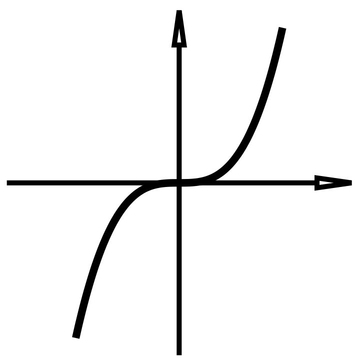
We have, now, several possibilities,



(a) The osculating plane



(b) The normal plane



(c) The rectifying plane

Figure 1.8: The projection of a space curve on the planes of the Frenet frame

a) If $a \neq 0$ (i.e. the plane does not contain the tangent), that the plane has a contact of order zero with the curve (intersection, they have a single point in common).

b) If $a = 0$, then the intersection equation has $s = 0$ as a double solution, which means that the curve and the plane have a contact of order 1 (tangency contact). It can be observed immediately that in this case the plane Π passes through the tangent at the curve at the point M_0 (which is, in fact, the straight line of equations $y = 0, z = 0$ in the coordinate frame considered by us).

c) If we want a contact of order at least 2 (osculation contact), then the coefficient of s^2 in the intersection equation should also vanish. This may happen only if $b = 0$ (as the point M_0 is biregular, the curvature is not zero at this point). Thus, we have osculation contact if we impose $a = 0$ and $b = 0$. But this choice leads us to the equation $z = 0$ for the plane Π, i.e. the plane is already completely determined (the osculating plane).

d) As the plane Π is completely determined by the condition of having a second order contact with the curve, we cannot specialize more, in order to have higher order contact (superosculation). Anyway, looking at the intersection equation, we conclude that the osculating plane has a contact or superosculation with the curve at the planar points of the curve, where the torsion vanishes. At all other points, the contact is just of second order (osculation).

1.13 The contact between a space curve and a sphere. The osculating sphere

As in the previous paragraph, we consider a naturally parameterized curve $(I, \mathbf{r})$, we assume that 0 is an interior point of the interval I and that $\mathbf{r}(0)$ is a biregular point of the curve. We choose as coordinate frame the Frenet frame of the curve $\mathbf{r}$ at the point M_0, $\mathcal{R}_0 = \{M_0; \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$. Then an arbitrary sphere passing through the point M_0 will have, with respect to this coordinate system, the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0, \tag{1.13.1}$$

where $\Omega(a, b, c)$ is the center of the sphere. Then the condition of the intersection will be, again, $F(x(s), y(s), z(s)) = 0$, where $x = x(s), y = y(s), z = z(s)$ are the local equations of the curve with respect to the Frenet frame at M_0. Thus, in our case, we have

$$\begin{aligned}F(x(s),y(s),z(s)) &= \left(s-\frac{1}{6}k^2(0)s^3+o(s^3)\right)^2+\left(\frac{1}{2}k(0)s^2+\frac{1}{6}k'(0)s^3+o(s^3)\right)^2+\\&+\left(\frac{1}{6}k(0)\chi(0)s^3+o(s^3)\right)^2-2a\left(s-\frac{1}{6}k^2(0)s^3+o(s^3)\right)-\\&-2b\left(\frac{1}{2}k(0)s^2+\frac{1}{6}k'(0)s^3+o(s^3)\right)-2c\left(\frac{1}{6}k(0)\chi(0)s^3+o(s^3)\right)=0.\end{aligned} \tag{1.13.2}$$

or

$$-2as+(1-bk(0))s^2+\frac{1}{3}(ak^2(0)-bk'(0)-ck(0)\chi(0))s^3+o(s^3). \tag{1.13.3}$$

Now, the discussion that follows is, also, similar to that from the case of the contact between a curve and a plane. Still, here we have more cases to take into consideration.

a) If $a \neq 0$, then $s = 0$ is a simple solution of the intersection equation. Thus, in this case the sphere and the curve have a contact of order zero (intersection contact).

b) The sphere and the curve have a first order contact (tangency contact) if and only if the intersection equation has a double zero at the origin. This happens, obviously, iff $a = 0$. This means that the first coordinate of the center of the sphere is zero, i.e. this center lies in the normal plane of the curve at M_0. It is easy to see that in this case the tangent line at M_0 of the curve lies in the tangent plane of the sphere at the same point, which justifies once more the denomination of *tangency contact*.

c) For an osculation (second order) contact, the coefficient of s^2 in the intersection equation also has to vanish and we get

$$b=\frac{1}{k(0)}=R(0), \tag{1.13.4}$$

where $R(0)$ is the curvature radius of the curve at the point M_0. Thus, in this case, the center of the sphere lies on the intersection line of the planes $x = 0$ (the normal plane) and $y = R(0)$. This line, which, as one can easily see, is parallel to the binormal of the curve at M_0, is called *curvature axis* or *polar axis* of the curve. It intersects the osculating plane at M_0 of the curve at the center of curvature at M_0 of the curve. Thus, any sphere with the center on the curvature axis of the curve has an osculation contact with this one.

d) We shall say that the sphere has a contact of *superosculation* with the curve if they have a contact of order at least three, which means that in the intersection equation the coefficients of the of the powers up to three of s have to vanish simultaneously. It is usually claimed that there is a single sphere which has a contact of superosculation with the curve and this sphere is called the *osculating sphere*. In reality, the things are a little bit more subtle, as we shall see in a moment.

1) Let us assume, first, that the torsion of the curve at M_0 does not vanish, i.e. $\chi(0) \neq 0$. In this case, as one sees immediately, between the sphere and the curve there is a superosculation contact if and only if we have

$$\begin{cases} a = 0, \\ b = \frac{1}{k(0)}, \\ c = \frac{k'(0)}{k^2(0)\chi(0)}, \end{cases} \tag{1.13.5}$$

i.e. in this case the sphere is uniquely determined and we will call it *osculating sphere* [7].

2) If $\chi(0) = 0$, while $k'(0) \neq 0$, as one can see easily, the coefficient of s^3 is never zero, therefore there is no sphere which has a superosculation contact with the curve. However, as we have seen in the previous paragraph, in this case the osculating plane has a superosculation contact with the curve and we can think of this plane as playing also the role of the osculating sphere, of infinite radius, though.

3) If both $\chi(0)$ and $k'(0)$ vanish, then the coefficient of s^3 in the intersection equation vanishes for any c, in other words in this case any sphere which has an osculation contact with the curve also has a superosculation contact. Thus, now we have an *infinity* of osculating spheres.

1.14 Existence and uniqueness theorems for parameterized curves

1.14.1 The behaviour of the Frenet frame under a rigid motion

Definition 1.14.1. A *rigid motion* of $\mathbb{R}^3$ is a map $D : \mathbb{R}^3 \to \mathbb{R}^3$, $D(x) = \mathcal{A} \cdot x + \mathbf{b}$, where $\mathcal{A} \in M_{3\times 3}(\mathbb{R})$ is an orthogonal matrix, $\mathcal{A}^t \cdot \mathcal{A} = I_3$, with the determinant equal to one: $\det \mathcal{A} = 1$, while $\mathbf{b} \in \mathbb{R}^3$ is a constant vector. The linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A(x) = \mathcal{A} \cdot x$ is called the *homogeneous part* of the rigid motion.

[7] In some books all the spheres which have an osculation contact with the curve are called osculating spheres, while the one which has a superosculation contact is called, accordingly, *superosculating sphere*

Remarks. (i) A rigid motion in $\mathbb{R}^3$ is just a rotation around an arbitrary axis followed by a translation.

(ii) If we don't ask $\det \mathcal{A} = 1$, we get, equally, an isometry of $\mathbb{R}^3$. However, in this case (if $\det \mathcal{A} = -1$), a transformation $D(x) = \mathcal{A} \cdot x + b$ does not reduce anymore to a rotation and a translation, we have to add, also, a reflection with respect to a plane. In many books the term "motion" implies only that the matrix $\mathcal{A}$ is orthogonal and for what we call "rigid motion" it is used the term "proper motion".

Definition 1.14.2. Let $D : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion, with the homogeneous part $A : \mathbb{R}^3 \to \mathbb{R}^3$. The *image* of a parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ through D is, by definition, the parameterized curve $(I, \mathbf{r}_1 = (D \circ \mathbf{r})(t))$.

Remark. Since A is a nondegenerate linear map, the image of a regular parameterized curve is, also, a regular parameterized curve.

Theorem 1.14.1. *Let D be a rigid motion of $\mathbb{R}^3$, with the homogeneous part A, $(I, \mathbf{r} = \mathbf{r}(t))$ – a biregular parameterized curve, $(I, \mathbf{r}_1 = D \circ \mathbf{r})$ – its image through D and $\{\mathbf{r}(t); \boldsymbol{\tau}(t), \boldsymbol{\nu}(t), \boldsymbol{\beta}(t)\}$ – the Frenet frame of the parameterized curve $\mathbf{r}$ at t. Then the frame $\{\mathbf{r}_1(t); A(\boldsymbol{\tau}(t)), A(\boldsymbol{\nu}(t)), A(\boldsymbol{\beta}(t))\}$ is the Frenet frame of $\mathbf{r}_1$ at t.*

Proof. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$, $D(x, y, z) = (x_1, y_1, z_1)$, where

$$\begin{cases} x_1 = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + b_1 \\ y_1 = \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + b_2 \\ z_1 = \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + b_3 \end{cases} \tag{1.14.1}$$

Then

$$\mathbf{r}_1(t) = (x_1(t), y_1(t), z_1(t)),$$

hence

$$\mathbf{r}'_1(t) = A(\mathbf{r}'(t)), \quad \mathbf{r}''_1(t) = A(\mathbf{r}''(t)). \tag{1.14.2}$$

Since A is a linear isometry, preserving the orientation of $\mathbb{R}^3$, we have

$$\begin{gathered} \|\mathbf{r}'_1\| = \|A(\mathbf{r}')\| = \|\mathbf{r}'\|, \ \mathbf{r}'_1 \cdot \mathbf{r}''_1 = \mathbf{r}' \cdot \mathbf{r}'', \\ \mathbf{r}'_1 \times \mathbf{r}''_1 = A(\mathbf{r}' \times \mathbf{r}''), \ \|\mathbf{r}'_1 \times \mathbf{r}''_1\| = \|\mathbf{r}' \times \mathbf{r}''\|. \end{gathered}$$

Then:

$$
\begin{aligned}
\boldsymbol{\tau}_1 &= \frac{\mathbf{r}_1'}{||\mathbf{r}'||} = \frac{A(\mathbf{r}')}{||A(\mathbf{r}')||} = \frac{A(\mathbf{r}')}{||\mathbf{r}'||} = A\left(\frac{\mathbf{r}'}{||\mathbf{r}'||}\right) = A(\boldsymbol{\tau}) \\
\boldsymbol{\nu}_1 &= \frac{||\mathbf{r}_1'||}{||\mathbf{r}_1' \times \mathbf{r}_1''||}\mathbf{r}_1'' - \frac{\mathbf{r}_1' \cdot \mathbf{r}_1''}{||\mathbf{r}_1'|| \cdot ||\mathbf{r}_1' \times \mathbf{r}_1''||}\mathbf{r}_1' = \frac{||\mathbf{r}'||}{\mathbf{r}' \times \mathbf{r}''} \cdot A(\mathbf{r}'')- \\
&- \frac{\mathbf{r}' \cdot \mathbf{r}''}{||\mathbf{r}'|| \cdot ||\mathbf{r}' \times \mathbf{r}''||} \cdot A(\mathbf{r}') = A(\boldsymbol{\nu}) \\
\boldsymbol{\beta}_1 &= \frac{\mathbf{r}_1' \times \mathbf{r}_1''}{||\mathbf{r}_1' \times \mathbf{r}_1''||} = A(\boldsymbol{\beta}).
\end{aligned}
$$

□

Consequence. *The parameterized curves* $(I, \mathbf{r})$ *and* $(I, \mathbf{r}_1 = D \circ \mathbf{r})$ *have the same curvature and torsion.*

Proof. We have

$$k_1 = \frac{||\mathbf{r}_1' \times \mathbf{r}_1''||}{||\mathbf{r}_1'||} = \frac{||\mathbf{r}' \times \mathbf{r}''||}{||\mathbf{r}'||} = k.$$

For the torsion, the situation is slightly more complicated. We have, from the theorem,

$$\boldsymbol{\beta}_1(t) = A(\boldsymbol{\beta}(t)) \quad \text{and} \quad \boldsymbol{\nu}_1(t) = A(\boldsymbol{\nu}(t)).$$

A being a linear operator, a similar result holds for the derivatives of the two Frenet vectors, i.e. we have

$$\boldsymbol{\beta}_1'(t) = A(\boldsymbol{\beta}'(t)) \quad \text{and} \quad \boldsymbol{\nu}_1'(t) = A(\boldsymbol{\nu}'(t)).$$

Using the last Frenet equations for the two curves, we have the equalities

$$-\chi_1(t)\boldsymbol{\nu}_1(t) = A(-\chi_(t)\boldsymbol{\nu}(t)) = -\chi(t)A(\boldsymbol{\nu}(t)) = -\chi(t)\boldsymbol{\nu}_1(t),$$

and, thus, the two torsions are equal, as claimed. □

1.14.2 The uniqueness theorem

Theorem 1.14.2. *Let* $(I, \mathbf{r} = \mathbf{r}(t))$ *and* $(I, \mathbf{r}_1 = \mathbf{r}_1(t))$ *be two biregular parameterized curves. If* $k(t) = k_1(t)$, $\chi(t) = \chi_1(t)$ *and* $||\mathbf{r}'(t)|| = ||\mathbf{r}_1'(t)||$ $\forall t \in I$, *then there is a single rigid motion* D *of* $\mathbb{R}^3$ *such that* $\mathbf{r}_1 = D \circ \mathbf{r}$.

Proof. Let $t_0 \in I$ be an arbitrary point and D – the rigid motion of $\mathbb{R}^3$ sending the Frenet frame $\{\mathbf{r}(t_0); \boldsymbol{\tau_0}, \boldsymbol{\nu_0}, \boldsymbol{\beta_0}\}$ of the curve $\mathbf{r}$ at t_0 into the Frenet frame $\{\mathbf{r}_1(t_0); \boldsymbol{\tau_{10}}, \boldsymbol{\nu_{10}}, \boldsymbol{\beta_{10}}\}$ of the curve $\mathbf{r}_1$ at the same point. Obviously, there is a single rigid motion with this property. Let $(I, \mathbf{r}_2(t{=}D \circ \mathbf{r}(t))$ – the image of the curve $\mathbf{r}$ through D, and k_2, χ_2 – the curvature and the torsion, respectively, of the parameterized curve $\mathbf{r}_2$. Then

$$k_2(t) \equiv k(t) \equiv k_1(t)$$

$$\chi_2(t) \equiv \chi(t) \equiv \chi_1(t)$$

and, moreover,

$$\|\mathbf{r_2'}(t)\| \equiv \|\mathbf{r_1'}(t)\|.$$

Therefore, the vector functions $\boldsymbol{\tau_1}(t)$, $\boldsymbol{\nu_1}(t)$, $\boldsymbol{\beta_1}(t)$ and $\boldsymbol{\tau_2}(t)$, $\boldsymbol{\nu_2}(t)$, $\boldsymbol{\beta_2}(t)$ giving the Frenet frame are solutions of the same system of Frenet equations

$$\begin{cases} \boldsymbol{\tau}' = \|\mathbf{r_1'}\| k_1 \boldsymbol{\nu} \\ \boldsymbol{\nu}' = -\|\mathbf{r_1'}\| k_1 \boldsymbol{\tau} + \|\mathbf{r_1'}\| \chi_1 \boldsymbol{\beta} \\ \boldsymbol{\beta}' = -\|\mathbf{r_1'}\| \chi_1 \boldsymbol{\nu}. \end{cases}$$

Since for $t = t_0$ the solutions coincide, due to the uniqueness theorem for the solution of a Cauchy problem they have to coincide globally. In particular, we have

$$\boldsymbol{\tau_1}(t) \equiv \boldsymbol{\tau_2}(t) \quad \text{or} \quad \frac{\mathbf{r_1'}(t)}{\|\mathbf{r_1'}(t)\|} = \frac{\mathbf{r_2'}(t)}{\|\mathbf{r_2'}(t)\|},$$

hence

$$\mathbf{r_1'}(t) - \mathbf{r_2'}(t) = 0 \;\Rightarrow\; \mathbf{r}_1(t) - \mathbf{r}_2(t) = \text{const}.$$

But for $t = t_0$, $\mathbf{r}_1(t_0) - \mathbf{r}_2(t_0) = 0$, hence the two functions coincide for all t, thus $\mathbf{r}_1(t) \equiv \mathbf{r}_2(t) = D \circ \mathbf{r}(t)$.

As for the uniqueness of D, we notice that, for any other point $t_1 \in I$, since $\mathbf{r}_1 \equiv \mathbf{r}_2$, D sends the Frenet frame of the curve $\mathbf{r}$ at t_1 into the Frenet frame of the curve $\mathbf{r}_1$ at t_1. □

Remark. For naturally parameterized curves the condition $\|\mathbf{r}'(t)\| = \|\mathbf{r_1'}(t)\|$ is always fulfilled.

1.14.3 The existence theorem

Theorem 1.14.3. *Let $f(s)$ and $g(s)$ be two smooth functions, defined on an interval I, such that $f(s) > 0$, $\forall t \in I$. Then there is a single naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$ for which $f(s) = k(s)\ \forall s \in I$ and $g(s) = \chi(s)\ \forall s \in I$. This curve is uniquely defined, up to a rigid motion of $\mathbb{R}^3$.*

Proof. Let $\{\mathbf{r}_0; \mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}\}$ be a direct orthonormal frame in
$mathbbR^3$. We consider the system of linear ordinary differential equations

$$\begin{cases} \mathbf{T}'(s) = f(s)\mathbf{N}(s) \\ \mathbf{N}'(s) = -f(s)\mathbf{T}(s) + g(s)\mathbf{B}(s) \\ \mathbf{B}'(s) = -g(s)\mathbf{N}(s) \end{cases} \tag{1.14.3}$$

with respect to the vector functions $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$.

If we denote

$$X(s) = (\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)), \tag{1.14.4}$$

then the system (1.14.3) can be written as

$$X'(s) = A(s) \cdot X(s), \tag{1.14.5}$$

with

$$A(s) = \begin{pmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{pmatrix}.$$

In the theory of ordinary differential equations one proves that the system (1.14.5) has a single solution subject to

$$X(s_0) = (\mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}),$$

where $s_0 \in I$, while the columns of the matrix $X(s_0)$ are the vectors $\mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}$ of the initial orthonormal basis.

We are going to show first that, for any $s \in I$ the vectors din $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s))$ form an orthonormal basis. It is enough to show for any $s \in I$, $X(s)$ is orthogonal, i.e. $X^t(s) \cdot X(s) = I_3$, for any $s \in I$. We have

$$\frac{d}{dt}\left(X^t \cdot X\right) = \frac{d}{dt}\left(X^t(s)\right) \cdot X + X^t \cdot \frac{d}{dt}(X(s)) = X^t(A^t X + AX) = X^t(A^t + A)X.$$

but, as A is skew symmetrical, $A^t + A = 0$, therefore

$$\frac{d}{dt}\left(X^t \cdot X\right) = 0 \ \Rightarrow\ X^t \cdot X = const.$$

On the other hand, from the initial condition, $(X^t \cdot X)(s_0) = I_3$, hence $X^t(s) \cdot X(s) = I_3$ for any $s \in I$.

Let us define now

$$\mathbf{r} = \mathbf{r}_0 + \int_{s_0}^{s} \mathbf{T}(s) ds, \tag{sol}$$

where $\mathbf{r}_0$ is the origin of the original frame, while $\mathbf{T}(s)$ is the first column of $X(s)$. We are going to show that $(I, \mathbf{r}(s))$ is the curved searched for. We have, clearly:

$$\begin{aligned} \mathbf{r}'(s) &= \mathbf{T}(s), \\ \|\mathbf{r}'(s)\| &= \|\mathbf{T}(s)\| = 1, \\ \mathbf{r}''(s) &= \mathbf{T}^{'}(s) = f(s)\mathbf{N}(s). \end{aligned}$$

We note immediately that $\mathbf{r}'(s) \times \mathbf{r}''(s) \neq 0$, therefore $\mathbf{r}(s)$ is a biregular naturally parameterized curve. On the other hand,

$$\mathbf{r}'''(s) = (f(s)\mathbf{N})' = f'\mathbf{N} + f\mathbf{N}^{'} = f'\mathbf{N} + f(-f\mathbf{T} + g\mathbf{B}) = -f^2\mathbf{T} + f'\mathbf{N} + fg\mathbf{B},$$

hence

$$(\mathbf{r}', \mathbf{r}'', \mathbf{r}''') = (\mathbf{T}, f\mathbf{N}, -f^2\mathbf{T} + f'\mathbf{N} + fg\mathbf{B}) = (\mathbf{T}, f\mathbf{N}, fg\mathbf{B}) = f^2 g \underbrace{(\mathbf{T}, \mathbf{N}, \mathbf{B})}_{=1} = f^2 g.$$

Now we have all we need to compute the curvature and the torsion:

$$\begin{aligned} k(s) &= \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = |f(s)| = f(s), \\ \chi(s) &= \frac{f^2(s)g(s)}{f^2(s)} = g(s), \end{aligned}$$

hence the curve $\mathbf{r}$ fulfils the conditions of the theorem.

The uniqueness of $\mathbf{r}$, up to rigid motions, follows from the previous theorem. □

CHAPTER 2

Plane curves

2.1 Introduction

After the exposition of the theory of space curves in all the generality, we shall focus in this chapter on topics which are specific to the theory of plane curves. In particular, we shall discuss here notions which cannot be defined for space curves (such as the signed curvature), or which are easier to investigate and richer in contents for plane curves.

2.2 Envelopes of plane curves

In this section, if not mentioned otherwise, all the curves are *parameterized curves*, if not specified otherwise. Let

$$\mathbf{r} = \mathbf{r}(t, \lambda) \tag{2.2.1}$$

be a family of plane parameterized curves, depending smoothly on a real parameter λ.

Definition. The *envelope* of the family (2.2.1) is a parameterized curve (J, Γ) which, at each point, is tangent to a curve from the family.

Theorem. *The points of the envelope of the family* $\mathbf{r}(t, \lambda)$ *are subject to*

$$\mathbf{r} = \mathbf{r}(t, \lambda) \tag{2.2.2}$$

$$\mathbf{r}'_\lambda \times \mathbf{r}'_t = 0. \tag{2.2.3}$$

Proof. If Γ is the envelope of the family (γ_λ) and P is a point of Γ, then P is a tangency point between Γ and a curve from the family, corresponding to some value of the parameter λ. Thus, the equation of Γ will be of the form

$$\mathbf{r}_1 = \mathbf{r}_1(\lambda).$$

On the other hand, P is on a curve γ_λ and, therefore, it verifies

$$\mathbf{r}_1 = \mathbf{r}(t(\lambda), \lambda).$$

The tangency condition between Γ and γ_λ reads

$$\mathbf{r}'_{1\lambda} \parallel \mathbf{r}'_t$$

or

$$\mathbf{r}'_{1\lambda} \times \mathbf{r}'_t = 0$$

or, also, since $\mathbf{r}'_{1\lambda} = \mathbf{r}'_t \cdot t'_\lambda + \mathbf{r}'_\lambda$,

$$(\mathbf{r}'_t \cdot t'_\lambda + \mathbf{r}'_\lambda) \times \mathbf{r}'_t = 0,$$

and the theorem is proven, because $\mathbf{r}'_t \times \mathbf{r}'_t = 0$. □

Remarks. 1. The set of points described by the equations (2.2.2) and (2.2.3) is called the *discriminant set* of the family γ_λ. It includes not only the support of the envelope, but, also, the singular points of the curves from the family, for which $\mathbf{r}'_t = 0$.

2. The equation $\mathbf{r}'_\lambda \times \mathbf{r}'_t = 0$ can be written also as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'_\lambda & y'_\lambda & 0 \\ x'_t & y'_t & 0 \end{vmatrix} = 0 \;\Leftrightarrow\; x'_\lambda y'_t - x'_t y'_\lambda = 0 \;\Leftrightarrow$$

$$\Leftrightarrow \frac{x'_\lambda}{x'_t} = \frac{y'_\lambda}{y'_t}. \tag{2.2.4}$$

Example. Let us consider the family of curves

$$\mathbf{r}(t, \lambda) = (\lambda + a\cos t, \lambda + a\sin t), \qquad \lambda, t \in \mathbb{R}, a > 0.$$

Clearly, we are dealing with a family of circles of radius a, with the centres on the first bisector of the coordinate axes. Then we have

$$\begin{aligned}\mathbf{r'}_\lambda &= \{1, 1\},\\ \mathbf{r'}_t &= \{-a\sin t, a\cos t\},\end{aligned}$$

therefore, the points of the envelope (and only them, since the circles have no singular points) verify

$$\begin{cases} x = \lambda + a\cos t \\ y = \lambda + a\sin t \\ x'_\lambda \cdot y'_t = x'_t \cdot y'_\lambda \end{cases}$$

or

$$\begin{cases} x = \lambda + a\cos t \\ y = \lambda + a\sin t \\ a\cos t = -a\sin t. \end{cases}$$

Eliminating t, we get the parametric equations of the envelope:

$$\begin{cases} x(\lambda) = \lambda \pm \frac{a}{\sqrt{2}} \\ y(\lambda) = \lambda \mp \frac{a}{\sqrt{2}}, \end{cases}$$

i.e. the envelope is, in fact, a pair of straight lines, parallel to the first bissector of the coordinate axes.

2.2.1 Curves given through an implicit equation

Proposition 2.2.1. *The points of the envelope of a family of plane curves given through the implicit equation*

$$F(x, y, \lambda) = 0 \tag{2.2.5}$$

verify the system of equations

$$\begin{cases} F(x, y, \lambda) = 0 \\ F'_\lambda(x, y, \lambda) = 0 \end{cases} \tag{2.2.6}$$

Proof. Locally, around each point of a the curve of the family, we can parameterize the curve, i.e. we can represent it as

$$\begin{cases} x = x(t, \lambda) \\ y = y(t, \lambda) \end{cases} .$$

By substituting into the equation of the family, we get

$$F(x(t,\lambda), y(t,\lambda), \lambda) = 0,$$

whence, differentiating with respect to t and λ, respectively, we obtain the system:

$$\begin{cases} F'_x x'_t + F'_y y'_t = 0 \\ F'_x x'_\lambda + F'_y y'_\lambda + F'_\lambda = 0 \end{cases}.$$

But, from (2.2.4),

$$x'_\lambda = K x'_t, \quad y'_\lambda = K y'_t,$$

with $K = \text{const.}$, therefore, the second equation from above becomes

$$K\underbrace{(F'_x x'_t + F'_y y'_t)}_{=0} + F'_\lambda = 0$$

or

$$F'_\lambda = 0.$$

□

Example. We consider again the family of circles from the previous paragraph, this time given through their implicit equation

$$F(x,y,\lambda) \equiv (x-\lambda)^2 + (y-\lambda)^2 - a^2 = 0.$$

Then the second equation of the discriminant set will be

$$F'_\lambda(x,y,\lambda) = -2(x+y-2\lambda) = 0,$$

whence we get

$$\lambda = \frac{x+y}{2},$$

which, when substituted into the equation of the family, gives

$$(x-y)^2 = 2a^2,$$

i.e. we get, again, the same equations of the envelope as before, namely

$$y = x \pm a\sqrt{2}.$$

2.2.2 Families of curves depending on two parameters

Proposition 2.2.2. *Suppose we are given a family of curves depending smoothly on two parameters, λ and μ*

$$F(x, y, \lambda, \mu) = 0, \tag{2.2.7}$$

where the parameters λ and μ are connected through a relation

$$\varphi(\lambda, \mu) = 0, \tag{2.2.8}$$

then the points of the envelope verify the system

$$\begin{cases} F(x, y, \lambda, \mu) = 0 \\ \varphi(\lambda, \mu) = 0 \\ \dfrac{D(F, \varphi)}{D(\lambda, \mu)} = 0 \end{cases} . \tag{2.2.9}$$

Proof. From the equation

$$\varphi(\lambda, \mu) = 0,$$

we can assume, if, for instance, that

$$\mu = \mu(\lambda),$$

therefore, substituting in F, and φ,

$$\begin{cases} F(x, y, \lambda, \mu(\lambda)) = 0, \\ \varphi(\lambda, \mu(\lambda)) = 0. \end{cases}$$

Differentiating with respect to λ these two equations, we get:

$$\begin{cases} F'_\lambda + F'_\mu \mu'_\lambda = 0 \\ \varphi'_\lambda + \varphi'_\mu \mu'_\lambda = 0. \end{cases}$$

Eliminating the derivative μ'_λ between the two equations, we get the third equation from (2.2.9), as required. □

2.2.3 Applications: the evolute of a plane curve

Definition. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve. The *evolute* of $\mathbf{r}$ is, by definition, the envelope of the family of normals of the curve.

The following result holds:

Proposition 2.2.3. *The parametric equations of the evolute of the curve* $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t))$ *are*

$$\begin{cases} X = x - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'} \\ Y = y + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'} \end{cases} \tag{2.2.10}$$

Proof. As we know, the equation of the normal of a plane curve is

$$F(X, Y, t) = (X - x(t)) \cdot x'(t) + (Y - y(t)) \cdot y'(t) = 0.$$

The relations verified by the points of the envelope of the family of normals (and *only* by them, since in this case the curves of the family are straight line and they have no singular points) are (see (2.2.6)):

$$\begin{cases} F(X, Y, t) = 0 \\ F'_t(X, Y, t) = 0 \end{cases},$$

i.e.

$$\begin{cases} x'(t)X + y'(t)Y = x(t) \cdot x'(t) + y(t) \cdot y'(t) \\ x''(t)X + y''(t)Y = x'^2(t) + x''(t) \cdot x(t) + y'^2(t) + y(t) \cdot y''(t) \end{cases}.$$

The equations (2.2.10) follow now instantly, after solving this system of linear equations with respect to X and Y. □

Example. For the ellipse

$$\begin{cases} x = a \cos t \\ y = b \sin t \end{cases}$$

one gets, after the computations,

$$\begin{cases} X = \frac{a^2 - b^2}{a} \cos^3 t \\ Y = \frac{b^2 - a^2}{b} \sin^3 t \end{cases}$$

or, eliminating the parameter t,

$$a^{\frac{2}{3}} X^{\frac{2}{3}} + b^{\frac{2}{3}} Y^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

The curved described by this equation is called a *lengthened astroid* (see figure 2.2.3).

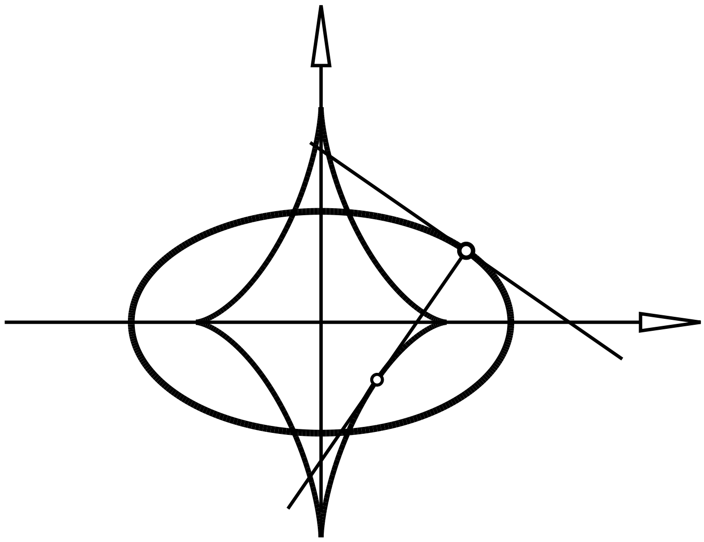


Figure 2.1: The evolute of an ellipse

2.3 The curvature of a plane curve

As we saw, in the case of an arbitrary space curve, the curvature is always a positive scalar. Of course, this concept of curvature can be equally applied for plane curves. It turns out, however that, in this particular case, we can obtain more information about the curve if we use a slightly different concept, allowing the curvature to have a sign. To define the curvature of a plane curve we shall use a little technical trick, which will allow us to give the definition in a coordinate independent way.

Definition. The *complex structure* on $\mathbb{R}^2$ is the map $J : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$J(x, y) = (-y, x).$$

Remark. Applying J simply means to rotate the vector $\{x, y\}$ by $\frac{\pi}{2}$ or multiplying the complex number $x + iy$ by the complex unit i (this is, actually, the origin of the name).

Some obvious properties of the complex structure are collected into the following proposition:

Proposition 2.3.1. *a)* $J\mathbf{v} \cdot J\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$.

b) $(J\mathbf{v}) \cdot \mathbf{v} = 0$.

c) $J(J\mathbf{v}) = -\mathbf{v}$ *(i.e.* $J^2 = -id$*).*

All these properties follow immediately from the geometrical interpretation of the complex structure.

Anticipating a little bit, we would like to say a few words about the kind of curvature we are going to define. We remember that the curvature of an arbitrary parameterized space curve $\mathbf{r} = \mathbf{r}(t)$ can be computed by

$$k(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Now, if $\mathbf{r}$ is a *plane* curve, with the support lying into the coordinate plane xOy, then the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are situated, also, into this plane. Therefore, their vector product is a vector which is directed along the z axis, and, therefore, the norm of this vector is just the absolute value of the z-component. Now, the idea of the definition of the signed curvature is just to replace this absolute value with the component itself. To do that, the following characterization of the vector product of two vectors in the plane will prove very useful.

Proposition 2.3.2. *Let* $\mathbf{u}(x_1, y_1), \mathbf{v}(x_2, y_2) \in \mathbb{R}^2$. *Then*

$$\mathbf{u} \times \mathbf{v} = [\mathbf{v} \cdot J\mathbf{u}] \cdot \mathbf{k}.$$

Proof. As it is known, the cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ can be computed by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = (x_1 y_2 - x_2 y_1) \cdot \mathbf{k}.$$

On the other hand,

$$[\mathbf{v} \cdot J\mathbf{u}] = \{x_2, y_2\} \cdot \{-y_1, x_1\} = -x_2 y_1 + x_1 y_2,$$

whence the equality announced. □

We are now ready to define the curvature of a plane curve.

Definition. Let $\mathbf{r} = \mathbf{r}(t)$ be a plane parameterized curve. The *signed curvature* of $\mathbf{r}$ is, by definition, the quantity

$$k_{\pm} = \frac{\mathbf{r}'' \cdot J\mathbf{r}'}{\|\mathbf{r}'\|^3}. \tag{2.3.1}$$

Remark. According to the proposition 2.3.2, the signed curvature is just the projection of the curvature vector on the z-axis. Now, as the curvature vector is parallel to the z-axis, we have, thus

$$|k_\pm| = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = k.$$

Another immediate, but important, for computational reasons, result, is the following:

Proposition 2.3.3. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve. If $\mathbf{r}(t) = (x(t), y(t))$, then the signed curvature of $\mathbf{r}$ can be expressed as*

$$k_\pm(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left(x'^2(t) + y'^2(t)\right)^{3/2}}.$$

Consequence. *If $y = f(x)$ is the explicit equation of a plane curve, then its signed curvature is given by*

$$k_\pm(x) = \frac{f''(x)}{(1 + f'^2)^{3/2}}.$$

Remark. The previous consequence show that, for an explicitly given curve (i.e. the graph of a real function of a single real variable) the sign of the signed curvature is, in fact, the sign of the second derivative of the function f, i.e., as it known from calculus, the signed of the signed curvature is a n indication of the convexity or concavity of the function.

Exactly as happens for the torsion of the space curves, the signed curvature of a plane curve is "almost" invariant at a parameter change, i.e. we have

Theorem. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ an equivalent parameterized curve, with the parameter change $\lambda : I \to J$, $u = \lambda(t)$. Then*

$$k_\pm[\boldsymbol{\rho}](u) = \operatorname{sgn}(\lambda') \cdot k_\pm[\mathbf{r}](t).$$

Proof. We have

$$\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)) \\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t) \\
\mathbf{r}'' \cdot J\boldsymbol{\rho}' &= (\boldsymbol{\rho}''\lambda'^2 + \boldsymbol{\rho}'\lambda'' J(\lambda'\boldsymbol{\rho}') = \\
&= \lambda'^3 \boldsymbol{\rho}'' \cdot J\boldsymbol{\rho}' + \lambda'\lambda'' \underbrace{\boldsymbol{\rho}' \cdot J\boldsymbol{\rho}'} = 0 = \lambda'^3 \boldsymbol{\rho}'' \cdot J\boldsymbol{\rho}' \\
\|\mathbf{r}'\|^3 &= |\lambda'^3| \|\boldsymbol{\rho}'\|^3
\end{aligned}$$

therefore

$$k_{\pm}[\mathbf{r}](t) = \frac{\mathbf{r}'' \cdot J\mathbf{r}'}{\|\mathbf{r}'\|^3} = \frac{\lambda'^3}{|\lambda'^3|} \cdot \frac{\boldsymbol{\rho}''(u) \cdot J\boldsymbol{\rho}'(u)}{\|\boldsymbol{\rho}'(u)\|^3} = \operatorname{sgn}(\lambda') \cdot k_{\pm}[\boldsymbol{\rho}](u),$$

whence

$$k_{\pm}[\boldsymbol{\rho}](u) = \operatorname{sgn}(\lambda') \cdot k_{\pm}[\mathbf{r}](t).$$

□

Remark. The previous theorem shows that the signed curvature is invariant under any *positive* parameter change, therefore it makes sense to define it also for regular *oriented* plane curves.

The curvature vector of a plane, naturally parameterized curve can be expressed easily as a function of the signed curvature:

Lemma. *Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a plane naturally parameterized curve. Then*

$$\mathbf{r}''(s) = k_{\pm}(s) \cdot J\mathbf{r}'(s).$$

Proof. We have $\mathbf{r}'^2(s) = 1$ (the curve is *naturally* parameterized), therefore $\mathbf{r}' \cdot \mathbf{r}'' = 0$, whence it follows that $\mathbf{r}'' \perp \mathbf{r}'$ or, which is the same, $\mathbf{r}'' \parallel J\mathbf{r}'$.

On the other hand, from the definition of the signed curvature, $k_{\pm}(s) = \mathbf{r}''(s) \cdot J\mathbf{r}'(s)$. If we put $\mathbf{r}''(s) = \alpha(s) \cdot J\mathbf{r}'(s)$, then we should have $\mathbf{r}''(s) \cdot J\mathbf{r}'(s) = \alpha(s) \cdot [J\mathbf{r}'(s)]^2 = \alpha(s)$, whence $\alpha(s) = k_{\pm}(s)$. □

2.3.1 The geometrical interpretation of the signed curvature

For the signed curvature of plane curve we have a geometrical interpretation which is similar to the geometrical interpretation of the curvature of a space curve, except that, this time the sign is also taken into account. We will need first the following definition.

Definition. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve. The *rotation angle* of $\mathbf{r}$ is the function $\theta[\mathbf{r}] : I \to \mathbb{R}$, defined through:

$$\boldsymbol{\tau}(t) = \{\cos\theta[\mathbf{r}](t), \sin\theta[\mathbf{r}](t)\} = \exp(i\theta[\mathbf{r}](t)), \tag{2.3.2}$$

where $\boldsymbol{\tau}(t)$ is the unit tangent, i.e. $\theta[\mathbf{r}]$ is the angle made by the unit tangent with the positive direction of the x axis.

Remark. This definition is very innocent looking and natural. After all, is should be clear for anyone that θ is just the angle made by the unit tangent vector with the positive direction of the x-axis. In reality, however, for an arbitrary plane curve, it is by no mean obvious that we can find a *continuous* angle function, let alone a smooth one. Such functions *do* exist (see, for instance, for a modern proof, the book of Bär, [1]) and any two such functions differ by an integer multiple of 2π.

The following lemma provides the connection between the signed curvature and the variation of the rotation angle. The contents of this lemma is quite similar to the geometrical interpretation of the curvature of a space curve. In fact, when the plane curve is regarded as a particular case of space curve, then the *variation* of the rotation angle is equal (as absolute value) with the variation of the contingency angle, therefore, in fact, the geometrical interpretation of the *absolute* curvature of a plane curve (regarded as a curve in space) is an immediate consequence of this lemma.

Lemma. *If* $(I, \mathbf{r} = \mathbf{r}(t))$ *is a plane regular parameterized curve,* θ *is its rotation angle and* $k_{\pm}$ *– its signed curvature, then:*

$$\frac{d\theta}{dt} = \|\mathbf{r}'(t)\| k_{\pm}(t).$$

Proof. From the definition of the unit tangent vector we have $\boldsymbol{\tau}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, whence

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} + \mathbf{r}'(t)\frac{d}{dt}\left(\frac{1}{\|\mathbf{r}'(t)\|}\right).$$

On the other hand, if we use the expression of $\boldsymbol{\tau}(t)$ as a function of the angle θ, given by (2.3.2), we get for $\frac{d\boldsymbol{\tau}}{dt}$ the formula

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{d\theta}{dt}\{-\sin\theta(t), \cos\theta(t)\} = \frac{d\theta}{dt}J\boldsymbol{\tau}(t).$$

Combining the two relations, we get the equality

$$\frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} + \mathbf{r}'(t)\frac{d}{dt}\left(\frac{1}{\|\mathbf{r}'(t)\|}\right) = \frac{d\theta}{dt}J\boldsymbol{\tau}(t) \equiv \frac{d\theta}{dt}\cdot\frac{J\mathbf{r}''(t)}{\|J\mathbf{r}'(t)\|}.$$

Multiplying both sides by $J\mathbf{r}'(t)$ and having in mind that $J\mathbf{r}'(t)\cdot\mathbf{r}'(t) = 0$ and $J\mathbf{r}'(t)\cdot J\mathbf{r}'(t) = \|\mathbf{r}'(t)\|^2$, we obtain:

$$\frac{\mathbf{r}''(t)\cdot J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{d\theta}{dt}\cdot\|\mathbf{r}'(t)\|,$$

whence, using the definition of the signed curvature, it follows that

$$\frac{d\theta}{dt} \cdot \|\mathbf{r}'(t)\| = k_{\pm}(t) \cdot \|\mathbf{r}'(t)\|^2$$

or, after simplification,

$$\frac{d\theta}{dt} = k_{\pm}(t) \cdot \|\mathbf{r}'(t)\|,$$

which is exactly what we had to prove. □

Corollary. *For a naturally parameterized curve,* $(I, \mathbf{r} = \mathbf{r}(s))$*, we have*

$$k_{\pm}(s) = \frac{d\theta}{ds}.$$

Remark. From the previous formula we get

$$k \equiv \|k_{\pm}\| = \left|\frac{d\theta}{ds}\right|,$$

which is exactly the formula for the curvature of arbitrary space curves, which, thus, remains valid, as expected, for the particular case of plane curves.

2.4 The curvature center. The evolute and the involute of a plane curve

Definition. A point $\Omega \in \mathbb{R}^2$ is called the *curvature center* at $\mathbf{r}_0 = \mathbf{r}(t_0)$ of a plane parameterized curve $\mathbf{r} : I \to \mathbb{R}^2$ if there is a circle (γ), centred at Ω, which is tangent to the curve at $\mathbf{r}_0 = \mathbf{r}(t_0)$, with $t_0 \in I$, such that the signed curvatures $\mathbf{r}$ and γ at $\mathbf{r}_0$ coincide, whence the position of the point Ω for an arbitrary $t \in I$:

$$\Omega(t) = \mathbf{r}(t) + \frac{1}{k_{\pm}(t)} \frac{J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Remark. The notion of curvature center is invariant at a parameter change: if $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ is equivalent to $\mathbf{r}$, with the parameter change $\lambda : I \to J$, then $\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t))\lambda'(t)$ and $k_{\pm}[\mathbf{r}](t) = \operatorname{sgn}(\lambda')k_{\pm}[\boldsymbol{\rho}](\lambda(t))$. Obviously, we could have problems only when $\lambda' < 0$, but in this case $J\mathbf{r}'$ changes the sense and $k_{\pm}$ changes the sign and, overall, the situation remains unchanged.

We defined the evolute of a plane curve as being the envelope of the normals of the curve. The following proposition provides a different approach.

Proposition 2.4.1. *The evolute of a plane curve is the geometrical locus of the curvature centers of the curve.*

Proof. The curvature center of the curve at an arbitrary value of the parameter is

$$\Omega(t) = \mathbf{r}(t) + \frac{\|\mathbf{r}'(t)\|^2}{\mathbf{r}''(t) \cdot J\mathbf{r}'(t)} \cdot J\mathbf{r}'(t) = (x(t), y(t)) + \frac{x'^2 + y'^2}{x'y'' - x''y'}\{-y', x'\}.$$

Thus, if $\Omega(t) = (X(t), Y(t))$, projecting the previous equation on the coordinate axes, we get the parametric equations of the locus of curvature centers:

$$\begin{cases} X(t) = x(t) - \frac{y'(x'^2+y'^2)}{x'y''-x''y'} \\ Y(t) = y(t) + \frac{x'(x'^2+y'^2)}{x'y''-x''y'} \end{cases},$$

which are exactly the equations of the evolute. □

From the previous remark, we get immediately:

Corollary. *The definition of the evolute makes sense also for regular curves (i.e. two equivalent parameterized curves have the same evolute).*

Exercise 2.4.1. Find the evolute of the astroid

$$\begin{cases} x(t) = a\cos^3 t, \\ y = a\sin^3 t \end{cases}.$$

Show that the evolute is also an astroid (see the figure **??**.

Exercise 2.4.2. Find the evolute of the cycloid

$$\begin{cases} x(t) = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}.$$

Show that the evolute is also an cycloid (see the figure **??**.

Another interesting plane curve associated to a given one is the so-called *involute*, which,a s we shall see immediately, is, in a sense, the inverse of the evolute.

Definition. Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parameterized curve $c \in I$. The *involute* of $\mathbf{r}$ with the origin at $\mathbf{r}(c)$ (or at c, for short) is the parameterized curve $(I, \boldsymbol{\rho}[\mathbf{r}, c] = \boldsymbol{\rho}[\mathbf{r}, c](s))$ where

$$\boldsymbol{\rho}[\mathbf{r}, c](s) = \mathbf{r}(s) + (c - s)\mathbf{r}'(s).$$

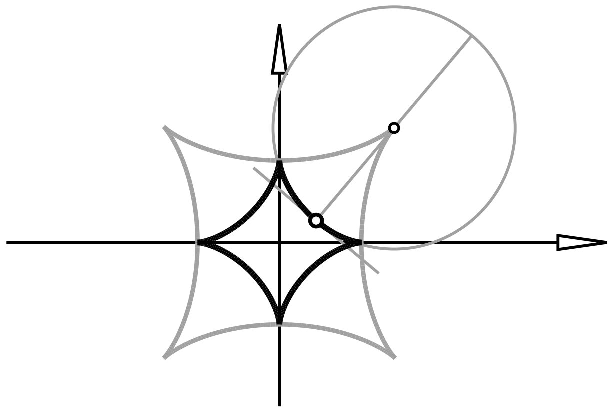


Figure 2.2: The evolute of an astroid

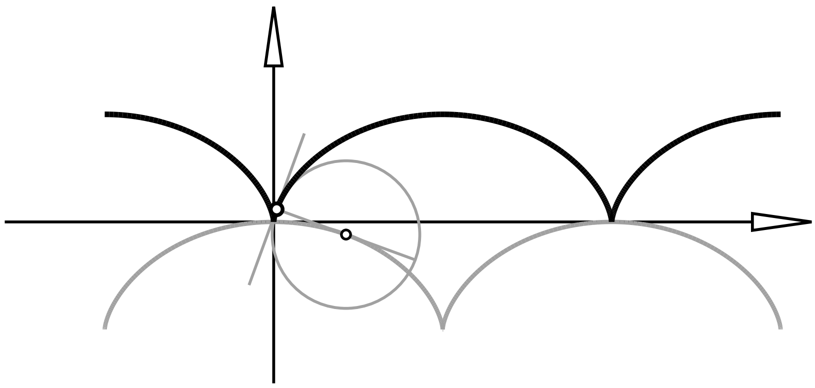


Figure 2.3: The evolute of an cycloid

Remark. Generally speaking, s is *not* a natural parameter along $\boldsymbol{\rho}$.

If $(I, \mathbf{r} = \mathbf{r}(t))$ is an arbitrary parameterized curve, the we can replace the parameter t by the arc length $s = \int\limits_0^t \|\mathbf{r}'(\tau)\| d\tau$ and define the involute of $\mathbf{r}$ as being the involute of the naturally parameterized curve equivalent to it, the natural parameter being the arc length. Is is easy to see that the following proposition holds true:

Proposition 2.4.2. *Let* $(I, \mathbf{r} = \mathbf{r}(t))$ *be a parameterized curve. The the involute of* $\mathbf{r}$ *with* *the origin at $c \in I$ is given by*

$$\boldsymbol{\rho}(t) = \mathbf{r}(t) + (c - s(t))\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

where $s = s(t)$ is the arc length of $\mathbf{r}$.

Example. Let $\mathbf{r}(t) = (a\cos t, a\sin t)$ be a circle. Then

$$\begin{cases} \mathbf{r}'(t) = \{-a\sin t, a\cos t\} \\ x'^2 + y'^2 = a^2 \\ s(t) = \int\limits_0^t a dt = at, \end{cases}$$

hence the equation of the involute is

$$\boldsymbol{\rho}(t) = (a\cos t, a\sin t) + \frac{(c - at)}{a}\{-a\sin t, a\cos t\}$$

or, in projection,

$$\begin{cases} X(t) = a\cos t - (c - at)\sin t \\ Y(t) = a\sin t + (c - at)\cos t \end{cases}.$$

We represented in the figure 2.4 an involute of a circle of radius 1.5, with the origin at the point of parameter 0.

The following lemma establishes the connection between the signed curvature of a parameterized curve and that of one of its involutes and will serve as a tool to establish the relation between evolute and involute.

Lemma. *Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parameterized curve and $\boldsymbol{\rho}$ the involute of* $\mathbf{r}$ *with the origin at $c \in I$. Then the signed curvature of $\boldsymbol{\rho}$ is given by*

$$k_\pm[\boldsymbol{\rho}](s) = \frac{\operatorname{sgn}(k_\pm[\mathbf{r}](s))}{|c - s|}.$$

Proof. We have

$$\begin{aligned} \boldsymbol{\rho}'(s) &= \mathbf{r}'(s) + (c - s)\mathbf{r}''(s) - \mathbf{r}'(s) = (c - s)\mathbf{r}''(s) = (c - s)k_\pm[\mathbf{r}](s) \cdot J\mathbf{r}'(s) \\ \boldsymbol{\rho}''(s) &= -k_\pm[\mathbf{r}](s.J\mathbf{r}'(s) + (c - s)(k_\pm[\mathbf{r}](s))' \cdot J\mathbf{r}'(s) + (c - s)k_\pm[\mathbf{r}](s) \cdot J\mathbf{r}''(s) \\ &= [-k_\pm[\mathbf{r}](s) + (c - s)(k_\pm[\mathbf{r}](s))'] \cdot J\mathbf{r}'(s) - (c - s)(k_\pm[\mathbf{r}](s))^2 \cdot \mathbf{r}'(s), \end{aligned}$$

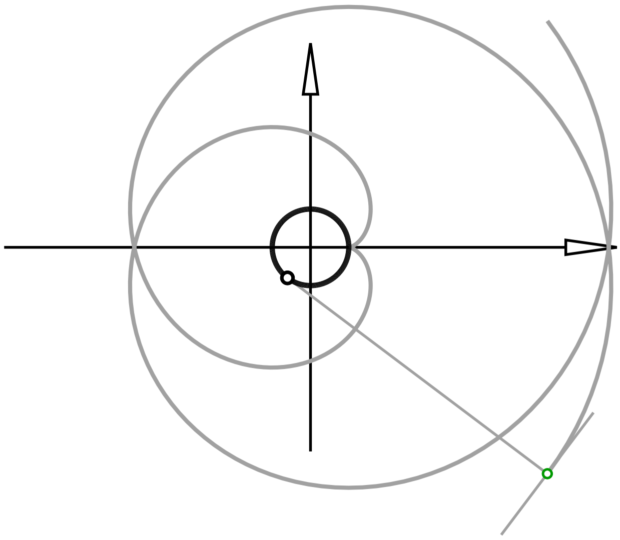


Figure 2.4: An involute of a circle

whence

$$J\boldsymbol{\rho}' = -(c-s)k_{\pm}[\mathbf{r}](s)\cdot\mathbf{r}'(s),$$

while

$$\boldsymbol{\rho}''(s)\cdot J\boldsymbol{\rho}'(s) = (c-s)^2\cdot(k_{\pm}[\mathbf{r}](s))^3.$$

The conclusion follows now from the definition of the signed curvature. □

The following theorem provides a connection between the involute and the evolute. In many textbooks this connection is taken, in fact, as the definition of the involute.

Theorem. *Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parameterized curve and $\boldsymbol{\rho}$ – its involute with the origin at $c \in I$. The the evolute of $\boldsymbol{\rho}$ is $\mathbf{r}$.*

Proof. The evolute of $\boldsymbol{\rho}$ is given, as known, by the equation

$$\boldsymbol{\rho}_1(s) = \boldsymbol{\rho}(s) + \frac{1}{k_{\pm}[\boldsymbol{\rho}](s)}\cdot\frac{J\boldsymbol{\rho}'(s)}{\|\boldsymbol{\rho}'(s)\|}.$$

Using the previous lemma to express the signed curvature of $\boldsymbol{\rho}$ as a function of the signed curvature of $\mathbf{r}$, we get

$$\boldsymbol{\rho}_1(s) = \mathbf{r}(s) + (c-s)\mathbf{r}'(s) + \frac{|c-s|}{\operatorname{sgn}(k_{\pm}[\mathbf{r}](s))}\cdot\frac{(c-s)k_{\pm}[\mathbf{r}](s)\cdot J^2\mathbf{r}'(s)}{\|(c-s)k_{\pm}[\mathbf{r}](s)\cdot J\mathbf{r}'(s)\|} = \mathbf{r}(s).$$

□

2.5 The osculating circle of a curve

Definition. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parameterized curve. The *osculating circle* of $\mathbf{r}$ at a point $t \in I$ is the circle centered at the curvature center $\Omega(t)$, with the radius equal to the curvature radius $\frac{1}{k(t)}$ of the curve at that point.

Exactly as the osculating plane at a point of a space curve can be regarded as a limit position of a plane determined by three neighboring points, when these points approach indefinitely the given one, the osculating circle is the limit position of a *circle* determined by three neighboring points, when these points approach indefinitely the given one. More precisely, we have:

Theorem. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve and $t_1 < t_2 < t_3 \in I$ Let $C(t_1, t_2, t_3)$ be the circle passing through $\mathbf{r}(t_1)$, $\mathbf{r}(t)2)$, $\mathbf{r}(t_3)$. We assume that, for a value $t \in I$ of the parameter we have $k_\pm(t) \neq 0$. Then the osculating circle of $\mathbf{r}$ at the point $\mathbf{r}(t)$ is the circle*

$$C = \lim_{\substack{t_1 \to t \\ t_2 \to t \\ t_3 \to t}} C(t_1, t_2, t_3).$$

Proof. Let $A(t_1, t_2, t_3)$ be the centre of the circle $C(t_1, t_2, t_3)$ and $f : I \to \mathbb{R}$ – the function defined through $f(t) = \|\mathbf{r}(t) - A\|^2$. Then, clearly, f is smooth and we have:

$$\begin{cases} f'(t) = 2\mathbf{r}' \cdot (\mathbf{r}(t) - A) \\ f''(t) = 2\mathbf{r}''(t) \cdot (\mathbf{r}(t) - A) + 2\|\mathbf{r}'(t)\|^2 \end{cases}.$$

Since f is differentiable and $f(t_1) = f(t_2) = f(t_3)$, from the mean value theorem it follows that there exist two points $u_1, u_2 \in I$, with $t_1 < u_1 < t_2 < u_2 < t_3$ such that

$$f'(u_1) = f(u_2) = 0.$$

On the other hand, applying once more the mean value theorem, this time to the derivative f', which is also differentiable , it follows that there is a $v \in (u_1, u_2)$ such that

$$f''(v) = 0.$$

Now, if we let $t_1, t_2, t_3 \to t$, then we will have, equally, $u_1, u_2, v \to t$, therefore, at the limit, we should get:

$$\begin{cases} \mathbf{r}'(t) \cdot (\mathbf{r}(t) - A(t)) = 0 \\ \mathbf{r}''(t) \cdot (\mathbf{r}(t) - A(t)) = -\|\mathbf{r}'(t)\|^2 \end{cases}, \tag{*}$$

where

$$A(t) = \lim_{\substack{t_1 \to t \\ t_2 \to t \\ t_3 \to t}} A(t_1, t_2, t_3).$$

From (*) and the definition of the signed curvature, it follows that

$$\mathbf{r}(t) - A(t) = \frac{-1}{k_\pm(t)} \cdot \frac{J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

hence C is the osculating circle of the curve $\mathbf{r}$ at the point $\mathbf{r}(t)$. □

2.6 The existence and uniqueness theorem for plane curves

The existence and uniqueness theorem for plane parameterized curves is similar to the analogue theorem for space curves and can be proved in the same manner, therefore we shall skip the proof here.

Theorem 2.6.1. *Let $f : I \to \mathbb{R}$ be a continuous function. Then there is a regular, naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$ such that $\forall s \in I$, $k_\pm(s) = f(s)$. $\mathbf{r}$ is unique, up to a proper motion of $\mathbb{R}^2$.*

To give an example, we will find the curve $\mathbf{r}$ for the particular case when the function f is a constant α, for any real value of the parameter s.

Starting from the geometrical interpretation of the signed curvature, we get

$$\alpha = k_\pm(s) = \frac{d\theta}{ds},$$

therefore θ (the rotation angle) will be a linear function of s:

$$\theta = \alpha s + \theta_0,$$

where θ_0 is a constant. On the other hand, from the definition of the rotation angle, we obtain:

$$\boldsymbol{\tau}(s) \equiv \left\{\frac{dx}{ds}, \frac{dy}{ds}\right\} = \{\cos\theta(s), \sin\theta(s)\} = \{\cos(\alpha s + \theta_0), \sin(\alpha s + \theta_0)\}$$

whence the system of differential equations:

$$\begin{cases} \frac{dx}{ds} = \cos(\alpha s + \theta_0) \\ \frac{dy}{ds} = \sin(\alpha s + \theta_0) \end{cases}.$$

Since the equations are separated, the system can be integrated very easy and we get the solution:

$$\begin{cases} x = \frac{1}{\alpha}[\sin \alpha s \sin \theta_0 + \cos \alpha s \sin \theta_0] + x_0, \\ y = \frac{1}{\alpha}[-\cos \alpha s \cos \theta_0 + \sin \alpha s \sin \theta_0] + y_0 \end{cases}, \tag{*}$$

where x_0 and y_0 are two integration constant. The solution (*) can be written in the matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha} \sin \alpha s \sin \theta_0 + \frac{1}{\alpha} \cos \alpha s \sin \theta_0 \\ -\frac{1}{\alpha} \cos \alpha s \cos \theta_0 + \frac{1}{\alpha} \sin \alpha s \sin \theta_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

or, also,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{2} - \theta_0\right) & \sin\left(\frac{\pi}{2} - \theta_0\right) \\ -\sin\left(\frac{\pi}{2} - \theta_0\right) & \cos\left(\frac{\pi}{2} - \theta_0\right) \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha} \cos \alpha s \\ \frac{1}{\alpha} \sin \alpha s \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

which shows that any plane curve of constant signed curvature, equal to α, can be obtained from the curve

$$\begin{cases} x = \frac{1}{\alpha} \cos \alpha s \\ y = \frac{1}{\alpha} \sin \alpha s \end{cases}$$

by applying a rotation followed by a translation, i.e. a *rigid motion*. But this is the circle of radius $1/\alpha$, centered at the origin. The conclusion is that *the only plane curve of constant positive curvature α is the circle of radius $1/\alpha$.*

CHAPTER 3

The integration of the natural equations of a curve

3.1 The Riccati equation associated to the natural equations of a curve

The general theory of differential equations guarantees the existence and uniqueness (up to a rigid motion) for the Frenet equations. However, finding *analytically* a curve when the curvature and torsion are given is quite another matter. This, of course, amounts to finding an analytical solution for the Frenet equations. Although they look quite innocently, it turns out that, in the general situation, it is not possible to integrate the system.

Clearly, if we manage to solve the Frenet system, the, in particular, we can find the tangent versor and, by another quadrature, we can find the curve. Apparently, the Frenet system should be equivalent to a third order vectorial differential equation or to a system of three third order scalar equations. Nevertheless, it turns out that the system (made of none scalar equations) actually contains three identical sets of three equations, therefore it should be equivalent to a single third order scalar equations. Moreover, the three vectors of the solution are subject to the orthonormality conditions, which should allow further reductions. In fact, we shall prove this indirectly, by showing that the Frenet system is equivalent to a second order Ricatti differential equation. As it is known from the theory of ordinary differential equations, we can find the general solution of a Ricatti equation if and only if a particular solution is already available (and there is no general procedure for finding such a solution).

Notice, first of all, that the Frenet system contains three copies of the scalar system:

$$\begin{cases} X' = k(s)Y, \\ Y' = -k(s)X + \chi(s)Z \\ Z' = -\chi(s)Y \end{cases} \tag{3.1.1}$$

and the solution of the system should verify, also, the condition

$$X^2 + Y^2 + Z^2 = 1. \tag{3.1.2}$$

The idea of the method we are going to describe (and which goes back to Sophus Lie and Gaston Darboux), relies exactly on the supplementary condition (3.1.2). Namely, it was observed that this equation can be decomposed (over the complex numbers) as

$$(X + iY)(X - iY) = (1 - Z)(1 + Z).$$

We introduce now the complex functions u and v by letting

$$u = \frac{X + iY}{1 - Z}; \quad -\frac{1}{v} = \frac{X - iY}{1 - Z}. \tag{3.1.3}$$

Clearly, w and $-1/v$ are conjugated to each other. It is now possible to express X, Y, Z in terms of u and v. One can check easily that

$$X = \frac{1 - uv}{u - v}; \quad Y = i\frac{1 + uv}{u - v}; \quad Z = \frac{u + v}{u - v}. \tag{3.1.4}$$

Thus, the solution of the Frenet system amounts to the finding of the complex functions u and v. Now, an easy manipulation of the formulas show that both u and v are solution of the Riccati equation

$$\frac{dw}{ds} = -\frac{i}{2}\chi(s) - ik(s)w + \frac{i\chi(s)}{2}w^2. \tag{3.1.5}$$

This is, as we mentioned earlier, an indirect prove of the fact that the natural equations of a space curve cannot be integrated through quadratures, in the general situation.

3.2 Examples for the integration of the natural equation of a plane curve

We saw, in the previous section, that the plane curves of constant curvature are the circles. We shall give, here, other interesting examples of natural equation for plane curves that can be integrated. We remind that in the case of plane curves the problem is not the *integrability* of the natural equation, but, rather, the possibility of expressing the solution in terms of elementary functions. Usually this is not the case, and this makes even more interesting the (few) cases in which this possibility does exist. For convenience, we shall use, for the rest of this paragraph, the *curvature radius* $R = 1/k$ instead of the curvature.

The logarithmic spiral. In this case the curvature radius is given by

$$R = a \cdot s, \tag{3.2.1}$$

where a is a constant. Let α be the contingency angle. Then, as we know, we have

$$\frac{d\alpha}{ds} = \frac{1}{R(s)} = \frac{1}{as},$$

therefore, by integrating and neglecting the integration constant, we get

$$\alpha = \frac{1}{a} \ln s, \quad \text{whence} \quad s = e^{a\alpha}.$$

To get the coordinates, we have to integrate the system of differential equations

$$\frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \sin\alpha.$$

We have (neglecting, again, the integration constants)

$$x = \int \cos\alpha ds = a \int \cos\alpha e^{a\alpha} d\alpha = \frac{ae^{a\alpha}}{a^2+1}(a\cos\alpha + \sin\alpha)$$

and, analogously,

$$y = \int \sin\alpha \, ds = \frac{ae^{a\alpha}}{a^2+1}(a\sin\alpha - \cos\alpha).$$

Hence, the parametric equations of the logarithmic spiral are

$$\begin{cases} x = \dfrac{ae^{a\alpha}}{a^2+1}(a\cos\alpha + \sin\alpha) \\ y = \dfrac{ae^{a\alpha}}{a^2+1}(a\sin\alpha - \cos\alpha) \end{cases}. \tag{3.2.2}$$

The logarithmic spiral (see figure 3.1) is most conveniently described through its *polar*

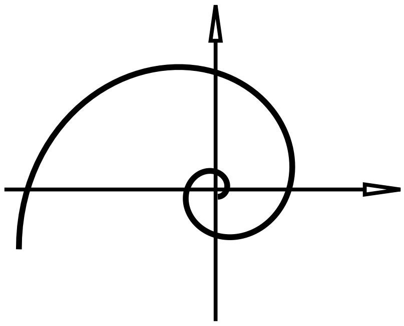


Figure 3.1: The logarithmic spiral

equation, $\mathbf{r} = \mathbf{r}(\varphi)$. We shall indicate now how can one obtain that equation starting from the parametric equations. First of all, we notice that

$$x^2 + y^2 = \frac{a^2 e^{2a\alpha}}{a^2 + 1},$$

whence

$$\mathbf{r} = \sqrt{x^2 + y^2} = \frac{ae^{a\alpha}}{\sqrt{a^2 + 1}}.$$

We put $a = \tan\psi$. Then, for the polar angle we get

$$\tan\varphi = \frac{y}{x} = \frac{a\sin\alpha - \cos\alpha}{\sin\alpha + a\cos\alpha} = -\cot(\alpha + \psi),$$

therefore $\varphi = \alpha + \psi + \frac{\pi}{2}$, hence $\alpha = \varphi - \psi - \frac{\pi}{2}$. Thus

$$\mathbf{r} = \sin\psi e^{\varphi - \psi - \frac{\pi}{2}},$$

i.e. the polar equation of the logarithmic spiral is of the form

$$r = C \cdot e^{\varphi}, \tag{3.2.3}$$

where C is a constant.

Cycloidal curves. They correspond to the natural equation

$$\frac{s^2}{a^2} + \frac{R^2}{b^2} = 1, \tag{3.2.4}$$

where a and b are nonvanishing constants. A possibility would be to express R in terms of s and then integrate. However, this would lead to complications, due to the the presence of the square root and the sign ambiguity. We prefer, instead, to introduce a new parameter t, through the relations

$$s = a \sin t, \quad R = b \cos t.$$

It is, then, very easy to find the contingency angle in terms of this new parameter. Indeed, we have

$$\alpha = \int \frac{1}{R} ds = \int \frac{1}{b \cos t} a \cos t \, dt = \frac{at}{b}.$$

We can proceed now with the determination of the coordinates x and y, in terms of the parameter t:

$$x = \int \cos \alpha \, ds = \int \cos \frac{at}{b} \cdot a \cdot \cos t \, dt = \frac{a}{2} \left(\frac{b}{a-b} \sin \frac{(a-b)t}{b} + \frac{b}{a+b} \sin \frac{(a+b)t}{b} \right),$$

$$y = \int \sin \alpha \, ds = \int \sin \frac{at}{b} \cdot a \cdot \cos t \, dt = -\frac{a}{2} \left(\frac{b}{a-b} \cos \frac{(a-b)t}{b} + \frac{b}{a+b} \cos \frac{(a+b)t}{b} \right).$$

The clothoid. This is the curve whose natural equation is

$$R = \frac{a^2}{s}. \tag{3.2.5}$$

Thus, for the clothoid (also known as the *Cornu's spiral*), the radius of curvature is proportional to the *inverse* of the arc length. From this point of view, it is, to some extent, similar to the logarithmic spiral, where the radius of curvature was proportional to the arc length, rather than to its inverse. We include this curve here to show that even in the case of a very simple expression of the curvature in terms of the arc length (in this case the curvature is proportional to the arc length, i.e. it is a linear function), we might not be able to find a parametric representation in terms of elementary functions.

The contingency angle is easily found:

$$\alpha = \frac{s^2}{2a}, \tag{3.2.6}$$

therefore the coordinates are

$$x = \int \cos \frac{s^2}{2a} ds, \quad x = \int \sin \frac{s^2}{2a} ds. \tag{3.2.7}$$

Unfortunately, it is a well known fact from analysis that the integrals from (3.2.7) cannot be expressed in terms of elementary functions. They carry the name of *Fresnel integrals*, after the French physicists who used them first in his works on optics (the theory of diffraction, to be more specific)[1].

The catenary. For the catenary, the natural equation is

$$R = a + \frac{s^2}{a}, \tag{3.2.8}$$

where a is a non-vanishing constant. Again, instead of integrating directly, to get the contingency angle, we introduce, first, the new parameter t through the relations

$$s = a \tan t, \quad R = \frac{a}{\cos^2 t}. \tag{3.2.9}$$

Then the contingency angle will be

$$\alpha \equiv \int \frac{1}{R(s)}\, ds = \int \frac{\cos^2 t \cdot a}{a \cdot \cos^2 t}\, dt = t.$$

The coordinates are, now, easy to find in terms on the new parameter:

$$x = \int \cos \alpha ds = \int \frac{a \cos t}{\cos^2 t}\, dt = a \int \frac{1}{\cos t}\, dt = \ln\left(\frac{1 + \sin t}{\cos t}\right),$$

and, analogously,

$$y = a \frac{1}{\cos t}.$$

It is not difficult to check that one can eliminate the parameter t from the previous two equation and one gets the usual explicit equation of the catenary, i.e.

$$y = a \cosh \frac{x}{a}. \tag{3.2.10}$$

[1] According to Gino Loria, in fact, these integrals were first considered by Euler, in 1781.

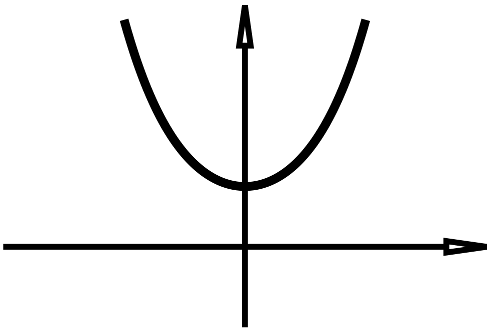


Figure 3.2: The catenary

The involute of the circle. Let us start with the natural equation

$$R^2 = 2as. \tag{3.2.11}$$

We introduce a new parameter, t, such that

$$s = \frac{at^2}{2}, \quad R = at.$$

Then the contingency angle is

$$\alpha = \int \frac{1}{R}\, ds = \int \frac{1}{at} \cdot at\, dt = t,$$

therefore

$$x = \int \cos\alpha\, ds = \int at \cos t\, dt = a(\cos t + t \sin t)$$

and

$$y = \int \sin\alpha\, ds = \int at \sin t\, dt = a(\sin t - t \cos t),$$

which are, indeed, the parametric equations of the involute of the circle.

The tractrix. Finally, we start with the natural equation

$$R^2 + a^2 = a^2 e^{-2s/a}, \tag{3.2.12}$$

where a is a non-vanishing constant. We introduce the parameter t such that

$$e^{-\frac{s}{a}} = \frac{1}{\cos t}, \quad R = a \tan t.$$

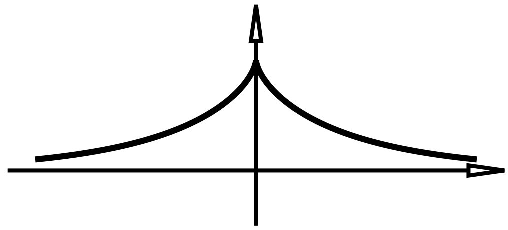


Figure 3.3: The tractrix

Then

$$\alpha = \int \frac{1}{R}\, ds = \int \frac{1}{a \tan t}(-a \tan t)\, dt = -t,$$

therefore

$$x = \int \cos \alpha \, ds = \int \cos t(-a \tan t)\, dt = a \cos t$$

and

$$y = \int \sin \alpha \, ds = \int \sin t \cdot a \cdot \tan t \, dt = a\left(\ln \frac{1 + \sin t}{\cos t} - \sin t\right).$$

Problems

1. Write the equations of the geometric locus of the points for which the product of their distances to two fixed points F_1 and F_2, with $d(F_1, F_2) = 2b$, is a constant quantity, equal to a^2 (the curve is called a *Cassini oval*).

2. We consider a circle of diameter $OA = 2a$ and its tangent at the point A. Through the point O passes a half-line OC, where C is the intersection point between the half-line and the circle. On this half-line one takes a segment OM, of length BC, where B is the intersection point between the half-line and the tangent to the circle at A. If we let the half-line OC rotate around the point O, then the point M describe a curve, called the *Diocles' cissoid*. Find the equations of this curve.

3. An arbitrary half-line OE, starting from the origin O, intersects at the points D and E the circle

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

and its tangent at C, the opposite of O on the circle. We draw, through the points D and E, two straight line, parallel to the axes Ox and Oy, respectively. The two lines intersect at M. Find the equations of the geometrical locus of M (*the witch of Maria Agnesi*).

4. A point M is moving uniformly on a straight line ON, which is rotating with constant angular velocity around the point O. Find the equations of the trajectory of M (*Archimede's spiral*).

5. A straight line OL rotates around the point O with a constant angular velocity ω. A point M moves along the line OL with a speed proportional to the distance OM. Find the equation of the curve described by the point M (*the logarithmical spiral*).

6. A straight line segment, of constant length, equal to $2a$, moves on the plane, keeping its ends, A and B, on the axes of a rectangular. coordinate system. We construct the perpendicular OM from the origin of coordinates on this ($M \in AB$). Write the equation of the geometrical locus of the point M (*the four petals rosette*).

7. On a fixed circle of radius a, one takes an invariable point O. A ray with the origin at O rotates around O and intersects the circle in a variable point A. On this ray one takes two points M_1 and M_2, symmetrical with respect to A such that $AM_1 = AM_2 = 2a$. Find the equations of the curve described by the points M_1 and M_2 (*the cardioid*).

8. We take a fixed point O on the circle $r = 2a\cos\varphi$ and a variable ray OM, with the origin at O. Let A be the point of intersection between the ray and the circle. We take on the ray two points M_1 and M_2, symmetrical with respect to A, such that $AM_1 = AM_2 = 2b$. Write down the equations of the geometrical locus described by M_1 and M_2 when the ray rotates around O (*the Pascal's limaçon*). What curve do we get for $a = b$?

9. The straight line $x = a$ cuts the Ox-axis at the point A and an arbitrary ray OB at the point B. On the ray we take two points M_1 and M_2, symmetrical with respect to B, such that $BM_1 - BM_2 = AB$. Write down the equations of the geometrical locus described by the points M_1 and M_2 when the ray rotates around the origin (*the strophoid*).

10. Through the point $E\left(a, \frac{\pi}{2}\right)$, given through its polar coordinates, one constructs a straight line parallel to the polar axis. A variable ray OK intersects this line at the point K. On the ray one takes two points M_1 and M_2, symmetrical with respect to the point K, such that $KM_1 = KM_2 = l$. Write down the equations of the geometrical locus of the points M_1 and M_2 (*the Nicomede's conchoid*).

11. The ends of a straight line segment of constant length $AB = a$ slide on the axes of a rectangular coordinate system. The straight lines AC and BC are parallel to the coordinate axes and they intersect each other at the point C. Find the equations of the geometrical locus of the foot M of the perpendicular drawn from C on the straight line AB (*the astroid*).

12. A circle of radius a rules without sliding on a straight line. Find the parametrical equations of the curve described by a point invariably linked to the circumference of the circle (*the cycloid*).

13. A circular disk of radius a rules without sliding on a straight line. Find the parametrical equations of the curve described by a point M, invariably linked to the disk, lying at a distance d from the centre of the circle (for $d < a$ the curve is called *the shortened cycloid*, while for $d > a$ — *the lengthened cycloid*).

14. A circular disk of radius r rules without sliding on a circle of radius R, at its exterior. Find the parametrical equations of a point M, invariably linked to the circumference of the mobile disk (*the epicycloid*).

15. A circular disk of radius r rules without sliding on a circle of radius R, at its interior. Find the parametrical equations of a point M, invariably linked to the circumference of the mobile disk (*the hypocycloid*).

16. Find the equation of the tangent and the equation of the normal for the following curves:

a) $\mathbf{r} = a\cos t\mathbf{i} + b\sin t\mathbf{j}$ (the ellipse);

b) $\mathbf{r} = \frac{a}{2}\left(t + \frac{1}{t}\right)\mathbf{i} + \frac{b}{2}\left(t - \frac{1}{t}\right)\mathbf{j}$ (the hyperbola);

c) $\mathbf{r} = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$ (the astroid);

d) $\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}$ (the cycloid);

e) $\mathbf{r} = a\varphi\cos\varphi\mathbf{i} + a\varphi\sin\varphi\mathbf{j}$ (the Archimede's spiral);

f) $\mathbf{r} = \dfrac{at}{1 + t^3}\mathbf{i} + \dfrac{at^2}{1 + t^3}\mathbf{j}$ (Descartes' folium);

g) $\mathbf{r} = \dfrac{a}{1 + t^2}\mathbf{i} + \dfrac{a}{t(1 + t^2)}\mathbf{j}$ (Diocles' cissoid).

17. Find the equation of the tangent and the equation of the normal to the curves:

a) $x^2(x + y) - a^2(x - y) = 0$ at the point $(0, 0)$;

b) $2x^2 - x^2y^2 - 3x + y + 7 = 0$ at the point $(1, -2)$;

18. The sides of a right angle are tangent to the astroid

$$\begin{cases} x = \frac{3}{4}R\cos\frac{t}{4} + \frac{1}{4}R\cos\frac{3t}{4} \\ y = \frac{3}{4}R\sin\frac{t}{4} - \frac{1}{4}R\sin\frac{3t}{4} \end{cases}.$$

Find the geometrical locus of the vertex of this angle when its sides occupy all the possible positions, remaining perpendicular to each other and tangent to the astroid.

19. Show that the Bernoulli's lemniscate

$$(x^2+y^2)^2-2a^2(x^2-y^2)=0$$

is the set of the symmetrical points of the centre of an equilateral hyperbola with respect to the tangents to this hyperbola.

20. At which point the tangent to the parabola $y=x^2$ makes an angle of $45°$ with the x-axis?

21. Show that the inclination angle φ of the tangent to the curve

$$y=x^5+2x^3+x-1$$

on the x-axis belongs to the interval $\frac{\pi}{4}\le\varphi\le\frac{\pi}{2}$.

22. Find the equations of the tangents to the curve

$$y=\frac{1}{1+x^2}$$

at the points where the curve intersects the hyperbola

$$y=\frac{1}{1+x}$$

23. Show that the curves

$$y=a\sin\frac{x}{a},\quad y=a\tan\frac{x}{a},\quad y=a\ln\frac{x}{a},\quad a\in\mathbb{R},$$

intersect the x-axis under the same angle, for any value of the parameter a, provided all the functions involved are defined for the respective value.

24. Find the tangents to the astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

which are the most distant from the origin of the coordinates.

25. Find the intersection angle between two coaxial parabolas if the vertex of each one coincides with the focus of the other one.

26. Show that all the normals to the curve

$$\begin{cases} x = a(\cos t + \sin t) \\ y = a(\sin t - \cos t) \end{cases}$$

lie at the same distance with respect to the origin.

27. Show that the segment from the normal to the curve

$$\begin{cases} x = 2a \sin t + a \sin t \cos^2 t \\ y = -a \cos^3 t \end{cases}$$

intercepted between the coordinate axes has constant length, equal to $2a$.

28. It is possible to draw two pairs of tangents to the Bernoulli's lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, parallel to a given direction. Show that the straight lines connecting the tangency points of each pair intercept an angle of $120°$.

29. Prove that the circles

$$\begin{aligned} x^2 + y^2 + a_1 x + b_1 y + c_1 &= 0 \\ x^2 + y^2 + a_2 x + b_2 y + c_2 &= 0 \end{aligned}$$

intersect each other under a right angle iff

$$a_1 a_2 + b_1 b_2 = 2(c_1 + c_2).$$

30. Show that if all the normals to a given curve are passing through the same point, then the curve is an arc of circle.

31. Find the equation of the curve from whose points a given parabola is seen under a constant angle, α. Which curve do we get for $\alpha = \frac{\pi}{2}$?.

32. Find the envelope of a family of straight lines passing through the extremities of a pair of conjugates diameters of an ellipse.

33. Find the envelope of a family of straight lines which determine on a right angle a a triangle of constant area.

34. Find the envelope of a family of straight lines which determine on a right angle a a triangle of constant perimeter.

35. Find the envelopes of the families of curves

a) $\mathbf{r} = (t^2 + \lambda)\,\mathbf{i} + (t^3 + \lambda)\,\mathbf{j}$;

b) $\mathbf{r} = (t^2 + \lambda)\,\mathbf{i} + t^3\,\mathbf{j}$.

36. Let O be the centre of an equilateral hyperbola M – an arbitrary point on the hyperbola. The segment OM is projected on the assymptotes of the hyperbola and on the projections, as semiaxes, one constructs ellipses. Find the envelope of the family of ellipses obtained this way.

37. There are given two concentrical circles, of radii a abd b, respectively. One constructs a family of circles, with the centres on the circle of radius b, their radii being, all of them, equal to a, and a second family of circles, this time with the centres on the circle of radius a and having the radii equal to b. Show that the two families of circles have the same envelope and find this envelope.

38. Find the envelope of the family of circles having as diameters the chords of a parabola, passing through the focus of the parabola.

39. A radius vector of an ellipse projects on two conjugate diameters of the ellipse and on the projections, as semiaxes, one builds ellipses. Find the envelope of the family of ellipses obtained this way.

40. Find the envelope of a family of circles with the centers on a given ellipse and passing through the same focus of the ellipse.

41. Find the envelope of the family of curves

$$\frac{x^2}{\lambda} + \frac{y^2}{\mu} = 1,$$

where $\lambda + \mu = 1$.

42. A vertex of a plane curve is a point of the curve at which the radius of the osculating circle is stationary. Find the vertices of the curve $y = \ln x$.

43. Find the osculating circle of minimum radius of the equilateral hyperbola $xy = 1$.

44. Find the radius of the osculating circle at an arbitrary point of the conical section

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a = 0.$$

45. Let M be an arbitrary point of the cycloidcycloid and P – the corresponding point of the generating circle. We extend the segment MP with a distance $PC = MP$. Show that C is the center of the osculating circle of the cycloid at M.

46. Compute the radius of the osculating circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at one of its vertices and give a geometrical method for the construction of this osculating circle. Consider the particular case of the equilateral hyperbola $x^2 - y^2 = a^2$.

47. We consider a curve (γ) given through the vectorial equation $\mathbf{r} = \mathbf{r}(t)$, such that the function $\mathbf{r}$ is defined and at least three times continuously differentiable on an interval $[a, b]$, while $\mathbf{r}' \times \mathbf{r}'' \neq 0$ on this interval. The vector

$$\boldsymbol{\nu}_{\mathbf{a}} = 3\mathbf{r}'' \times (\mathbf{r}' \times \mathbf{r}'') - \mathbf{r}' \times (\mathbf{r}' \times \mathbf{r}''')$$

is called the *directing vector of the affine normal* of the curve (γ) at the considered point. Show that if at a point M_0 of the curve (γ) this vector is not perpendicular on the tangent to the curve at this point, then the osculating circle of the curve at the point M_0 intersects the curve.

48. Find the directing vector of the affine normal for the curves:

a) $\mathbf{r} = a\cos t\mathbf{i} + b\sin t\mathbf{j}$ (the ellipse);

b) $\mathbf{r} = \dfrac{a}{2}\left(t + \dfrac{1}{t}\right)\mathbf{i} + \dfrac{b}{2}\left(t - \dfrac{1}{t}\right)\mathbf{j}$ (the hyperbola);

c) $y = ax^2$ (the parabola);

d) $y = \ln x$

and find the position of the osculating circle with respect to the curve in each case.

49. Show that the circle cercul $(x-a)^2 + (y-a)^2 - \frac{8a^2}{9} = 0$ and the parabola $\sqrt{3x} + \sqrt{3y} = 2\sqrt{a}$ have a third order contact at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$.

50. Find the curvature radii of the following curves:

a) $y = \sin x$ at the vertex;

b) $\begin{cases} x = (1+m)\cos mt - am\cos(1+m)t \\ y = (1+m)\sin mt - am\sin(1+m)t \end{cases}$ (the epicycloid);

c) $y = \cosh\frac{x}{a}$ (the catenary);

d) $x^2y^2 = (a^2 - y^2)(b + y)^2$ (the conchoid);

e) $\mathbf{r} = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}$ (the astroid).

51. Find the evolutes of the following curves:

a) $\mathbf{r} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j}$ (the ellipse);

b) $\mathbf{r} = x\,\mathbf{i} + ax^2\,\mathbf{j}$ (the parabola);

c) $\mathbf{r} = \dfrac{a}{2}\left(t + \dfrac{1}{t}\right)\mathbf{i} + \dfrac{b}{2}\left(t - \dfrac{1}{t}\right)\mathbf{j}$ (the hyperbola);

d) $\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}$ (the cycloid);

e) $\mathbf{r} = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$ (the astroid).

52. Find the equation of the evolute of the curve

$$\mathbb{R} = \mathbf{r} - \frac{\mathbf{r}'}{|\mathbf{r}'|}\int\limits_0^t |\mathbf{r}'|dt.$$

53. Find the involute of a circle.

54. Find the involute of the parabola $x^2 = 2py$.

55. Find the parametrical equations of the curves having the following natural equations:

a) $R = as$;

b) $\dfrac{s^2}{a^2} + \dfrac{R^2}{b^2} = 1$;

c) $RS = a^2$;

d) $R = a + \dfrac{s^2}{a}$;

56. Show that the curve

$$\begin{cases} x = \sin 2\varphi, \\ y = 1 - \cos 2\varphi, \\ z = 2\cos\varphi \end{cases}$$

lies on a sphere.

57. Consider the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = a^2$ with the cylinder $x^2 + y^2 = ay$ (*the Viviani's window*).

a) Find a set of parametrical equations of the curve.

b) Find the curvature and the torsion at an arbitrary point of the curve.

c) Find the projection of the curve on the coordinate plane yOz.

58. Consider the curve

$$\begin{cases} x = p\sqrt{p^2 - q^2}\cos t \\ y = q\sqrt{p^2 - q^2}(1 + \sin t) \\ z = \sqrt{p^2 - q^2}(1 + \sin t) \end{cases}$$

where p, q are constant, while $p > q$.

1. Show that this curve lies on the elliptical paraboloid $\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} = 2z$.

2. Show that this curve is a circle, by computing its curvature and torsion.

3. Check that the osculating plane to the curve is one of the planes of circular section in the paraboloid.

59. Find the projection of the coordinate plane xOy of the intersection curve between the paraboloid $z = x^2 + y^2$ and the plane $x + y + z = 1$.

60. Find the equation of the tangent to the curve

$$\begin{cases} x = \frac{t^4}{4}, \\ y = \frac{t^3}{3}, \\ z = \frac{t^2}{2} \end{cases}$$

parallel to the plane $x + 3y + 2z = 0$.

61. Write the equation of the tangent to the curve

$$\begin{cases} x^2 + y^2 = 10, \\ y^2 + z^2 = 25 \end{cases}$$

at the point $(1, 3, 4)$.

62. Show that the tangents to the curve

$$x = a(\sin t + \cos t),\ \ y = a(\sin t - \cos t),\ \ z = be^{-t}$$

intersects the coordinate plane xOy after the circle

$$x^2 + y^2 = 4a^2$$

63. Find the equation of the normal plane at an arbitrary point of the curve

$$\begin{cases} z = x^2 + y^2, \\ y = x. \end{cases}$$

64. Show that the normal planes to the curve

$$\begin{cases} x = a\cos t, \\ y = a\sin\alpha\sin t, \\ z = a\cos\alpha\sin t \end{cases}$$

are passing through the straight line

$$\begin{cases} x = 0, \\ z + y\tan\alpha = 0. \end{cases}$$

65. Show that $(\boldsymbol{\tau}, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \chi$.

66. Show that $(\dot{\boldsymbol{\beta}}, \ddot{\boldsymbol{\beta}}, \dddot{\boldsymbol{\beta}}) = \chi^5 \left(\dfrac{k}{\chi}\right)$.

67. Show that $(\dot{\boldsymbol{\tau}}, \ddot{\boldsymbol{\tau}}, \dddot{\boldsymbol{\tau}}) = k^5 \left(\dfrac{\chi}{k}\right)$.

68. Show that if the principal normals to a curve form a constant angle with the direction of the vector **e**, then

$$\frac{d}{ds} = \frac{k^2 + \chi^2}{k \dfrac{d}{ds} \dfrac{\chi}{k}} + \chi = 0$$

and conversely: if the previous relation is verified, then the principal normals to the curve form a constant angle with the direction of a vector. Find this vector.

69. Show that if all the osculating planes to a given curve (which is not a straight line) contain the same vector, then the curve is a plane curve.

Part II

Surfaces

CHAPTER 4

General theory of surfaces

4.1 Parameterized surfaces (patches)

Definition. A regular parameterized surface (patch) in $\mathbb{R}^3$ is a smooth map $\mathbf{r} : U \to \mathbb{R}^3$, $(u, v) \to \mathbf{r}(u, v)$, where U is a domain (an open, connected subset) in $\mathbb{R}^2$, while $\mathbf{r}$ is subject to

$$\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}} \neq 0. \tag{4.1.1}$$

The condition (4.1.1) is called the *regularity condition*.

A parameterized surface is usually denoted by $(U, \mathbf{r})$, $(U, \mathbf{r} = \mathbf{r}(u, v))$ or just $\mathbf{r} = \mathbf{r}(u, v)$.

Definition. The set $\mathbf{r}(U) \subset \mathbb{R}^3$ is called the *support* of the parameterized surface $(U, \mathbf{r})$.

Remark. Usually, one and the same point of the support of a parameterized surface $(U, \mathbf{r})$ may correspond to several distinct pairs (u, v), since the map $\mathbf{r}$ is not assumed to be injective.

Definition. Two parameterized surfaces $(U, \mathbf{r})$ and $(V, \mathbf{r}_1)$ are called *equivalent* if there is a diffeomorphism $\lambda : U \to V$ such that $\mathbf{r} = \mathbf{r}_1 \circ \lambda$.

Remark. The supports of two equivalent parameterized surfaces always coincide.

Examples. 1. If $U = \mathbb{R}^2$, while $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$, the the support of the parameterized surface is the plane passing through $\mathbf{r}_0$, perpendicular to the vector $\mathbf{a} \times \mathbf{b}$. This parameterized surface is called a *plane*.

2. Let $U = \{(u, v) \in \mathbb{R}^2 | \pi/2 < u < \pi/2, 0 < v < 2\pi\}$ and

$$\mathbf{r}(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u).$$

The support of this parameterized surface is the sphere of radius R, centred at the origin of $\mathbb{R}^3$, with a meridian removed. The parameters u and v are analogues of the geographical coordinates.

3. Let $U = \mathbb{R}^2$, $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v^3\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$. The support of this parameterized surface is identical to the support of the parameterized surface from the example 1), although the two surfaces are not equivalent, since the map $(u, v) \to (u, v^3)$ is not a diffeomorphism.

4.2 Surfaces

Definition. A subset $S \subset \mathbb{R}^3$ is called a *regular surface* if each point $a \in S$ has an open neighbourhood W in S such that there is a parameterized surface $(U, \mathbf{r})$ with $\mathbf{r}(U) = W$, while the map $\mathbf{r} : U \to W$ is a homeomorphism. The pair is called a *local parameterization* of the surface S around the point a, while the support $\mathbf{r}(U)$ is called the *domain* of the parameterization. A surface S which has a *global* parameterization (i.e. a local parameterization $(U, \mathbf{r})$ for which $\mathbf{r}(U) = S$) is called a *simple* surface.

4.2.1 Representations of surfaces

The same kind of representations we used for the case of curves are, equally, available for surfaces.

Parametrical representation. If S is a surface and $(U, \mathbf{r})$ is a local parameterization of S, then, if $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, then the equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in U,$$

are called the *parametric equations* of the surface. We would like to emphasize, once more, that these are only *local* equations, they cannot be used to describe all the points of the surface, unless we are dealing with a global parameterization of a simple surface.

Explicit representation. If $f : U \to \mathbb{R}$ is a smooth function, where $U \subset \mathbb{R}^2$ is a domain, then its graph, $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$ is a simple surface. Indeed, we have the global parameterization $\mathbf{r} : U \to \mathbb{R}^3$, $\mathbf{r}(u, v) = (u, v, f(u, v))$.

Implicit representation. Let $F : V \to \mathbb{R}$, with $V \subset \mathbb{R}^3$ an open set, be a smooth function. We denote by $S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$ the 0-level set of F. If, at each point of S, the vector

$$\operatorname{grad} F = \left\{\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\}$$

is different from zero, then S is a surface. Indeed, for instance, at $(x_0, y_0, z_0) \in S$, we have $F'_z(x_0, y_0, z_0) \neq 0$, then, from the implicit function theorem, there is an open (in the topology of $\mathbb{R}^3$) neighbourhood M of (x_0, y_0, z_0) such that the set $M \cap S$ (which is an open neighbourhood of (x_0, y_0, z_0), this time in the topology of S) is the graph of a smooth function $z = f(x, y)$, therefore, according to the previous paragraph, there is a local parameterization of S around the point (x_0, y_0, z_0). We note that this parameterization is global for $M \cap S$ but, generally, it is not so for the entire S. Even if F'_z is different from zero on the entire S, it is still not sure that the function f can be defined globally. Thus, in general, unlike that case of the surfaces given explicitly, the implicitly given surfaces are *not* simple.

Examples. 1. The plane Π passing through $\mathbf{r}_0$ and having the direction given by the vectors $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a} \times \mathbf{b} \neq 0$ is a simple surface with the global parameterization $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$. In projection on the coordinate axes, the parametric equations of the plane will be

$$\begin{cases} x = x_0 + ua_x + vb_x \\ y = y_0 + ua_y + vb_y \\ z = z_0 + ua_z + vb_z \end{cases}, \quad (u, v) \in \mathbb{R}^2.$$

2. *Revolution surfaces*
Let C be a curve in the plane xOz, which does not intersect the z-axis, and S – the subset of $\mathbb{R}^3$ obtained by rotating C around the z-axis. Let v be the rotation angle of the plane xOz and a' – the point of S obtained by rotating the point $a \in C$ by an angle v_0. Let $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$ be a local parameterization of the curve C around the point a, $\boldsymbol{\rho}(t) = (x(t), z(t))$. The we get the local parameterization os S around a',

$$\mathbf{r}(t, v) = (x(t)\cos(v + v_0), x(t)\sin(v + v_0), z(t)),$$

defined on the domain $U = I \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3. *The sphere.* Let $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x, y, z) = x^2 + y^2 + z^2 - R^2$. F is, obviously, a smooth function and its 0-level set

$$S^2_R = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$$

is the sphere of radius R, with the centre at the origin. The gradient of F is

$$\operatorname{grad} F = \{2x, 2y, 2z\}$$

and, obviously, it does not vanish on the sphere S^2_R, therefore this sphere is a regular surface. Note that S^2_R is not simple since it is compact subset of $\mathbb{R}^3$, therefore it cannot be homeomorphic to an open subset of $\mathbb{R}^2$, which is not compact.

4. *The torus.* We choose now $F : \mathbb{R}^3 \to \mathbb{R}$,

$$F(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2 - b^2, \quad 0 < b < a.$$

Its 0-level set,

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 | F(x, y, z) = 0\}$$

is called the *2-dimensional torus*. If we compute the derivatives of F with respect to the coordinates, we get

$$\begin{cases} F'_x = 2\left(\sqrt{x^2 + y^2} - a\right) \frac{x}{\sqrt{x^2+y^2}} \\ F'_y = 2\left(\sqrt{x^2 + y^2} - a\right) \frac{y}{\sqrt{x^2+y^2}} \\ F'_z = 2z \end{cases},$$

therefore the gradient of F vanishes iff

$$\begin{cases} x = y = z = 0 \quad \text{or} \\ x^2 + y^2 = 0, y = 0, z = 0 \quad \text{or} \\ x = 0, x^2 + y^2 = a^2, z = 0 \quad \text{or} \\ x^2 + y^2 = a^2, z = 0 \end{cases}.$$

It is easy to check that $\operatorname{grad} F$ is different from zero on T^2, therefore the torus is a surface (again, it is compact, therefore it cannot be simple).

The torus can also be obtained by rotating the circle $(x - a)^2 + z^2 = b^2$ (lying in the plane xOz) around the axis Oz.

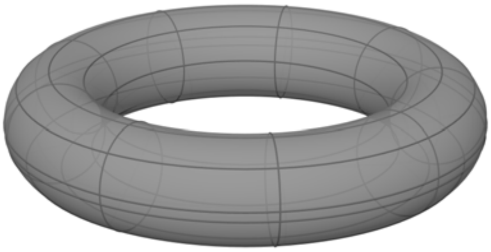


Figure 4.1: The torus

4.3 The equivalence of local parameterizations

Definition. Let S be a surface, $(U, \mathbf{r})$ – a local parameterization and $W = \mathbf{r}(U)$. Then the map $\mathbf{r}^{-1} : W \to U$ is a bijection, associating to each point of W a pair of real numbers $(u, v) \in U$. This correspondence is called a *curvilinear coordinate system* on S or a *chart* on S.

Remark. We ought to mention here that in many books the fundamental objects used to describe a surface are not the local parameterizations, but the charts. Of course, the two approaches are completely equivalent, but they came from different directions. The description of surfaces by using local parameterizations has, probably, the origin in mathematical analysis, where the surfaces are thought off either as images of functions or as graphs of functions, in any case, the central objects is *the function*. The "charts's approach" has the origin in chartography. In fact, assigning a local chart to a surface simply means to apply that surface on a portion of a plane, in other words, what is meant is the construction of a "map" of the surface and, actually, in the advanced differential geometry, a collection of charts that cover the entire surface is called an *atlas*, exactly as happens in chartography.

Theorem 4.3.1. *(of the parameters' change) Let* $(U, \mathbf{r})$ *and* $(U_1, \mathbf{r}_1)$ *be two local parameterizations of a surface* S *and* $\mathbf{r}(U) = \mathbf{r}(U_1)$. *Then there is a diffeomorphism* $\lambda : U \to U_1$ *such that* $\mathbf{r} = \mathbf{r}_1 \circ \lambda$. *The diffeomorphism* λ *is called a* parameters' change.

Before proving the theorem, some remarks are in order. If a change of parameters λ does exist, then, from the relation $\mathbf{r} = \mathbf{r}_1 \circ \lambda$, we should have, of course, $\lambda = \mathbf{r}_1^{-1} \circ \mathbf{r}$. The real difficulty is to show that λ and its inverse are smooth. Although $\mathbf{r}$ is smooth, $\mathbf{r}^{-1}$ is not[1], because its domain, $\mathbf{r}_1(U_1)$, is not an open subset of the Euclidean space $\mathbb{R}^3$.

[1]not in the classical sense, at least

We will show, however, in the following lemma, that, locally, $\mathbf{r}_1^{-1}$ is the *restriction* of a smooth function. We ought to emphasize, however, that this representation of $\mathbf{r}_1^{-1}$ is only local: in general, $\mathbf{r}_1^{-1}$ cannot be written, globally, as the restriction of a single smooth function, defined on an open set (with respect to the topology of $\mathbb{R}^3$), which contains the set $\mathbf{r}_1(U_1)$.

Lemma. *Let* $(U, \mathbf{r})$ *be a local parameterization of the surface* S*,* $\mathbf{r}(U) = W$ *and* $\mathbf{r}^{-1} : W \to U$ *– the inverse map. Then, for each point* $a \in W$ *there is an open set (in the topology of* $\mathbb{R}^3$*)* $B \ni a$ *and a smooth map* $G : B \to U$ *such that* $\mathbf{r}^{-1}|_{W\cap B} = G|_{W\cap B}$.

Proof of the lemma. Let $\mathbf{r}(u, v) = (f_1(u, v), f_2(u, v).f_3(u, v))$ and $a = \mathbf{r}(u_0, v_0)$. Due to the regularity of $\mathbf{r}$, the Jacobi matrix

$$\begin{pmatrix} f'_{1u} & f'_{1v} \\ f'_{2u} & f'_{2v} \\ f'_{3u} & f'_{3v} \end{pmatrix}$$

has the rank two. Without restricting the generality, we may assume that

$$\begin{vmatrix} f'_{1u} & f'_{1v} \\ f'_{2u} & f'_{2v} \end{vmatrix} \neq 0.$$

Then, from the inverse function's theorem for the map

$$f : (u, v) \longrightarrow (x = f_1(u, v), y = f_2(u, v)),$$

there is an open neighbourhood V of the point (u_0, v_0) in U and an open neighbourhood $\widetilde{V}$ of the point $(x_0 = f_1(u_0, v_0), y_0 = f_2(u_0, v_0))$ in the plane xOy such that $f : V \to \widetilde{V}$ is a diffeomorphism. Since the map $\mathbf{r} : U \to W$ is a homeomorphism, $\mathbf{r}(V)$ is an open neighbourhood in S of the point $a = \mathbf{r}(u_0, v_0)$, therefore, in $\mathbb{R}^3$ there is an open neighbourhood B of the point a such that $\mathbf{r}(V) = B \cap S = B \cap W$. Let $p : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, y)$ be the orthogonal projection on the coordinate plane xOy. We are going to show that the map $G = (f^{-1} \circ p)|_B : B \to U$ is the map searched for. Indeed, G is smooth, as a composition of smooth maps. Moreover, to each point (x, y, z) from $B \cap W$ corresponds a single point $(u, v) = \mathbf{r}^{-1}(x, y, z)$ from V, and to each point $(x, y) \in \widetilde{V}$ – the point $(u, v) = f^{-1}(x, y)$ from V. Thus, for the points $(x, y, z) \in B \cap W$, we have

$$\mathbf{r}^{-1}(x, y, z) = f^{-1}(x, y) = f^{-1}(p(x, y, z)) = G(x, y, z).$$

$\square$

Proof of the theorem. Let $(U, \mathbf{r})$ and $(U_1, \mathbf{r}_1)$ – two local parameterizations of the surface S such that $\mathbf{r}(U) = \mathbf{r}_1(U_1) = W$. We consider the map $\lambda = \mathbf{r}_1^{-1} \circ \mathbf{r} : U \to U_1$. Since $\mathbf{r}_1 : U_1 \to W$ is a homeomorphism, the same is true for $\mathbf{r}_1^{-1} : W \to U_1$ and, thus, λ is a homeomorphism, as a composition of two homeomorphisms. We have to prove only that λ and λ^{-1} are smooth. To prove the smoothness of λ este it is enough to prove that each point $(u_0, v_0) \in U$ has an open neighbourhood $V \subset U$ such that $\lambda|_V$ is smooth. We apply the previous lemma to the parameterization $(U_1, \mathbf{r}_1)$ at a point $a = \mathbf{r}_1(\lambda(u_0, v_0))$. Let $G : B \to U_1$ be the smooth map for which $\mathbf{r}_1^{-1}|_{B\cap W} = G|_{B\cap W}$ and $V = \mathbf{r}^{-1}(B \cap W)$. Then $\lambda|_V = \mathbf{r}_1^{-1} \circ \mathbf{r}|_V = (G \circ \mathbf{r})|_V$ and, therefore, $\lambda|_V$ is smooth, as a restriction of a smooth map. The smoothness of λ^{-1} is proved in the same way, replacing the parameterization $\mathbf{r}_1$ by the parameterization $\mathbf{r}$. □

Locally, each surface is the support of a parameterized surface. The converse is not true, i.e. the support of an arbitrary parameterized surface is *not* a surface. Still, if we choose an arbitrary parameterized surface, by restricting the domain, we can obtain a parameterized surface whose support is a regular surface. Thus, we have:

Theorem 4.3.2. *Let $(U, \mathbf{r})$ be a regular parameterized surface. Then each point $(u_0, v_0) \in U$ has an open neighbourhood $V \subset U$ such that the $\mathbf{r}(V)$ is a simple surface in $\mathbb{R}^3$, for which the pair $(V, \mathbf{r}|_V)$ is a global parameterization.*

Proof. The only extra condition we have to impose on V is that the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$ to be a homeomorphism. Let $\mathbf{r}(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$. Without restricting the generality, we may assume that the Jacobian of the map $f : (u, v) \to (x = f_1(u, v), y = f_2(u, v))$ is different from zero at (u_0, v_0). Then, from the inverse function's theorem, there is an open neighbourhood $V \subset U$ of the point (u_0, v_0) and an open neighbourhood $\widetilde{V}$ of the point $(x_0, y_0) = f(u_0, v_0)$ such that the map $F : V \to \widetilde{V}$ is a diffeomorphism. We shall prove first the injectivity of the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$. Let $(u_1, v_1), (u_2, v_2) \in V$, such that $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$. Then, in particular,

$$f_1(u_1, v_1) = f_1(u_2, v_2) \quad \text{şi} \quad f_2(u_1, v_1) = f_2(u_2, v_2),$$

i.e. $f(u_1, v_1) = f(u_2, v_2)$. Or, f is diffeomorphism, so it is, in particular, an injective map, therefore $(u_1, v_1) = (u_2, v_2)$. The map $\mathbf{r} : U \to \mathbb{R}^3$ is continuous, so it is continuous also its restriction $\mathbf{r}|_V : V \to \mathbb{R}^3$. Since on $\mathbf{r}(V)$ we take the subspace topology, the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$ is, also, continuous. To prove the continuity of the inverse map, we note that its is a composition of the following continuous maps: $(x, y, z) \in \mathbf{r}(V) \to (x, y) \in \widetilde{V} \to (u, v) = f^{-1}(x, y) \in V$, as seen in the proof of the lemma. □

4.4 Curves on a surface

We shall say that a smooth parameterized curve $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$ lies on a surface S if its support $\boldsymbol{\rho}(I)$ is included in S. It is easy to describe the parameterized curves whose support is contained into the domain of a parameterization $(U, \mathbf{r})$ of the surface S.

Theorem 4.4.1. *Let $(U, \mathbf{r})$ be a parameterization of the surface S and $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$ a smooth parameterized curve whose support is included in $\mathbf{r}(U)$. Then there is a single smooth parameterized curve $(I, \tilde{\boldsymbol{\rho}})$ on U such that*

$$\boldsymbol{\rho}(t) \equiv \boldsymbol{\rho}(\tilde{\boldsymbol{\rho}}(t)). \tag{4.4.1}$$

Conversely, any smooth parameterized curve $\tilde{\boldsymbol{\rho}}$ on U defines, through the formula (4.4.1), a smooth parameterized curve on $\mathbf{r}(U)$. The regularity of $\boldsymbol{\rho}$ at t is equivalent to the regularity of $\tilde{\boldsymbol{\rho}}$ at t.

Proof. Since the map $\mathbf{r} : U \to \mathbf{r}(U)$ is a homeomorphism, while $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$, from the formula (4.4.1), we can obtain $\tilde{\boldsymbol{\rho}}$, by putting

$$\tilde{\boldsymbol{\rho}} = \mathbf{r}^{-1} \circ \boldsymbol{\rho}.$$

Clearly, $\tilde{\boldsymbol{\rho}}$ is continuous, as a composition of two continuous maps. We shall check now that $\tilde{\boldsymbol{\rho}}$ is, actually, smooth. Let $t \in I$, then $\boldsymbol{\rho}(t) \in \mathbf{r}(U)$. According to the lemma from the previous paragraph there is an open neighbourhood B of the point $\boldsymbol{\rho}(t)$ in $\mathbb{R}^3$ and a smooth map $G : B \to U$ such that $\mathbf{r}^{-1}|_{B \cap \mathbf{r}(U)} = G|_{B \cap \mathbf{r}(U)}$. Therefore, the map $\tilde{\boldsymbol{\rho}}$, can be represented, in the neighbourhood of the point t, as a composition $G \circ \boldsymbol{\rho}$ of smooth maps and, thus, it is smooth. The converse affirmation can be proved even simpler, because we have

$$\boldsymbol{\rho} = \mathbf{r} \circ \tilde{\boldsymbol{\rho}}$$

and, as $\mathbf{r}$ and $\tilde{\boldsymbol{\rho}}$ are smooth, so is $\boldsymbol{\rho}$.

To verify the equivalence of the regularity conditions for $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{\rho}}$, we consider the components of the path $\tilde{\boldsymbol{\rho}}$:

$$\tilde{\boldsymbol{\rho}} = (u(t), v(t)).$$

Then the equality (4.4.1) becomes

$$\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t)).$$

Differentiating this relation, we get

$$\boldsymbol{\rho}'(t) = \mathbf{r}'_{\mathbf{u}} \cdot u'(t) + \mathbf{r}'_{\mathbf{v}} \cdot v'(t).$$

Since the vectors $\mathbf{r}'_\mathbf{u}$ and $\mathbf{r}'_\mathbf{v}$ are not colinear (because the surface, as always, is supposed to be regular), from the previous relation it follows that $\boldsymbol{\rho}'(t) = 0$ iff $u'(t) = 0$ and $v'(t) = 0$, i.e. iff $\tilde{\boldsymbol{\rho}}'(t) = 0$. □

Definition. The parameterized curve $\tilde{\boldsymbol{\rho}}(t)$ on the domain U is called the *local representation* of the parameterized curve $\boldsymbol{\rho}(t)$ in the local parameterization $(U, \mathbf{r})$, while the equations

$$\begin{cases} u = u(t) \\ v = v(t) \end{cases}$$

are called the *local equations* of $\boldsymbol{\rho}(t)$ in the considered parameterization.

Example. Let $(U, \mathbf{r} = \mathbf{r}(u, v))$ a local parameterization S and $(u_0, v_0) \in U$. We consider, in $\mathbf{r}(U) \subset S$ the paths defined by the local equations

$$\begin{cases} u = u_0 + t \\ v = v_0 \end{cases} \tag{4.4.2}$$

and

$$\begin{cases} u = u_0 \\ v = v_0 + t \end{cases} \tag{4.4.3}$$

It is easy to see that the supports of these paths lie, indeed, on S. Through each $\mathbf{r}(u_0, v_0) \in \mathbf{r}(U)$ pass exactly two such curves, one of each kind. The parameterized curves are called *coordinate lines or coordinate curves* on the surface S, in the local parameterization $(U, \mathbf{r})$.

4.5 The tangent vector space, the tangent plane and the normal to a surface

Let us denote, for any $a \in \mathbb{R}^3$, by R^3_a the space of bound vectors with the origin at a. This is, obviously, a 3-dimensional vector space, naturally isomorphic to $\mathbb{R}^3$.

Definition 4.5.1. A vector $\mathbf{h} \in \mathbb{R}^3_a$ is called a *tangent vector* to the surface S at the point a if there is a parameterized curve $(I, \boldsymbol{\rho}(t))$ on S and a $t_0 \in I$ such that $\boldsymbol{\rho}(t_0) = a$ and $\boldsymbol{\rho}'(t_0) = \mathbf{h}$. Thus, a tangent vector to a surface is just a tangent vector to a curve on the surface.

We shall denote by T_aS the set of all the tangent vectors to the surface S at $a \in S$. The following lemma is trivial, but it plays an important role in the sequel.

Lemma. *Let $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$ be a parameterized curve on S, given by the local equations $u = u(t)$, $v = v(t)$, with respect to a local parameterization $(U, \mathbf{r})$ of S. Then we have the relation*

$$\boldsymbol{\rho}'(t) = u'(t)\mathbf{r}'_{\mathbf{u}}(u(t), v(t)) + v'(t)\mathbf{r}'_{\mathbf{v}}(u(t, v(t)). \tag{4.5.1}$$

Proof. We just differentiate the relation $\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t))$ with respect to t. □

Theorem 4.5.1. *The set T_aS is a 2-dimensional vector subspace of $\mathbb{R}^3$. If $(U, \mathbf{r})$ is a local parameterization of S, while $a = \mathbf{r}(u_0, v_0)$, then the vectors $\mathbf{r}'_{\mathbf{u}}(u_0, v_0)$ and $\mathbf{r}'_{\mathbf{v}}(u_0, v_0)$ make up a basis of this subspace, called the* natural basis *or the* coordinate basis *of the tangent space.*

Proof. Let $(U, \mathbf{r})$ be a local parameterization of S, with $a = \mathbf{r}(u_0, v_0)$. if the parameterized curve $(I, \boldsymbol{\rho}(t))$ is on the surface and $\boldsymbol{\rho}(t_0) = a$, then, restricting, if necessary, the interval I, we may assume that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$, while its local equations in this parameterization of the surface are $u = u(t), v = v(t)$. Then, from the formula (4.5.1), it follows that

$$\boldsymbol{\rho}'(t_0) = u'(t_0)\mathbf{r}'_{\mathbf{u}}(u_0, v_0) + v'(t_0)\mathbf{r}'_{\mathbf{v}}(u_0, v_0).$$

Conversely, any vector of the form

$$\mathbf{h} = \alpha\mathbf{r}'_{\mathbf{u}}(u_0, v_0) + \beta\mathbf{r}'_{\mathbf{v}}(u_0, v_0)$$

is tangent to the parameterization curve given by the local equations

$$\begin{cases} u = u_0 + \alpha t \\ v = v_0 + \beta t \end{cases},$$

which is a curve on S, passing through a for t_0, therefore $\mathbf{h} \in T_aS$. □

The vector space T_aS is called the *tangent space* to S at a. As we mentioned before, $\mathbb{R}^3_a$ is naturally[2] isomorphic to $\mathbb{R}^3$. Based on this isomorphism, we may consider, when is convenient, that T_aS is, in fact, a subspace of $\mathbb{R}^3$ rather than of $\mathbb{R}^3_a$. In this case, T_aS is a *vectorial plane*, in the sense that it passes through the origin of $\mathbb{R}^3_a$; then the plane passing through a and having T_aS as directing plane (i.e. the plane parallel to T_aS, passing through a), is called the tangent *plane* to S at the point a and it is denoted by Π_aS.

[2]In this context, *naturally* means that there is an isomorphism which is independent of the choice of bases in the two vector spaces.

If $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a local parameterization of the surface S and $a = \mathbf{r}(u_0,v_0) = (x_0,y_0,z_0) \in S$, then, clearly, the equation of the tangent plane of S at a should be

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix} = 0.$$

Let, now, $(U, \mathbf{r} = \mathbf{r}(u,v))$ be a parameterization of S and $(u_0,v_0) \in U$. If we modify the arguments by $\Delta u = \alpha\Delta t$, $\Delta v = \beta\Delta t$, with $\alpha,\beta \in \mathbb{R}$ fixed, the Taylor's formula gives:

$$\mathbf{r}(u_0 + \Delta u, v_0 + \Delta v) = \mathbf{r}(u_0,v_0) + \Delta t \cdot (\alpha\mathbf{r}'_\mathbf{u}(u_0,v_0) + \beta\mathbf{r}'_\mathbf{v}(u_0,v_0)) + \Delta t \cdot \boldsymbol{\varepsilon},$$

with $\lim\limits_{\Delta t\to 0} \boldsymbol{\varepsilon} = 0$. Using this formula, we shall give another characterization of the tangent plane. Let Π be a plane in $\mathbb{R}^3$, passing through $a = \mathbf{r}(u_0,v_0)$, d – the distance from the point $\Delta a = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v)$ to the plane Π, and h – the distance between the points a and Δa.

Theorem 4.5.2. *The plane Π is the tangent plane to the surface S at the point a iff for any modification of the arguments of the form $\Delta u = \alpha\Delta t$, $\Delta v = \beta\Delta t$, with $\alpha,\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$, we have*

$$\lim_{\Delta t\to 0} \frac{d}{h} = 0, \tag{4.5.2}$$

i.e. the plane and the surface have a first order contact at a.

Proof. Let $\mathbf{n}$ be the versor of the normal to the plane Π, $\Delta\mathbf{r} = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v) - \mathbf{r}(u_0,v_0)$. Then $d = \Delta\mathbf{r} \cdot \mathbf{n}$, $h = \|\Delta\mathbf{r}\|$. Replacing $\Delta\mathbf{r}$ by its expression, we get:

$$\lim_{\Delta t\to 0} \frac{d}{h} = \lim_{\Delta t\to 0} \frac{\Delta t \cdot (\alpha\mathbf{r}'_\mathbf{u}(u_0,v_0) + \beta\mathbf{r}'_\mathbf{v}(u_0,v_0) + \boldsymbol{\varepsilon}) \cdot \mathbf{n}}{\|\alpha\mathbf{r}'_\mathbf{u}(u_0,v_0) + \beta\mathbf{r}'_\mathbf{v}(u_0,v_0) + \boldsymbol{\varepsilon}\|} = \pm\frac{(\alpha\mathbf{r}'_\mathbf{u}(u_0,v_0) + \beta\mathbf{r}'_\mathbf{v}(u_0,v_0)) \cdot \mathbf{n}}{\|\alpha\mathbf{r}'_\mathbf{u}(u_0,v_0) + \beta\mathbf{r}'_\mathbf{v}(u_0,v_0)\|},$$

therefore

$$\lim_{\Delta t\to 0} = 0 \iff (\alpha\mathbf{r}'_\mathbf{u}(u_0,v_0) + \beta\mathbf{r}'_\mathbf{v}(u_0,v_0)) \cdot \mathbf{n} = 0. \tag{4.5.3}$$

Necessity. If Π is the tangent plane, then the vectors $\mathbf{r}'_\mathbf{u}$ and $\mathbf{r}'_\mathbf{v}$, as directing vectors of Π, are orthogonal to $\mathbf{n}$, therefore the relation (4.5.3) is fulfilled.
Sufficiency. Let us assume, now, that (4.5.3) is fulfilled. Choosing $\alpha = 1, \beta = 0$, we get $\mathbf{r}'_\mathbf{u} \cdot \mathbf{n} = 0$. In the same way, for $\alpha = 0, \beta = 1$, one obtains $\mathbf{r}'_\mathbf{v} \times \mathbf{n} = 0$. Thus, $\mathbf{n}$ is orthogonal to the tangent plane, i.e. Π is the tangent plane. □

Definition 4.5.2. The straight line passing through a point of a surface, perpendicular to the tangent plane to the surface at that point, is called the *normal to the surface* at the considered point.

Thus, if $(U, \mathbf{r})$ is a parameterization of the surface around a point $a = \mathbf{r}(u_0, v_0) = (x_0, y_0, z_0) \in S$, then the a directing vector of the normal will be $\mathbf{r}'_{\mathbf{u}}(u_0, v_0) \times \mathbf{r}'_{\mathbf{v}}(u_0, v_0)$, which means that the equations of the normal at the point a will be given by:

$$\frac{X - x_0}{\begin{vmatrix} y'_u(u_0, v_0) & z'_u(u_0, v_0) \\ y'_v(u_0, v_0) & z'_v(u_0, v_0) \end{vmatrix}} = \frac{Y - y_0}{\begin{vmatrix} z'_u(u_0, v_0) & x'_u(u_0, v_0) \\ z'_v(u_0, v_0) & x'_v(u_0, v_0) \end{vmatrix}} = \frac{Z - z_0}{\begin{vmatrix} x'_u(u_0, v_0) & y'_u(u_0, v_0) \\ x'_v(u_0, v_0) & y'_v(u_0, v_0) \end{vmatrix}}. \tag{4.5.4}$$

To construct the tangent plane and the normal to a surface given in an implicit representation, the following result is very useful.

Theorem 4.5.3. *At the point (x_0, y_0, z_0) of the surface given through the equation*

$$F(x, y, z) = 0$$

the vector $\operatorname{grad} F_0 = \{F'_x(x_0, y_0, z_0), F'_y(x_0, y_0, z_0), F'_z(x_0, y_0, z_0)\}$ *is perpendicular to the tangent plane to the surface at this point.*

Proof. Let $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ be a local parameterization of the surface around the point $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$. Then we have the identity

$$F(x(u, v), y(u, v), z(u, v)) = 0,$$

whence, by differentiation, we get

$$\begin{cases} \underline{0} = F'_u = F'_x \cdot x'_u + F'_y \cdot y'_u + F'_z \cdot z'_u \equiv \underline{\operatorname{grad} F \cdot \mathbf{r}'_{\mathbf{u}}} \\ \underline{0} = F'_v = F'_x \cdot x'_v + F'_y \cdot y'_v + F'_z \cdot z'_v \equiv \underline{\operatorname{grad} F \cdot \mathbf{r}'_{\mathbf{v}}} \end{cases},$$

i.e. $\operatorname{grad} F \perp L(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) \equiv T_{(x_0, y_0, z_0)} S$. □

Consequence 1. *The equation of the tangent plane to the surface given by the implicit equation $F(x, y, z) = 0$ at the point (x_0, y_0, z_0) has the form*

$$(X - x_0)F'_x(x_0, y_0, z_0) + (Y - y_0)F'_y(x_0, y_0, z_0) + (Z - z_0)F'_z(x_0, y_0, z_0) = 0,$$

while the equations of the normal to the surface at the same point are

$$\frac{X - x_0}{F'_x(x_0, y_0, z_0)} = \frac{Y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{Z - z_0}{F'_z(x_0, y_0, z_0}.$$

Consequence 2. *For any point a of the sphere S^2_R the tangent space $T_a S^2_R$ is orthogonal to the radius vector of the point a.*

Proof. The sphere S_R^2 can be described by the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - R^2 = 0,$$

whence it follows

$$\operatorname{grad} F = 2\{x, y, z\} = 2\mathbf{a},$$

therefore the radius vector is parallel to the gradient of the function F, hence it is perpendicular to the tangent space. □

4.6 The orientation of surfaces

Definition 4.6.1. An *orientation* of a surface S is a choice of an orientation in each tangent space T_aS, i.e. a choice of the unit normal vector of T_aS, $\mathbf{n}(a)$. It is assumed, in this context, that the map $\mathbf{n} : S \to \mathbb{R}^3$, $a \to \mathbf{n}(a)$ is continuous. The surfaces on which is is possible to define an orientation are called *orientable*, while those on which an orientation has been already chosen – *oriented*.

Examples. a) We can define an orientation on the sphere S_R^2 using the versor of the exterior normal. It is not difficult to see that, if $\mathbf{a}$ is the radius point of the point $a \in S_R^2$, then $\mathbf{n}(a) = \frac{1}{R}\mathbf{a}$. Therefore, the map

$$\mathbf{n} : S_R^2 \to \mathbb{R}^3,$$

defining the orientation of the sphere can be represented as a composition of continuous maps:

$$S_R^2 \xrightarrow{i} \mathbb{R}^3 \xrightarrow{\frac{1}{R}} \mathbb{R}^3 : a \longrightarrow \mathbf{a} \longrightarrow \frac{1}{R}\mathbf{a}.$$

b) Let S be a simple surface, with the global parameterization $(U, \mathbf{r})$. This surface can be oriented by using the vector field

$$\mathbf{n}(u, v) = \frac{\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}}{\|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|}.$$

c) Let S be a surface given by the implicit equation $F(x, y, z) = 0$. Then the surface can be oriented through the gradient vector field:

$$\mathbf{n}(x, y, z) = \frac{\operatorname{grad} F}{\|\operatorname{grad} F\|}.$$

If the orientation of a surface S is given by the vector function (vector field) $\mathbf{n}(a)$, then the vector field $-\mathbf{n}(a)$ also defines an orientation on S, called the *opposite* orientation of the orientation given by $\mathbf{n}$. If the orientable surface S is connected, then each orientation of S should coincide to one of the two orientations just mentioned. Indeed, if $\mathbf{N}(a)$ is an orientation of the surface S, then we must have $\mathbf{N}(a) = \lambda\,\mathbf{n}(a)$, where λ is a continuous function on S, taking values into the finite set $\{-1, 1\}$, therefore, if S is connected, λ has to be a constant function. Thus, a connected orientable surface has only two distinct orientations. Of course, if the surface is not connected, there are more orientations, corresponding to different combinations of the two possible orientations on each connected component of the surface.

Remark. Not any surface is orientable. We consider, for instance, the support of the parameterized surface

$$\mathbf{r}(u, v) = (\cos u + v\cos\frac{u}{2}\cos u, \sin u + v\cos\frac{u}{2}\sin u, v\sin\frac{u}{2}),$$

with $u, v \in \mathbb{R}$ (the *Möbius's band*, see the next figure). It will be shown bellow that S is not orientable. (It has a single side: it is possible to move continuously the origin of the unit normal along a close path on S such that, after the "trip", the unit normal will change into its opposite.) Notice that S is not simple, as it might seem, because $\mathbf{r}$ is not a parameterization, since it is not a homeomorphism on the image.

The non-orientability of the Möbius's band. We consider two local parameterizations to describe the Möbius's band:

$$\mathbf{r} : T = \left\{(s, t) \mid -\frac{1}{2} < s < \frac{1}{2},\ 0 < t < 2\pi\right\},$$

$$\mathbf{r}(s, t) = \left(\cos t\left(1 + s\cos\frac{t}{2}\right), \sin t\left(1 + s\cos\frac{t}{2}\right), s\sin\frac{t}{2}\right)$$

and

$$\boldsymbol{\rho} : V = \left\{(u, v) \mid -\frac{1}{2} < u < \frac{1}{2},\ -\pi < v < \pi\right\},$$

$$\boldsymbol{\rho}(u, v) = \left(\cos v\left(1 + u\cos\frac{v}{2}\right), \sin v\left(1 + u\cos\frac{v}{2}\right), u\sin\frac{v}{2}\right)$$

It is not difficult to see that the domain of the diffeomorphism (parameters' change) $\Phi = \boldsymbol{\rho}^{-1} \circ \mathbf{r}$ is the set T^*, equal to T, with the segment $t = \pi$ removed. We can write Φ explicitly as $\Phi(s, t) \equiv (u, v) = (\varphi_1(s, t), \varphi_2(s, t))$, where

$$u = \varphi_1(s, t) = s, \quad \forall (s, t) \in T^*$$

and

$$v = \varphi_2(s,t) = \begin{cases} t & \text{if} \quad (s,t) \in T^*, 0 < t < \pi \\ \pi - t & \text{if} \quad (s,t) \in T^*, \pi < t < 2\pi \end{cases}.$$

The Jacobi matrix of the map Φ is, as one can readily see,

$$J(\Phi)(s,t) \equiv \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0 < t < \pi \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \pi < t < 2\pi \end{cases}$$

The supports of $\mathbf{r}$ and $\boldsymbol{\rho}$ are, clearly, orientable, as simple surfaces. However, admitting that the Möbius's band is orientable, the two local parameterizations do not define the same orientation on it, since, as one can see from the computation we made before, the determinant of the Jacobi matrix of the parameters' change is not always positive.

On the other hand, as our surface is connected, if it is orientable, it can have only two distinct orientations, in other words, any local parameterization of S should be positively equivalent either to $\mathbf{r}$ or to $\boldsymbol{\rho}$.

Let us suppose, now, that S is orientable. This means that there is a family of local parameterizations $\mathbf{r}_1, \mathbf{r}_2, \ldots$, which are pairwise positively equivalent and their supports cover S. We may assume, without restricting the generality, that on the support of $\mathbf{r}$ all this parameterizations are positively equivalent to $\mathbf{r}$. There should be a local parameter-

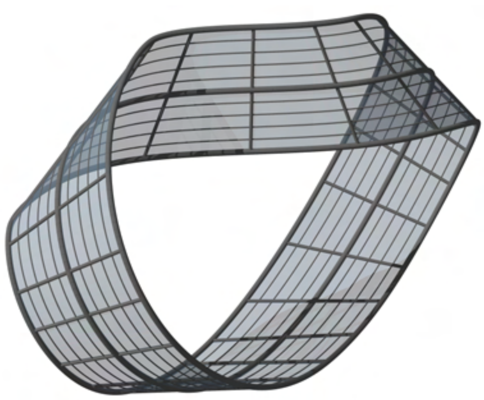


Figure 4.2: The Möbius's band

ization in the family such that its support intersects the segment $t = 0$. We shall suppose that this parameterization is $\mathbf{r}_1$. We may assume that the support of $\mathbf{r}_1$ is included into the support of $\boldsymbol{\rho}$ (otherwise, if necessary, we may shrink the domain of $\mathbf{r}_1$). It follows then that the Jacobi determinant of the map $\boldsymbol{\rho}^{-1} \circ \mathbf{r}_1$ should be either always positive or always negative on the domain of $\mathbf{r}_1$. On the other hand, we have, obviously, from the chain rule, that

$$J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}) = J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}_1) J(\mathbf{r}_1^{-1} \circ \mathbf{r}),$$

whence

$$\det J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}) = \det J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}_1) \det J(\mathbf{r}_1^{-1} \circ \mathbf{r}).$$

Now, in the right hand side, the last determinant is always positive, since the two parameterizations are assumed to be positively equivalent. The first determinant, from the hypothesis, is always positive or always negative. Thus, the right hand side has constant sign. On the other hand, as we saw previously, the left hand side has opposed signs on the two sides of the segment $t = 0$, whence the contradiction which shows that the Möbius's band is not orientable.

In the figure 4.6 we indicate how one can construct a Möbius band, from a strip of paper. Another example of a non-orientable surface is the so-called *Klein's bottle* (see figure **??**)

Definition. Let S be an oriented surface with the orientation $\mathbf{n}(a)$. A local parameterization $(U, \mathbf{r})$ of S is said to be *compatible* with the orientation $\mathbf{n}(a)$ if for any point $a = \mathbf{r}(u, v)$ we have

$$\mathbf{n}(a) = \frac{\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}}{\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|}$$

or, which is the same, if the frame $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{n}(a)\}$ is right-handed.

4.7 Differentiable maps on a surface

Definition 4.7.1. Let S be a surface in $\mathbb{R}^3$. A map $f : S \to \mathbb{R}^k$ is called *differentiable* or *smooth* if for any parameterization $(U, \mathbf{r})$ of S the map $f \circ \mathbf{r} : U \to \mathbb{R}^k$ is smooth. The map $f_{\mathbf{r}} \equiv f \circ \mathbf{r}$ is called the *expression of f în in the curvilinear coordinates* (u, v) or the *local representation* of f with respect to the parameterization $(U, \mathbf{r})$.

Remarks.

1. On can define similarly the differentiability of maps defined on any open subset of a surface S.
2. Any differentiable map $f : S \to \mathbb{R}^k$ is continuous, since, locally, it can be written as a composition of continuous maps: $f = f \circ (\mathbf{r} \circ \mathbf{r}^{-1}) = (f \circ \mathbf{r}) \circ \mathbf{r}^{-1} = f_{\mathbf{r}} \circ \mathbf{r}^{-1}$.

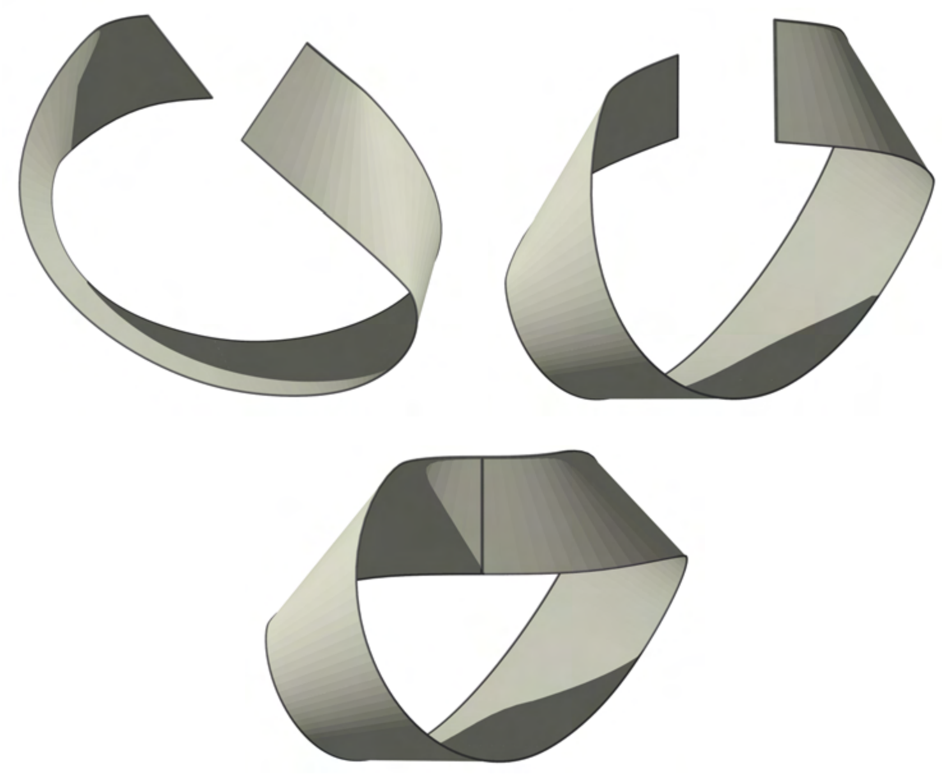


Figure 4.3: Construct your own Möbius band!

Examples. a) Any constant map $f : S \to \mathbb{R}^k : a \to A_0$, $A_0 \in \mathbb{R}^k$ is smooth, because its local representation with respect to any local parameterization of S is, equally, a constant map, hence it is differentiable.

b) If $F : \mathbb{R}^3 \to \mathbb{R}^k$ is a smooth map, then, for any surface S the map $f = F|_S : S \to \mathbb{R}^k$ is a smooth map. Indeed, for any local parameterization $(U, \mathbf{r})$ of S the local representation of f is $f_{\mathbf{r}} = F \circ \mathbf{r}$, where F and $\mathbf{r}$ are smooth maps, in the usual sense. In particular, the orthogonal projections of a surface S on the coordinate axes and planes are, all of them, smooth maps.

c) The inclusion $i : S \to \mathbb{R}^3 : a \to a$ is smooth, since for any local parameterization $(U, \mathbf{r})$ of S, the local representation of i is $i_{\mathbf{r}} = i \circ \mathbf{r} = \mathbf{r}$. (In fact, i is just the restriction of the identity map, $1_{\mathbb{R}^3}$ and, therefore, we can also apply the previous example).

Apparently, it is quite difficult to check whether a map defined on a surface is smooth,

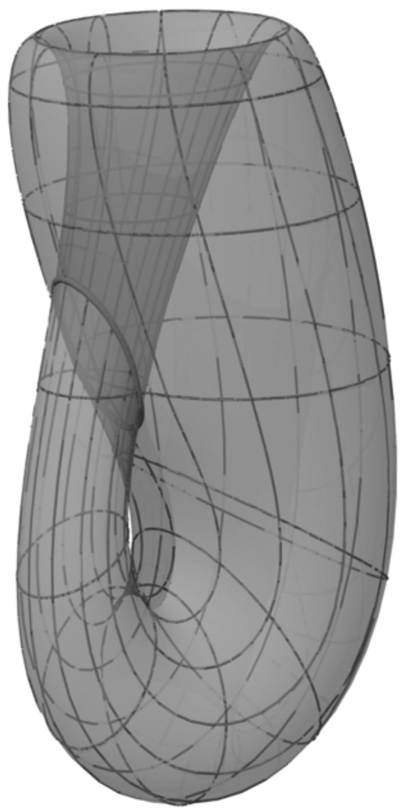


Figure 4.4: Klein's bottle

because we have to check the smoothness of its local representations with respect to *all* the local parameterizations of the surface, which are, of course, infinitely many. Fortunately, as the following theorem shows, it is enough to take only some of the local parameterizations, such that their domains cover the surface. In particular, if the surface is simple, it is enough to check for the global parameterization.

Theorem 4.7.1. *A map $f : S \to \mathbb{R}^k$ is smooth iff for any point $a \in S$ there is a local parameterization $(U, \mathbf{r})$ of the surface S with $a \in \mathbf{r}(U)$, such that the local representation $f_{\mathbf{r}} = f \circ \mathbf{r} : U \to \mathbb{R}^k$ is smooth.*

Proof. The necessity is obvious, since, if F is smooth, then it local representation $f_{\mathbf{r}}$ is smooth for *any* local parameterization $(U, \mathbf{r})$ of S.

Conversely, let's suppose that that $a \in S$ is an arbitrary point of the surface and $(U, \mathbf{r})$

is a local parameterization of S around a such that the local representation $f_{\mathbf{r}} = f \circ \mathbf{r}$ is smooth. Obviously, it is enough to show that then the local representation of f in any other parameterization of S around a is, also, smooth. We choose, thus, another parameterization, $(U_1, \mathbf{r}_1)$, around a and let $W = \mathbf{r}(U) \circ \mathbf{r}_1(U_1)$. Then in $\mathbf{r}^{-1}(W) \subset U_1$, $f_{\mathbf{r}_1}$ can be represented as

$$f_{\mathbf{r}_1} = f \circ \mathbf{r}_1 = f \circ (\mathbf{r} \circ \mathbf{r}^{-1}) \circ \mathbf{r}_1 = (f \circ \mathbf{r}) \circ (\mathbf{r}^{-1} \circ \mathbf{r}_1) = f_{\mathbf{r}} \circ (\mathbf{r}^{-1} \circ \mathbf{r}_1).$$

Since both $f_{\mathbf{r}}$ (from the hypothesis) and $\mathbf{r}^{-1} \circ \mathbf{r}_1$ (from the theorem 4.3.1) are smooth, it follows that $f_{\mathbf{r}_1}$ is, equally smooth. □

Example. Let S be a surface and $(U, \mathbf{r})$ a local parameterization of S. As we explained earlier, the map $\mathbf{r}^{-1} : \mathbf{r}(U) \to \mathbb{R}^2$ is nor smooth in the classical sense. The reason is that its domain is not an open set of an Euclidean space, so the notion itself doesn't make sense for it. We also showed that, however, locally, $\mathbf{r}^{-1}$ is the restriction of a smooth map defined on an open set of $\mathbb{R}^3$. The notion we just defined is, actually, the natural frame of discussing this important map, which, in fact, assigns to each point of the surface (lying in $\mathbf{r}(U)$, of course), a pair of coordinates. Indeed, as a map defined on an open set of S, $\mathbf{r}^{-1}$ is smooth, as we can see easily, because the local representation of $\mathbf{r}^{-1}$ in the parameterization $(U, \mathbf{r})$ is

$$\left(\mathbf{r}^{-1}\right)_{\mathbf{r}} \equiv \mathbf{r}^{-1} \circ \mathbf{r} = 1_U.$$

The next natural step will be to define the notion of a smooth map between two surface rather then from a surface to an Euclidean space. The idea is the following. Let S_1, S_2 be two surfaces In $\mathbb{R}^3$. Then any map $F : S_1 \to S_2$ can be regarded as a map $F : S_1 \to \mathbb{R}^3$. More specifically, one can associate to F the map $i \circ F : S \to \mathbb{R}^3$, where $i : S_2 \hookrightarrow \mathbb{R}^3$ is the inclusion.

Definition 4.7.2. Let $S_1, S_2 \subset \mathbb{R}^3$ be two surfaces. A map $F : S_1 \to S_2$ is called *smooth* if the map $F_1 = i \circ F : S_1 \to \mathbb{R}^3$ is smooth.

Remarks. 1. It easy to see that any smooth between surfaces is continuous.

2. Let $S_1 \subset \mathbb{R}^3$ a surface and $G : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism. Then $S_2 = G(S_1)$ is, also, a surface, while the map $G|_{S_1} : S_1 \to S_2$ is smooth.

3. Let $S_1, S_2 \subset \mathbb{R}^3$ two surfaces and $F : S_1 \to S_2$ a map. Then F is smooth iff for any $a \in S_1$, any local parameterization $(U_1, \mathbf{r}_1)$ of S_1 around a and any local parameterization $(U_2, \mathbf{r}_2)$ of S_2 around $F(a)$, the map

$$F_{\mathbf{r}_1, \mathbf{r}_2} \equiv \mathbf{r}_2^{-1} \circ F \circ \mathbf{r}_1 : U_1 \to U_2$$

is smooth (in the usual sense). $F_{\mathbf{r}_1,\mathbf{r}_2}$ is called the *local representation* of F with respect to the local parameterizations $(U_1, \mathbf{r}_1)$ and $(U_2, \mathbf{r}_2)$.

Definition 4.7.3. A map $F : S_1 \to S_2$ is called a *diffeomorphism* if F is bijective and both F and F^{-1} are smooth maps.

4.8 The differential of a smooth map between surfaces

The notion of a smooth map between surfaces is a natural generalization of the notion of smooth map between open sets in Euclidean spaces. The same should hold true, of course, also for the notion of differential. We shall emphasize, therefore, a property of the differential of map between Euclidean spaces that will be used to construct the generalization. Let $G : B \to \mathbb{R}^3$ be a smooth map, with $B \subset \mathbb{R}^3$ – an open set, while $G(x, y, z) = (g_1(x, y, z), g_2(x, y, z), g_2(x, y, z))$. Then, for any point $a = (x, y, z) \in B$, the differential of G at a,

$$d_a G : \mathbb{R}^3_a \to \mathbb{R}^3_{G(a)}$$

is a linear map, whose matrix is the Jacobi matrix

$$\left.\frac{D(g_1, g_2, g_3)}{D(x, y, z)}\right|_{(x_0,y_0,z_0)} = (\alpha_{ij}),\ 1 \le i, j \le 3,$$

where

$$\alpha_{ij} = \partial_i g_i(x_0, y_0, z_0).$$

For any vector $\mathbf{h} \in \mathbb{R}^3_a$, of components $\{h_1, h_2, h_3\}$, the vector $d_a G(\mathbf{h})$ has the components

$$\left\{\sum_{j=1}^{3} \alpha_{1j} h_j, \sum_{j=1}^{3} \alpha_{2j} h_j, \sum_{j=1}^{3} \alpha_{3j} h_j\right\}.$$

Let us suppose now that the vector $\mathbf{h}$ is tangent to the parameterized curve $\boldsymbol{\rho}(t) = (x(t), y(t), z(t))$ at point $t = t_0$, i.e. $\mathbf{h} = \boldsymbol{\rho}'(t_0)$. We are going to show that the vector $d_a G(\mathbf{h})$ is the tangent vector to the parameterized curve $(G \circ \boldsymbol{\rho})(t)$ at $t = t_0$. To this end, we differentiate the relation

$$(G \circ \boldsymbol{\rho})(t) = (g_1(x(t), y(t), z(t)), g_2(x(t), y(t), z(t)), g_3(x(t), y(t), z(t)))$$

and we obtain:

$$\begin{aligned}\overrightarrow{(G \circ \boldsymbol{\rho})}'(t_0) &= \left\{\sum_{k=1}^{3} \frac{\partial g_1}{\partial x^k}(x_0, y_0, z_0) h_k, \sum_{k=1}^{3} \frac{\partial g_2}{\partial x^k}(x_0, y_0, z_0) h_k, \sum_{k=1}^{3} \frac{\partial g_3}{\partial x^k}(x_0, y_0, z_0) h_k\right\} = \\ &= \left\{\sum_{k=1}^{3} \alpha_{1k} h_k, \sum_{k=1}^{3} \alpha_{2k} h_k, \sum_{k=1}^{3} \alpha_{3k} h_k\right\},\end{aligned}$$

where

$$\{h_1, h_2, h_3\} = \{x'(t_0), y'(t_0), z'(t_0)\} = \boldsymbol{\rho}'(t_0).$$

Thus, the differential d_aG assigns to a tangent vector to the path $\boldsymbol{\rho}(t)$ at $t = t_0$ the tangent vector to the path $G(\boldsymbol{\rho}(t))$ at $t = t_0$.

Let now $F : S_1 \to S_2$ be a smooth map between the surfaces S_1 and S_2 and $a \in S_1$. Then to any smooth path $(I, \boldsymbol{\rho})$ on S_1 corresponds a smooth path $(I, F \circ \boldsymbol{\rho})$ on S_2. If $\boldsymbol{\rho}(t)$ passes through a for $t = t_0$, then the path $F \circ \boldsymbol{\rho}(t)$ will pass through $F(a)$ for $t = t_0$.

Definition 4.8.1. The map $T_aS_1 \to T_{F(a)}S_2$, assigning to each tangent vector $\boldsymbol{\rho}'(t_0)$ to a parameterized curve $\boldsymbol{\rho}(t)$ on S_1, with $\boldsymbol{\rho}(t_0) = a$, the tangent vector $(\overrightarrow{F \circ \boldsymbol{\rho}})'(t_0)$ to the parameterized curve $F \circ \boldsymbol{\rho}$ at $t = t_0$ is called the *differential* of the smooth map $F : S_1 \to S_2$ at the point a and it is denoted by d_aF.

Now, there might be a little difficulty here. Namely, we could have on S_1 two different curves, which have the same tangent vector at a contact point. As the images of the two parameterized curves through F are, generally, distinct, it may happen that the images do not have the same tangent vector at the contact point. Well, as we shall see in a moment, this not actually the case. Indeed, let $a \in S_1$ and $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$, $(I_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(s))$ – two parameterized curves on S_1 such that $\boldsymbol{\rho}(t_0) = \boldsymbol{\rho}_1(s_0) = a$ and $\boldsymbol{\rho}'(t_0) = \boldsymbol{\rho}'_1(s_0)$. We choose an arbitrary local parameterization $(U, \mathbf{r})$ on S_1, around a. As we are interested only on the local phenomena that happen around a, we may assume, without restricting the generality, that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$ and $\boldsymbol{\rho}_1(I_1) \subset \mathbf{r}(U)$. Let's suppose that the local equation of the curves in the parameterization $(U, \mathbf{r})$ are

$$(\boldsymbol{\rho})\begin{cases} u = u(t) \\ v = v(t) \end{cases},$$

and,

$$(\boldsymbol{\rho}_1)\begin{cases} u = u_1(s) \\ v = v_1(s) \end{cases},$$

respectively. Then the vectors $\boldsymbol{\rho}'(t_0)$ and $\boldsymbol{\rho_1}'(s_0)$ have, in the natural basis $\{\mathbf{r}'_\mathbf{u}, \mathbf{r}'_\mathbf{v}\}$ the expressions

$$\boldsymbol{\rho}'(t_0) = \{u'(t_0), v'(t_0)\}$$
$$\boldsymbol{\rho_1}'(s_0) = \{u'_1(s_0), v'_1(s_0)\}.$$

Moreover, in the chosen parameterization,

$$(F \circ \boldsymbol{\rho})(t) = F_\mathbf{r}(u(t), v(t))$$
$$(F \circ \boldsymbol{\rho}_1)(s) = F_\mathbf{r}(u_1(s), v_1(s)),$$

with $F_{\mathbf{r}} = F \circ \mathbf{r}$. Thus,

$$\overrightarrow{(F \circ \boldsymbol{\rho})}(t_0) = \frac{d}{dt}(F_{\mathbf{r}}(u(t), v(t)))(u_0, v_0) = \overrightarrow{(F_{\mathbf{r}})'_u}(u_0, v_0)u'(t_0) + \overrightarrow{(F_{\mathbf{r}})'_v}(u_0, v_0)v'(t_0)$$
$$\overrightarrow{(F \circ \boldsymbol{\rho}_1)}(s_0) = \frac{d}{ds}(F_{\mathbf{r}}(u(s), v(s)))(u_0, v_0) = \overrightarrow{(F_{\mathbf{r}})'_u}(u_0, v_0)u'_1(s_0) + \overrightarrow{(F_{\mathbf{r}})'_v}(u_0, v_0)v'_1(s_0).$$

Now, since $\boldsymbol{\rho}'(t_0) = \boldsymbol{\rho_1}'(s_0)$, it follows that

$$\overrightarrow{(F \circ \boldsymbol{\rho})}'(t_0) = \overrightarrow{(F \circ \boldsymbol{\rho}_1)}'(s_0),$$

i.e. the definition of the differential of F makes sense.

Theorem. *The map $dF : T_aS_1 \to T_{F(a)}S_2$ is linear.*

Proof. From the computations made above, it follows immediately that if a vector $\mathbf{h} \in T_aS_1$ has in the natural basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$ of the linear space T_aS_1, the components $\{h_1, h_2\}$, then

$$d_aF(\mathbf{h}) = \overrightarrow{F'_{\mathbf{r}u}}(u_0, v_0)h_1 + \overrightarrow{F'_{\mathbf{r}v}}(u_0, v_0)h_2, \tag{4.8.1}$$

whence the linearity. □

Example. Let S_1, S_2 be two surfaces in $\mathbb{R}^3$, $D : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism such that $D(S_1) = S_2$, $F = D_{S_1} : S_1 \to S_2$, $a \in S \subset \mathbb{R}^3$. Then we have

$$d_aF = d_aD|_{T_aS_1}, \tag{4.8.2}$$

where $d_aD : \mathbb{R}^3_a \to \mathbb{R}^3_{D(a)}$ is the differential of the map D at a. In particular, let S^2_R and $S^2_{\mathbf{r}}$ be the spheres of radii R and r, respectively, centred at the origin and $D : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \to \frac{r}{R}(x, y, z)$ – a homothety, which is, obviously, a diffeomorphism such that $D(S^2_R) = S^2_r$. Then, for the map $F = D|_{S^2_R} : S^2_R \to S^2_r$, we have $d_aF(\mathbf{h}) = \frac{r}{R}\mathbf{h}$.

4.9 The spherical map and the shape operator of an oriented surface

Let $S \subset \mathbb{R}^3$ be an oriented surface and S^2 – the unit sphere centred at the origin. If the orientation of S is given by the unit normal $\mathbf{n}(a)$, $a \in S$, then we can build a map $\Gamma : S \to S^2$, associating to each $a \in S$ the point on S^2 has as radius vector $\boldsymbol{\Gamma}(a) = \mathbf{n}(a)$. The map Γ is called the *spherical map* of the surface S. This map place a central role in the theory of surfaces. We are going to show, first of all, that Γ is smooth:

Theorem 4.9.1. *The spherical map $\Gamma : S \to S^2$ of an oriented surface S into the unit sphere S^2 is a smooth map between surfaces.*

Proof. Let $a \in S$. We choose a local parameterization $(U, \mathbf{r})$ of the surface S around a, compatible with the orientation. Clearly, since S is orientable, such a parameterization always exists. Indeed, if we choose a parameterization $(U_1, \mathbf{r}_1)$ which is not compatible with the orientation, i.e. we have

$$\frac{\mathbf{r}'_{1u} \times \mathbf{r}'_{1v}}{\|\mathbf{r}'_{1u} \times \mathbf{r}'_{1v}\|} = -\mathbf{n}(u, v),$$

then we replace the domain U_1 by U_1^-, the symmetric of U_1 with respect to the Ov-axis, and the map $\mathbf{r}_1(u, v)$ by $\mathbf{r}_1^-(u, v) = \mathbf{r}_1(-u, v)$. It is easy to see that the pair $(U_1^-, \mathbf{r}_1^-)$ is a parameterization of the surface, compatible with the orientation.

Now we have

$$(\Gamma \circ \mathbf{r})(u, v) = \Gamma(\mathbf{r}(u, v)) = \Gamma_{\mathbf{r}}(u, v) = \mathbf{n}(u, v) = \frac{\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}}{\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|}.$$

Thus, the local representation of Γ is smooth, therefore Γ itself is smooth. □

Examples. (i) For a plane Π the spherical map is constant.

(ii) For the sphere S_R^2 the map $\Gamma : S_R^2 \to S$ has the expression $\Gamma(x, y, z) = \frac{1}{R}(x, y, z)$, with $x^2 + y^2 + z^2 = R^2$.

As we saw, the tangent space to the sphere, $T_{\Gamma(a)}S^2$, is orthogonal to the radius vector $\mathbf{n}(a)$ of the point $\Gamma(a)$. On the other hand, $\mathbf{n}(a)$ is orthogonal to T_aS. Thus, if we identify $\mathbb{R}_a^3$ and $\mathbb{R}_{\Gamma(a)}^3$ to $\mathbb{R}^3$, then the subspaces, T_aS and $T_{\Gamma(a)}S^2$ coincide. Therefore, we may think of the differential $d_a\Gamma : T_aS \to T_{\Gamma(a)}S^2$ as being, in fact, a linear operator $T_aS \to T_aS$.

Definition 4.9.1. The linear operator $d_a\Gamma : T_aS \to T_aS$ is called the *shape operator* of the oriented surface S at the point a and it is denoted by A or A_a.

Remark. There is no general agreement regarding the definition, nor the name of the shape operator. In some books, the shape operator carries an extra minus sign. Sometimes it is called the *Weingarten mapping*, or, also the *principal, or fundamental operator*. Historically, it is true, indeed, that Weingarten was the first to write down the formulae for the differentiation of the spherical map (in other words, he was the one to find the partial derivatives of the unit normal vectors in terms of the derivatives of the radius vector). Nevertheless, the shape operator, as a linear map, was introduced in differential geometry by the Italian mathematician Cesare Burali-Forti ([6]), in 1912, under the name *omografia fondamentale*, i.e. fundamental homography.

Example. (i) For a plane, the shape operator vanishes.

(ii) For a sphere, the shape operator is a homothety.

Let now $(U, \mathbf{r})$ be a local parameterization of S, with the orientation. Then the local representation of the spherical map Γ of S will be given by

$$\mathbf{n}(u, v) = \frac{\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}}{\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|},$$

therefore, for the shape operator we will have:

$$A(\mathbf{h}) = \mathbf{n}'_u h_1 + \mathbf{n}'_v h_2, \tag{4.9.1}$$

where $\mathbf{h} \in T_{\mathbf{r}(u,v)}S$; (h_1, h_2) are the components of $\mathbf{h}$ with respect to the natural basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$. In particular, we have

$$A(\mathbf{r}'_{\mathbf{u}}) = \mathbf{n}'_u,\ A(\mathbf{r}'_{\mathbf{v}}) = \mathbf{n}'_v. \tag{4.9.2}$$

Theorem. *The shape operator A is self-adjoint, i.e. $\forall \mathbf{h}, \mathbf{p} \in T_a S$, we have*

$$A(\mathbf{h}) \cdot \mathbf{p} = \mathbf{h} \cdot A(\mathbf{p}). \tag{4.9.3}$$

Proof. It is enough to make the proof for the vectors, $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$. The case $\mathbf{h} = \mathbf{p}$ is trivial, therefore it is enough to check that

$$A(\mathbf{r}'_{\mathbf{u}}) \cdot \mathbf{r}'_{\mathbf{v}} = \mathbf{r}'_{\mathbf{u}} \cdot A(\mathbf{r}'_{\mathbf{v}}),$$

i.e.

$$\mathbf{n}'_u \cdot \mathbf{r}'_{\mathbf{v}} = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v. \tag{4.9.4}$$

To prove this, we start from the obvious equalities:

$$\mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n} = 0 \tag{*}$$

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$$\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n} = 0 \tag{**}$$

We differentiate (*) with respect to u and (**) with respect to v and we get

$$\begin{cases} \mathbf{r}''_{uv} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_u = 0 \\ \mathbf{r}''_{uv} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v = 0 \end{cases}$$

whence, subtracting the two equalities, we get the conclusion. □

Consequence. *In each tangent space T_aS there is an orthonormal basis, made up of eigenvectors of the shape operator A.*

Proof. Since A is self-adjoint, the two eigenvalues λ_1, λ_2 are real. If $\lambda_1 \neq \lambda_2$, then the eigenvectors corresponding to λ_1 and λ_2 are orthogonal, hence it is enough to choose a unit vector in each eigenspace. If $\lambda_1 = \lambda_2$, then A is just a homothety and any orthonormal basis will do (since, in this case, any tangent space is an eigenvector of A). □

4.10 The first fundamental form of a surface

Let S be a surface in $\mathbb{R}^3$. Then the scalar product in $\mathbb{R}^3$ induces a scalar product in each $\mathbb{R}^3_a$ and, hence, it also induces a scalar product in each tangent space T_aS, $a \in S$.

Definition 4.10.1. The first fundamental form of a surface S is, by definition, the function φ_1, associating to each $a \in S$ the restriction of the scalar product of $\mathbb{R}^3_a$ to T_aS. We shall say usually, informally, that the first fundamental form is the restriction itself, but it should be kept in mind, however, what really happens. Thus, for any $a \in S$ and any $\mathbf{p}, \mathbf{q} \in T_aS$, we will have

$$\varphi_1(\mathbf{p}, \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}. \tag{4.10.1}$$

Remark. In many textbooks, especially the older ones, the first fundamental form is not defined as the restriction of the scalar product to T_aS, but rather as the quadratic form associated to this restriction.

If $(U, \mathbf{r})$ is a local parameterization of S, then for any $(u, v) \in U$ the tangent space $\mathbb{R}^2_{(u,v)}$ of the domain U at the point (u, v) can be identified with the space $T_{\mathbf{r}(u,v)}S$, associating to the vectors $\{1, 0\}$, $\{0, 1\}$, making up a basis of $\mathbb{R}^2_{(u,v)}$, the vectors $\mathbf{r}'_{\mathbf{u}}(u, v)$ and $\mathbf{r}'_{\mathbf{v}}(u, v)$. It is easy to see that, in fact, this identification is just the linear isomorphism $dr_{(u,v)} : \mathbb{R}^2_{(u,v)} \to T_{\mathbf{r}(u,v)}S$. Using this identification, one can transport the first fundamental form φ_1 of S to the domain S (which can be seen, as a matter of fact, as being a simple surface, with the global parameterization given by the identical map of U). Thus, for any $(u, v) \in U$, in the tangent space $T_{\mathbf{r}(u,v)}U \equiv \mathbb{R}^2_{(u,v)}$ at the domain U, the scalar product of two vectors is defined by the rule

$$\widetilde{\varphi}_1(\xi, \eta) = \varphi_1(dr_{(u,v)}(\xi), dr_{(u,v)}(\eta)) = dr_{(u,v)}(\xi) \cdot dr_{(u,v)}(\eta).$$

It easy to see that, by construction, the map $dr_{(u,v)} : \mathbb{R}^2_{(u,v)} \to T_{\mathbf{r}(u,v)}S$ is isometrical with respect to the scalar products $\widetilde{\varphi}_1$ and φ_1 respectively. We introduce the notations

$$\begin{cases} E(u,v) = \mathbf{r}'_{\mathbf{u}}(u,v) \cdot \mathbf{r}'_{\mathbf{u}}(u,v) \\ F(u,v) = \mathbf{r}'_{\mathbf{u}}(u,v) \cdot \mathbf{r}'_{\mathbf{v}}(u,v) \\ G(u,v) = \mathbf{r}'_{\mathbf{v}}(u,v) \cdot \mathbf{r}'_{\mathbf{v}}(u,v) \end{cases}.$$

Then the functions E, F, G are smooth on U, while the matrix $\mathcal{G} = \left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right)$ is the matrix of the scalar product φ_1 on the tangent space $T_{\mathbf{r}(u,v)}S$ with respect to the basis $\{\mathbf{r}'_{\mathbf{u}}(u,v), \mathbf{r}'_{\mathbf{v}}(u,v)\}$, but it is also the matrix of the scalar product $\widetilde{\varphi}_1$ on the the tangent space $\mathbb{R}^2_{(u,v)} = T_{(u,v)}U$ with respect to the basis $\{\{1,0\},\{0,1\}\}$.

Examples. 1. For the plane Π given by the global parameterization $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$, we have:

$$\begin{cases} \mathbf{r}'_{\mathbf{u}} = \mathbf{a} \\ \mathbf{r}'_{\mathbf{v}} = \mathbf{b} \end{cases} \qquad \text{hence} \qquad \begin{cases} E = \mathbf{a}^2 \\ F = \mathbf{a} \cdot \mathbf{b} \\ G = \mathbf{b}^2 \end{cases}.$$

If Π is the coordinate plane xOy, then we may set $\mathbf{r}_0 = 0$, $\mathbf{a} = \imath$, $\mathbf{b} = \jmath$, therefore the first fundamental form has the matrix $\mathcal{G} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

2. For the sphere S^2_R, we choose the local parameterization $(U, \mathbf{r})$, with

$$\mathbf{r}(u,v) = (R\cos u \cos v, R\cos u \sin v, R\sin u),$$

and $U = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We get immediately $E = R^2, F = 0, G = R^2\cos^2 u$, hence the matrix of the first fundamental form is given by

$$\mathcal{G} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$$

4.10.1 First applications

The length of a segment of curve on a surface

Let S be a surface, $(U, \mathbf{r})$ – a local parameterization of S and $(I, \boldsymbol{\rho})$ – a parameterized curve with $\boldsymbol{\rho}(I) \subseteq \mathbf{r}(U)$, given by the local equations $u = u(t), v = v(t)$. Then, in the natural basis, the tangent vector of $\boldsymbol{\rho}$, $\boldsymbol{\rho}'(t)$ has the components $\{u'(t), v'(t)\}$ and we can compute its length using the matrix $\mathcal{G}$. Therefore, we have, for the length of the segment of $\boldsymbol{\rho}$ between t_1 and t_2:

$$l_{t_1,t_2} = \int_{t_1}^{t_2} \|\boldsymbol{\rho}'(t)\| dt = \int_{t_1}^{t_2} \sqrt{E(t)u'^2 + 2F(t)u'v' + G(t)v'^2} dt,$$

where

$$\begin{cases} E(t) = E(u(t), v(t)) \\ F(t) = F(u(t), v(t)) \\ G(t) = G(u(t), v(t)) \end{cases}.$$

Example. We take, on the sphere S_R^2, the curve given by the local equations (in the parameterization described in the previous example) $u = 0, v = t$, where $t \in (0, 2\pi)$ (the equator with a point removed). As we saw above, the first fundamental form of the sphere has the matrix $\mathcal{G} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & cos^2 u \end{pmatrix}$. As along the curve we have $u = 0$, it follows that $u'(t) = 0$, $v'(t) = 1$. On the other hand, $\cos^2 u = \cos^2 0 = 1$, hence, along the curve, the matrix $\mathcal{G}$ will be the identity matrix, multiplied by R^2. If we want to compute, for instance, the length of the segment of curve between $t_1 = \frac{\pi}{2}$ and $t_2 = \pi$ we will get

$$l_{\frac{\pi}{2},\pi} = \int_{\frac{\pi}{2}}^{\pi} \sqrt{R^2 \cdot 0 + 2 \cdot 0 + R^2 \cdot 1} dt = R \cdot t \Big|_{\frac{\pi}{2}}^{\pi} = \frac{\pi R}{2},$$

which was to be expected (the segment of curve is a quarter of a great circle on the sphere).

The angle of two curves on a surface

Let $(U, \mathbf{r})$ be a parameterization of a surface S, $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$, $(I_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(s))$ – two curves on S such that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$, $\boldsymbol{\rho}_1(I_1) \subset \mathbf{r}(U)$. We assume that the supports of the two curves intersect at $\mathbf{r}(u_0, v_0)$, i.e. there are $t_0 \in I$, $s_0 \in I_1$ such that:

$$\boldsymbol{\rho}(t_0) = \boldsymbol{\rho}_1(s_0) = \mathbf{r}(u_0, v_0).$$

If the local equations of the two curves are

$$(\boldsymbol{\rho}) \begin{cases} u = u_1(t) \\ v = v_1(t) \end{cases},$$

and

$$(\boldsymbol{\rho}_1)\begin{cases} u = u_2(s) \\ v = v_2(s) \end{cases},$$

respectively, then the decomposition of the tangents vector at the intersection point with respect to the natural basis will be:

$$\begin{cases} \boldsymbol{\rho}'(t_0) = \{u_1'(t_0), v_1'(t_0)\} \\ \boldsymbol{\rho_1}'(s_0) = \{u_2'(s_0), v_2'(s_0)\} \end{cases},$$

therefore the cosine of the angle of the curves[3] at the contact point is, as it is well known,

$$\cos\theta = \frac{\boldsymbol{\rho}'(t_0)\cdot\boldsymbol{\rho_1}'(s_0)}{\|\boldsymbol{\rho}'(t_0)\|\cdot\|\boldsymbol{\rho_1}'(s_0)\|} = \frac{Eu_1'u_2' + F(u_1'v_2' + u_2'v_1') + Gv_1'v_2'}{\sqrt{Eu_1'^2 + 2Fu_1'v_1' + Gv_1'^2}\cdot\sqrt{Eu_2'^2 + 2Fu_2'v_2' + Gv_2'^2}},$$

where

$$\begin{cases} E = E(u_0, v_0) \\ F = F(u_0, v_0) \\ G = G(u_0, v_0) \end{cases} \quad \text{and} \quad \begin{cases} u_1' = u_1'(t_0) \\ v_1' = v_1'(t_0) \\ u_2' = u_2'(s_0) \\ v_2' = v_2'(s_0) \end{cases}.$$

The area of a parameterized surface

Let $(U, \mathbf{r})$ be an oriented parameterized surface. There are many ways of introducing the notion of area. All of them are more or less connected to integral calculus, so we are not going to enter into any details here. Basically, as in the case of plane geometric figures, the area should be a function associating to each oriented patch a positive number, subject to some restrictions. We choose, following Stoker, the following three restrictions:

a) The area should be given by an integral of the form

$$A = \iint_U f du dv,$$

where f should depend only upon $u, v, \mathbf{r}, \mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}'$ (no higher derivatives of $\mathbf{r}$ should be involved!).

[3] We are speaking, of course, about the angle of the tangent vectors.

b) It is invariant with respect to rigid motions of the space and to parameter transformations that preserve the orientation.

c) A square of side length 1 has area 1.

It can be proved that the only formula for area that verifies these three axioms is

$$A = \iint\limits_U \sqrt{EG - F^2}dudv \equiv \iint\limits_U \|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|dudv. \tag{4.10.2}$$

We will give a heuristic motivation for the formula (4.10.2). It should not be taken, however, as being a "proof", because no such claim is being currently made.

The "classical" approach. Let $(U, \mathbf{r})$ be a parameterized surface and $D \subset U$ – a compact subset of U such that $\mathbf{r}(\partial U)$ is a piecewise smooth curve in $\mathbb{R}^3$. We want to define the *area* of $\mathbf{r}(D) \subset \mathbf{r}(U)$. The basic idea is that we already have a notion of area for *plane* figures, in particular for parallelograms. Thus, let $(u, v) \in D$ and $M = \mathbf{r}(u, v)$. Through M pass two coordinate lines, one from each family. Let $M_1 = \mathbf{r}(u + \Delta u, v)$, $M_2 = \mathbf{r}(u, v + \Delta v)$ be two points on these lines, shifted from M with the parameter shifts Δu and Δv, respectively, and $M' = \mathbf{r}(u+\Delta u, v+\Delta v)$. If Δu and Δv are small enough, then the projection of the curvilinear parallelogram $MM_1M'M_2$ on the tangent plane to the surface at the point M is (approximately, of course), a plane parallelogram in the tangent plane. The sides of this parallelogram are $\mathbf{r}'_\mathbf{u}\Delta u$ and $\mathbf{r}'_\mathbf{v}\Delta v$ and its area will be, then

$$\Delta\sigma = \|\mathbf{r}'_\mathbf{u}\Delta u \times \mathbf{r}'_\mathbf{v}\Delta v\| = \|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|\Delta u\Delta v = \sqrt{EG - F^2}\Delta u\Delta v,$$

where, of course, the coefficients of the first fundamental form have been computed at the point of M. It is only natural, then, to define the area of $\mathbf{r}(D)$ as

$$A = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Sigma\Delta\sigma = \iint\limits_D \sqrt{EG - F^2}dudv,$$

where the sum in the middle term is taken after all the small curvilinear parallelograms that cover $\mathbf{r}(U)$.

Remark. We might expect to obtain for the area of domain on a surface an interpretation similar to the one we got for the length of a segment of curve. Namely, we may discretize the domain of the parameterization and consider the images of the selected points. They will determine a polygonal surface inscribed into the surface. Then we may consider that the area of the polygons that make up this polygonal surface goes to zero and define the area of our surface as being the limit of the area of the polygonal surface. Unfortunately, as a celebrated example of H.A. Schwartz shows, it just doesn't work, because the limit is not independent on the type of polygons we consider and, in particular, for some "polygonizations" of the surface, the area may be infinite and for other ones finite. Of course, the things can be fixed, with a little care, but this is a subject that belongs to theory of integration rather than to differential geometry, therefore we shall not insist.

4.11 The matrix of the shape operator of a surface in the natural basis

Let $(U, \mathbf{r})$ be a local parameterization of the oriented surface S, compatible with the orientation. We shall denote by $\mathcal{A}$ the matrix of the shape operator A with respect to the natural basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$. Since, as we saw earlier,

$$A(\mathbf{r}'_{\mathbf{u}}) = \mathbf{n}'_u,\ A(\mathbf{r}'_{\mathbf{v}}) = \mathbf{n}'_v,$$

we have

$$(\,\mathbf{n}_u\ \ \mathbf{n}'_v) = (\mathbf{r}'_{\mathbf{u}}\ \mathbf{r}'_{\mathbf{v}}) \cdot \mathcal{A}. \tag{4.11.1}$$

We multiply from the left with the matrix $\begin{pmatrix} \mathbf{r}'_{\mathbf{u}} \\ \mathbf{r}'_{\mathbf{v}} \end{pmatrix}$ and we get:

$$\begin{aligned}
\begin{pmatrix} \mathbf{r}'_{\mathbf{u}} \\ \mathbf{r}'_{\mathbf{v}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{n}'_u & \mathbf{n}'_v \end{pmatrix} &= \begin{pmatrix} \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u & \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v \\ \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_u & \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v \end{pmatrix} = \\
&= \begin{pmatrix} \mathbf{r}'_{\mathbf{u}} \\ \mathbf{r}'_{\mathbf{v}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{r}'_{\mathbf{u}} & \mathbf{r}'_{\mathbf{v}} \end{pmatrix} \cdot \mathcal{A} = \begin{pmatrix} \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}'_{\mathbf{u}} & \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}'_{\mathbf{v}} \\ \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{r}'_{\mathbf{u}} & \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{r}'_{\mathbf{v}} \end{pmatrix} \cdot \mathcal{A} = \mathcal{G} \cdot \mathcal{A}.
\end{aligned}$$

We introduce the functions

$$\begin{cases} L(u, v) = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u \\ M(u, v) = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v \\ N(u, v) = \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v \end{cases}, \tag{4.11.2}$$

and the matrix $\mathcal{H}$, defined by

$$\mathcal{H} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Then the last equality reads

$$\mathcal{H} = \mathcal{G} \cdot \mathcal{A}.$$

Since the scalar product on $\mathbb{R}^3$ is nondegenerate, the same is true for its restriction to any subspace and, as a consequence, the matrix $\mathcal{G}$ is invertible. If $\mathcal{G}^{-1}$ be its inverse, then for the matrix of the shape operator we get

$$\mathcal{A} = \mathcal{G}^{-1} \cdot \mathcal{H}, \tag{4.11.3}$$

where, as one can see very easily,

$$\mathcal{G}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

If we perform the computation, we obtain:

$$\mathcal{A} = \frac{1}{EG - F^2} \begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} \tag{4.11.4}$$

All we have to do now is to express the quantities L, M, N in terms of the derivatives of the function $\mathbf{r}$. To this end, we differentiate the relations $\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n} = 0$ and $\mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n} = 0$ with respect to u and v and we get:

$$\begin{cases} \mathbf{r}''_{\mathbf{u}^2} \cdot n + \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u = 0 \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v = 0 \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_u = 0 \\ \mathbf{r}''_{\mathbf{v}^2} \cdot n + \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v = 0 \end{cases},$$

whence we obtain from L, M, N the expressions:

$$\begin{cases} L = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u = -\mathbf{n} \cdot \mathbf{r}''_{\mathbf{u}^2} \\ M = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v = -\mathbf{n} \cdot \mathbf{r}''_{\mathbf{uv}} \\ N = \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v = -\mathbf{n} \cdot \mathbf{r}''_{\mathbf{v}^2} \end{cases} \tag{4.11.5}$$

or, having in mind that

$$\mathbf{n} = \frac{\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}}{\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|},$$

while

$$\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\| = H(= \sqrt{EG - F^2}),$$

$$\begin{cases} L = -\frac{1}{H}(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{r}''_{\mathbf{u}^2}) \\ M = -\frac{1}{H}(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{r}''_{\mathbf{uv}}) \\ N = -\frac{1}{H}(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{r}''_{\mathbf{v}^2}) \end{cases}. \tag{4.11.6}$$

Example. For the helicoid

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = bv \end{cases}, \qquad (u,v)\in\mathbb{R}^2, \quad b>0,$$

we can define the orientation by putting

$$\mathbf{n}(u,v) = \left\{ \frac{b\sin v}{\sqrt{b^2+u^2}}, -\frac{b\cos v}{\sqrt{b^2+u^2}}, \frac{u}{\sqrt{b^2+u^2}} \right\}.$$

On gets, after a straightforward computation:

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ 0 & b^2+u^2 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & \frac{b}{\sqrt{b^2+u^2}} \\ \frac{b}{\sqrt{b^2+u^2}} & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \frac{b}{(b^2+u^2)^{3/2}} \\ \frac{b}{(b^2+u^2)^{3/2}} & 0 \end{pmatrix}.$$

4.12 The second fundamental form of an oriented surface

Definition 4.12.1. The second fundamental form of an oriented surface S is a map associating to each $a \in S$ the application $\varphi_2(a) : T_aS \times T_aS \to \mathbb{R}$ given by

$$\varphi_2(\xi,\eta) = -\varphi_1(A(\xi),\eta), \quad \forall \xi,\eta \in T_aS. \tag{4.12.1}$$

Remark. The minus sign in the previous definition is a consequence of our particular choice of sign in the definition of the shape operator. We found natural to choose the shape operator to be the differential of the spherical map rather then the opposite of the differential, but then in the definition of the second fundamental form we had to introduce an extra minus sign, in order to be consistent with the generally accepted definition of the second fundamental form.

Proposition 4.12.1. *For each $a \in S$, $\varphi_2(a)$ is a symmetrical bilinear form.*

Proof. We take two arbitrary tangent vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in T_aS$ and two arbitrary real numbers $\alpha, \beta \in \mathbb{R}$. Then we have, first of all:

$$\varphi_2(\eta,\xi) = -\varphi_1(A(\eta),\xi) \underset{\text{self-adjoint}}{\overset{A}{=}} -\varphi_1(\eta, A(\xi)) \underset{\text{symmetrical}}{\overset{\varphi_1}{=}} -\varphi_1(A(\xi),\eta) = \varphi_2(\xi,\eta),$$

which means that φ_2 is symmetrical. Due to the symmetry, it is enough to prove the linearity in the first variable only. We have

$$\varphi_2(\alpha\xi_1+\beta\xi_2,\eta) = -\varphi_1(A(\alpha\xi_1+\beta\xi_2),\eta) \underset{\text{linear}}{\overset{A}{=}} -\varphi_1(\alpha A(\xi_1)+\beta A(\xi_2),\eta) \underset{\text{bilinear}}{\overset{\varphi_1}{=}}$$
$$\underset{\text{bilinear}}{\overset{\varphi_1}{=}} -\alpha\varphi_1(A(\xi_1),\eta) - \beta\varphi_1(A(\xi_2),\eta) = \alpha\varphi_2(\xi_1,\eta)+\beta\varphi_2(\xi_2,\eta),$$

which shows the linearity in the first variable and concludes the proof. □

Let $(U, \mathbf{r})$ be a local parameterization of the oriented surface S, compatible with the orientation. Then the matrix $[\varphi_2]$ of the second fundamental form with respect to the canonical basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$ has the form:

$$[\varphi_2] = \begin{pmatrix} \varphi_2(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{u}}) & \varphi_2(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) \\ \varphi_2(\mathbf{r}'_{\mathbf{v}}, \mathbf{r}'_{\mathbf{u}}) & \varphi_2(\mathbf{r}'_{\mathbf{v}}, \mathbf{r}'_{\mathbf{v}}) \end{pmatrix}.$$

But

$$\begin{cases} \varphi_2(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{u}}) = -\varphi_1(A(\mathbf{r}'_{\mathbf{u}}), \mathbf{r}'_{\mathbf{u}}) = -\varphi_1(\mathbf{n}'_u, \mathbf{r}'_{\mathbf{u}}) = -\mathbf{n}'_u \cdot \mathbf{r}'_{\mathbf{u}} \\ \varphi_2(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) = \varphi_2(\mathbf{r}'_{\mathbf{v}}, \mathbf{r}'_{\mathbf{u}}) = -\mathbf{n}'_u \cdot \mathbf{r}'_{\mathbf{v}} = -\mathbf{n}'_v \cdot \mathbf{r}'_{\mathbf{u}} \;\; \varphi_2(\mathbf{r}'_{\mathbf{v}}, \mathbf{r}'_{\mathbf{v}}) = -\mathbf{n}'_v \cdot \mathbf{r}'_{\mathbf{v}} \end{cases},$$

and, thus, we get the matrix

$$[\varphi_2] = -\begin{pmatrix} \mathbf{n}'_u \cdot \mathbf{r}'_{\mathbf{u}} & \mathbf{n}'_u \cdot \mathbf{r}'_{\mathbf{v}} \\ \mathbf{n}'_v \cdot \mathbf{r}'_{\mathbf{u}} & \mathbf{n}'_v \cdot \mathbf{r}'_{\mathbf{v}} \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \overset{\text{not.}}{=} \begin{pmatrix} D & D' \\ D' & D'' \end{pmatrix}.$$

It follows, this way, that the matrix of the second fundamental form in the canonical basis is just $-\mathcal{H}$. Thus, for the coefficients of the second fundamental form with respect to the natural basis, we have

$$\begin{cases} D = \mathbf{n} \cdot \mathbf{r}''_{\mathbf{u}^2} \\ D' = \mathbf{n} \cdot \mathbf{r}''_{\mathbf{uv}} \\ D'' = \mathbf{n} \cdot \mathbf{r}''_{\mathbf{v}^2} \end{cases} \tag{4.12.2}$$

Remark. The reader should be careful with the notations for the coefficients of the second fundamental form. For them there is, also, used the notation e, f, g. Also, in some books, the letters L, M, N are used to denote the coefficients of the second fundamental form themselves. The notations D, D', D'' are usually credited to Gauss (in *Disquisitiones*). It should be noted, however, that for Gauss, the meaning of the symbols is a little different: they are not the coefficients of the second fundamental form as we know it, but rather these coefficients multiplied by $\sqrt{EG - F^2}$.

Example. For the sphere $S = S^2_R$, we have, as we saw before, $\mathbf{n} = \frac{1}{R}\{x, y, z\}$, therefore, as we mentioned earlier, the shape operator A is a homothety of ratio $1/R$, i.e.

$$A(\mathbf{p}) = \frac{1}{R}\mathbf{p}, \quad , \forall \mathbf{p} \in T_a S^2 R.$$

Thus,

$$\varphi_2(\mathbf{p},\mathbf{q}) = -\varphi_1\left(\frac{1}{R}\mathbf{p},\mathbf{q}\right) = -\frac{1}{R}\varphi_1(\mathbf{p},\mathbf{q}) = -\frac{1}{R}\mathbf{p}\cdot\mathbf{q}.$$

therefore, for the sphere, the first two fundamental forms are proportional. Clearly, the same is true for the plane, when the second fundamental form vanishes identically. Surprisingly as it might seem, actually, it can be shown that these are the only two surfaces with this property.

4.13 The normal curvature. The Meusnier's theorem

Let S be an oriented surface and $\mathbf{n}$ – the unit normal. We take a regular parameterized curve, $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$, lying on S.

Definition 4.13.1. The projection of the curvature vector $\mathbf{k}(t)$ of $\boldsymbol{\rho}$ (as a signed scalar) on $\mathbf{n}(\boldsymbol{\rho}(t))$ is called the *normal curvature* of the curve $\boldsymbol{\rho}(t)$ at t and it is denoted by $k_n(t)$.

Let $\theta(t)$ be the angle between the osculating plane of $\boldsymbol{\rho}(t)$ and $\mathbf{n}(\boldsymbol{\rho}(t))$. Then, clearly,

$$k_n(t) = k(t)\cdot\cos\theta(t), \tag{4.13.1}$$

where $k(t)$ is the curvature of the curve $\boldsymbol{\rho}(t)$.

Examples.

1. The normal curvature of any plane curve is zero (in this case the angle $\theta(t)$ is always $\frac{\pi}{2}$, therefore $\cos\theta(t)\equiv 0$).
2. If the support of a parameterized curve lies on a straight line, then its normal curvature is always zero, no matter on what surface is the curve situated, because, this time, the curvature of the curve vanishes identically.

Remark. The relation (4.13.1) has a simple geometrical interpretation (the Meusnier's theorem): the center of curvature of a curve $\boldsymbol{\rho}$, lying on a surface S, is the orthogonal projection on its osculating plane of the center of curvature of the normal section tangent to $\boldsymbol{\rho}$ at that point.

The normal curvature of a curve on a surface can be expressed easily if we know the first two fundamental forms of the surface. Indeed, we have:

Theorem 4.13.1. *The normal curvature of a parameterized curve $\boldsymbol{\rho}(t)$, lying on an oriented surface S, is given by the formula*

$$k_n(t) = \frac{\varphi_2(\boldsymbol{\rho}'(t),\boldsymbol{\rho}'(t))}{\varphi_1(\boldsymbol{\rho}'(t),\boldsymbol{\rho}'(t))}. \tag{4.13.2}$$

Proof. As is many times the case with parameterized curves, the proof is simpler in the case of naturally parameterized curves. Since the curvature of any regular parameterized curve is invariant to a parameter change, we can replace the curve $\boldsymbol{\rho}(t)$ by a n equivalent, naturally parameterized curve, $\boldsymbol{\rho}_1(s)$, the natural parameter being the arc length. The curvature vector of the curve $\boldsymbol{\rho}_1(s)$ will be $\boldsymbol{\rho_1}''(s)$. We choose a local parameterization $(U, \mathbf{r})$ of the surface S and we assume that $\boldsymbol{\rho}_1(s)$ has in this parameterization the local equations $u = u(s),\ v = v(s)$, i.e. $\boldsymbol{\rho}_1(s) = \mathbf{r}(u(s), v(s))$. Then

$$\boldsymbol{\rho_1}''(s) = \mathbf{r}''_{\mathbf{u^2}} \cdot (u')^2 + 2\mathbf{r}''_{\mathbf{uv}} \cdot u'v' + \mathbf{r}''_{\mathbf{v^2}} \cdot (v')^2 + \mathbf{r}'_{\mathbf{u}} \cdot u'' + \mathbf{r}'_{\mathbf{v}} \cdot v''.$$

Thus, for the normal curvature of the curve $\boldsymbol{\rho}_1(s)$ we get the expression:

$$\begin{aligned}k_n(s) &= \mathbf{k}(s) \cdot\ \mathbf{n}(\boldsymbol{\rho}_1(s)) = \boldsymbol{\rho_1}''(s) \cdot\ \mathbf{n}(\boldsymbol{\rho}_1(s)) = \\ &= \mathbf{r}''_{\mathbf{u^2}} \cdot\ \mathbf{n} \cdot (u')^2 + 2\mathbf{r}''_{\mathbf{uv}} \cdot\ \mathbf{n} \cdot u'v' + \mathbf{r}''_{\mathbf{v^2}} \cdot\ \mathbf{n} \cdot (v')^2 = \\ &= -L \cdot (u')^2 - 2M \cdot u'v' - N \cdot (v')^2 = \varphi_2(\boldsymbol{\rho_1}'(s), \boldsymbol{\rho_1}'(s)).\end{aligned}$$

Now, we come back to the initial parameterized curve. We have

$$\boldsymbol{\rho}'(s) = \boldsymbol{\rho_1}'(s(t)) \cdot s'(t) \qquad \text{where} \quad s'(t) \equiv \|\boldsymbol{\rho}'(t)\|.$$

Thus,

$$\boldsymbol{\rho_1}'(s(t)) = \frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|},$$

therefore,

$$k_n(t) = \varphi_2\left(\frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|}, \frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|}\right) = \frac{1}{\boldsymbol{\rho}'(t) \cdot \boldsymbol{\rho}'(t)} \cdot \varphi_2(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t)) = \frac{\varphi_2(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t))}{\varphi_1(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t))}.$$

□

Consequence. *If two curves on an oriented surface have a common point and they have the same tangent line at this point, then the two curves have the same normal curvature at the contact point.*

Proof. Let $\mathbf{p}$ and $\mathbf{q}$ be the tangent vectors to the two curves at their common point. From the hypothesis, $\mathbf{p} = \alpha\mathbf{q}$, hence, from the theorem,

$$k_n = \frac{\varphi_2(\mathbf{p}, \mathbf{p})}{\varphi_1(\mathbf{p}, \mathbf{p})} = \frac{\varphi_2(\alpha\mathbf{q}, \alpha\mathbf{q})}{\varphi_1(\alpha\mathbf{q}, \alpha\mathbf{q})} = \frac{\alpha^2\varphi_2(\mathbf{q}, \mathbf{q})}{\alpha^2\varphi_1(\mathbf{q}, \mathbf{q})} = \frac{\varphi_2(\mathbf{q}, \mathbf{q})}{\varphi_1(\mathbf{q}, \mathbf{q})}$$

□

Remark. The previous consequence can be interpreted in another way. We take a family of biregular parameterized curves on the surface, $\{\boldsymbol{\rho}^{\alpha}(t)\}_{\alpha\in A}$, passing through the same point and having the same tangent line at the contact point. We denote by k^{α} the curvature of the curve $\boldsymbol{\rho}^{\alpha}$ and by θ^{α} the angle between the normal to the surface and the osculating plane of the curve $\boldsymbol{\rho}^{\alpha}$. The consequence is equivalent with the affirmation that the product $k_n = k^{\alpha}\cos\theta^{\alpha}$ does not depend on the choice of the curve from the family. It makes sense, thus, to choose an arbitrary straight line in the tangent plane, passing through the tangency plane, and to speak about the normal curvature of the *surface* in the direction of this line or, in other words, we can define a map k_n on the set of all non-vanishing tangent vectors to the surface with real values, by putting

$$k_n(\mathbf{h}) = \frac{\varphi_2(\mathbf{h},\mathbf{h})}{\varphi_1(\mathbf{h},\mathbf{h})}. \tag{4.13.3}$$

The quantity $k_n(\mathbf{h})$ is called the *normal curvature of the surface in the direction of the vector* $\mathbf{h}$ (since, clearly, it depends only on the *direction* of the vector $\mathbf{h}$, but not on its length or sense.) Thus, *the normal curvature* of an oriented surface in the direction of a vector $\mathbf{h}$ is the normal curvature of an arbitrary curve, lying on the surface, passing through the origin of $\mathbf{h}$ and whose tangent line is parallel to $\mathbf{h}$.

4.14 Asymptotic directions and asymptotic lines on a surface

We saw previously that the normal curvature of a surface at a given point and in a given direction can be expressed in terms of the first two fundamental forms of the surface and that, as stressed, although it is initially defined in terms of *curves* on surface, it actually depends only on the direction of the tangent vector of the curve. It is interesting for us to identify those directions for which the normal curvature vanish.

Definition 4.14.1. Let S be an oriented surface and $p \in S$. A non-vanishing vector $\mathbf{h} \in T_pS$ is said to have *asymptotic direction* if the normal curvature in its direction vanishes. Alternatively, based on the previous section, we can define a vector of asymptotic direction as being one for which

$$\varphi_2(\mathbf{h},\mathbf{h}) = 0.$$

Accordingly, an *asymptotic line or curve* on a surface is a curve on the surface for which all the tangent vectors have asymptotic direction.

Theorem 4.1. *Let $p \in S$ a point of an oriented surface. Then at this point we have asymptotic directions if and only if the quadratic form associated to the second fundamental form of S at p is negatively semi-defined. If we choose a local parameterization*

$(U, \mathbf{r})$ *of* S *such that* $p\mathbf{r}(u_0, v_0)$ *for some* $(u_0, v_0) \in U$, *then this condition simply means that*

$$D(u_0, v_0) \cdot D''(u_0, v_0) - D'^2(u_0, v_0) \le 0. \tag{4.14.1}$$

Proof. Let $\mathbf{h} = \{h_1, h_2\} \in T_pS$ be a non-vanishing tangent vector to the surface at the point p. Then, with respect to the local parameterization chosen, $\mathbf{h}$ has asymptotic direction if and only if

$$D(u_0, v_0)h_1^2 + 2D'(u_0, v_0)h_1h_2 + D''(u_0, v_0)h_2^2 = 0.$$

As $\mathbf{h} \neq 0$, we may assume, for instance, that $h_2 \neq 0$. Then the previous equation can be written as

$$D(u_0, v_0)\left(\frac{h_1}{h_2}\right)^2 + 2D'(u_0, v_0)\frac{h_1}{h_2} + D''(u_0, v_0) = 0.$$

This equation, obviously has *real* solutions if and only if discriminant vanishes, but this is exactly the condition 4.14.1. □

Remark. From the proof of the previous theorem it follows that when the second fundamental form is negatively defined we have two asymptotic directions, while at the points when it is degenerate we have only one asymptotic direction (or two confounded ones).

From the definition of the normal curvature of a curve on a surface, it follows immediately that

Proposition 4.14.1. *Any straight line lying on a surface is an asymptotic line.*

Proof. Indeed, the straight lines have zero curvature, therefore their normal curvature at each point also vanishes. □

The differential equation of the asymptotic lines on a surface can be obtained directly from the definition.

Theorem 4.2. *Let* S *be an oriented surface and* $\boldsymbol{\rho} : I \to S$ *– curve on the surface. We assume that there is a local parameterization of* S, $(U, \mathbf{r} = \mathbf{r}(u, v))$ *such that* $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$ *and the local equations of the curve with respect to this parameterization are* $u = u(t), v = v(t)$. *Then* $\boldsymbol{\rho}$ *is an asymptotic line on* S *if and only if*

$$D(u(t), v(t)) \cdot u'^2(t) + 2 \cdot D'(u(t), v(t)) \cdot u'(t) \cdot v'(t) + D''(u(t), v(t)) \cdot v'^2(t) = 0. \tag{4.14.2}$$

Proof. The equation (4.14.2) is just that condition for the tangent vector of the curve (which has, with respect to the natural basis of the tangent space, the components $\{u'(t), v'(t)\}$) to have asymptotic direction. □

Let us assume, now, that the curve $\boldsymbol{\rho}$ from the previous theorem is biregular, which means, in particular, that is curvature is always strictly positive. From the definition of the normal curvature it follows immediately that, if $\boldsymbol{\nu}$ is the unit principal normal vector of the curve, then the curve is anasymptotic line if and only if

$$\boldsymbol{\nu}(t) \cdot \mathbf{n}(u(t), v(t)) = 0,$$

where $\mathbf{n}$ is the unit normal vector to the surface. This actually means that the principal normal of the curve lies on the tangent plane of the surface, at each point of the curve. Thus, we obtain the following characterization of the asymptotic lines:

Theorem 4.3. *Let S be an oriented surface and $\boldsymbol{\rho}$ a biregular parameterized curve on S. Then $\boldsymbol{\rho}$ is an asymptotic line if and only if at each point its osculating plane coincide with the tangent plane to the surface at that point.*

Another immediate result concerning the asymptotic lines is the following:

Proposition 4.14.2. *Let S be an oriented surface and $(U, \mathbf{r} = \mathbf{r}(u, v))$ – a local parameterization of S. Then the coordinate lines $u = const$ and $v = const$ are asymptotic lines on $\mathbf{r}(U)$ if and only if $D = D'' = 0$.*

Example. Let us consider the helicoid, given by the parameterization

$$\begin{cases} x = u\cos v, \\ y = u\sin v \\ z = b \cdot v \end{cases}$$

where b is a constant. A straightforward computation leads to $D = D'' = 0$, $D'(u, v) = -b/\sqrt{b^2 + u^2}$. This means that, in this particular case, we have, at each point of the surface, two asymptotic lines and these are nothing but the coordinate lines, $u = const$ and $v = const$. We notice that the helicoid is a ruled surface (see the last chapter) and at each point, one of the coordinate lines is a straight line (or a line segment). This is, of course, the line $v = const$ (see the figure 4.14).

4.15 The classification of points on a surface

The first fundamental form of a surface is positively defined. The second one is not. This is fortunate, as it allows us to give a classification of the points of the surface, according to the sign of the second fundamental form or, more specifically, according to the sign of its discriminant $DD'' - D'^2$.

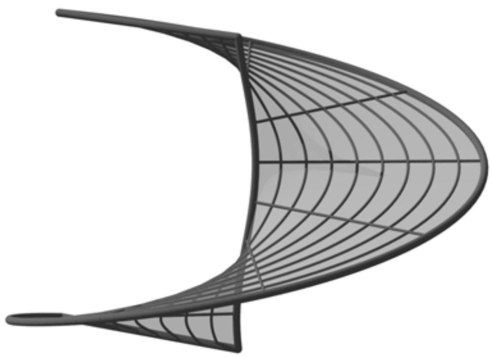


Figure 4.5: Asymptotic lines on the helicoid

Definition 4.15.1. A point $a \in S$ of an oriented surface is called

(i) *elliptic* if the second fundamental form is positively defined at a;

(ii) *parabolic* if the second fundamental form is zero, but at least one of the coefficients is different from zero;

(iii) *hyperbolic*, if the second fundamental form is negatively defined at a;

(iv) *flat*, or planar, if all the coefficients of the second fundamental form vanish at a.

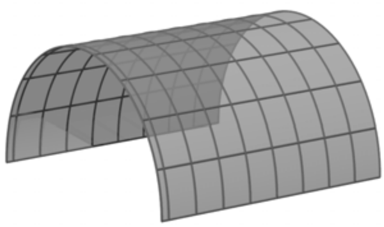


Figure 4.6: Parabolic points on a surface

It is not difficult to see that this definition do not depend on the choice of the local parameterization. Let us discuss, now separately, what happens in each particular case and, also, give some examples.

Elliptic points. At an elliptic point, the normal curvature has the same signs in all directions[4]. Applying the Meusnier's theorem, this means that the centers of curvature of all normal sections lie on the same side of the surface. An example of surface that has only elliptic points is the ellipsoid, given, for instance, by the parameterization

$$\mathbf{r}(u, v) = (a \cos u \cos v, b \sin u \cos v, c \sin v). \tag{4.15.1}$$

At an elliptic point there is no real asymptotic direction, therefore no asymptotic line passes through an elliptic point.

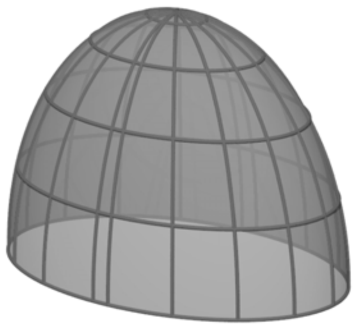


Figure 4.7: Elliptic points on a surface

Parabolic points. In this case the normal curvature does not change the sign, but there is exactly one direction where it vanishes. This is, clearly, an asymptotic direction. Thus, through a parabolic point of a surface passes only one asymptotic line. The cylinders and cones (with the apex removed) have only parabolic points.

Hyperbolic points. In the case of hyperbolic points, it is possible for k_n to change sign and there are exactly two direction where it vanishes. Thus, through a hyperbolic point of a surface pass two asymptotic lines. The hyperbolic points are also called *saddle points*. Such points we can find, for instance, on a hyperbolic paraboloid.

There are, of course, many surfaces on which we can meet all three kind of points (for instance on the torus).

[4] It is not necessarily positive.

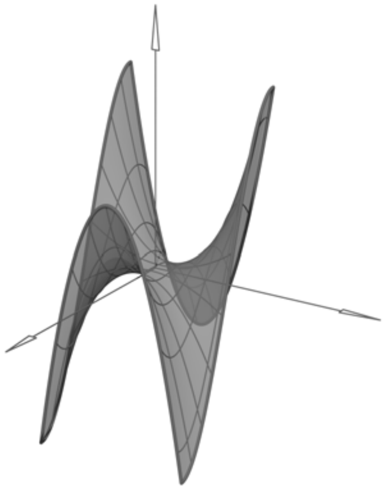


Figure 4.8: The monkey saddle

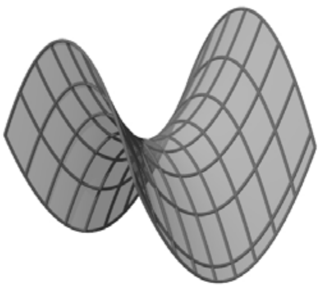


Figure 4.9: Hyperbolic points on a surface

Flat points. The shape of the surface around a flat point might be quite complicated and it is difficult to study it. In fact, in many cases, when proving a theorem in surface theory one explicitly assumes that the surface has no flat points. The surface from the figure 4.15 (the *monkey saddle* has a flat point at the origin of coordinates.

4.16 Principal directions, principal curvatures, Gauss curvature and mean curvature

Definition 4.16.1. The directions on the tangent plane to an oriented surface S at a point $a \in S$, T_aS, corresponding to the eigenvectors of the shape operators A are called the *principal directions* of the surface at the point a.

Remark. At each point, an oriented surface either has two orthogonal principal directions (if the eigenvalues of A are distinct), either all the directions are orthogonal (if the two eigenvalues coincide).

Definition 4.16.2. A curve (Γ) on a surface S is called a *principal line* or a *curvature line* if its tangent, at each point, has a principal direction.

Definition 4.16.3. A *principal curvature* of an oriented surface S at a point $a \in S$ is the normal curvature of S at a in a principal direction.

Proposition 4.16.1. *The principal curvatures of a surface are the eigenvalues of the shape operator, taken with opposite sign.*

Proof. If $\mathbf{e}$ is an eigenvector of A, then $A(\mathbf{e}) = \lambda \cdot \mathbf{e}$, where λ is the eigenvalue corresponding to $\mathbf{e}$, therefore

$$k_n(\mathbf{e}) = \frac{\varphi_2(\mathbf{e},\mathbf{e})}{\varphi_1(\mathbf{e},\mathbf{e})} = \frac{-A(\mathbf{e})\cdot\mathbf{e}}{\mathbf{e}\cdot\mathbf{e}} = \frac{-\lambda\cdot\mathbf{e}\cdot\mathbf{e}}{\mathbf{e}\cdot\mathbf{e}} = -\lambda.$$

□

Hereafter we shall denote by k_1 and k_2 the principal curvatures and we will always assume that $k_1 \geq k_2$.

Definition 4.16.4. An orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the tangent space at a point of a surface is called a *basis of principal directions* of the tangent space if the vectors of the basis have principal directions.

Thus, the vectors of a basis of principal directions are subject to

$$A(\mathbf{e}_i) = -k_i\mathbf{e}_i, \quad i = \overline{1,2}.$$

We shall fix now a point of the surface and ask the following question: find the normal curvature in the direction of a vector $\mathbf{e}$, such that $\angle(\mathbf{e}, \mathbf{e}_1) = \theta$.

As the length of $\mathbf{e}$ is not important, we shall assume that $\mathbf{e}$ is a unit vector: $\|\mathbf{e}\| = 1$. Then $\mathbf{e} = \mathbf{e}_1 \cdot \cos\theta + \mathbf{e}_2 \cdot \sin\theta$, therefore

$$k_n(\mathbf{e}) = \frac{\varphi_2(\mathbf{e},\mathbf{e})}{\varphi_1(\mathbf{e},\mathbf{e})} = \frac{-A(\mathbf{e})\cdot\mathbf{e}}{\underbrace{\mathbf{e}\cdot\mathbf{e}}_{=1}} = -A(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta)\cdot(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta) =$$
$$= (k_1\cos\theta\cdot\mathbf{e}_1 + k_2\sin\theta\cdot\mathbf{e}_2)\cdot(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta) = k_1\cos^2\theta + k_2\sin^2\theta.$$

Thus, we obtained:

Theorem 4.16.1. *Let S be an oriented surface. Then the normal curvature at a point of the surface, in the direction of a vector* $\mathbf{e}$*, is given by the* Euler's formula*:*

$$k_n(\mathbf{e}) = k_1\cos^2\theta + k_2\sin^2\theta, \tag{4.16.1}$$

where k_1 and k_2 are the principal curvature of the surface, while $\theta = \angle(\mathbf{e},\mathbf{e}_1)$.

An immediate consequence of the Euler's formula is:

Theorem 4.16.2. *The principal curvatures of a surface at a point are extreme values of the normal curvature of the surface in the direction of a vector* $\mathbf{e}$*, when the vector* $\mathbf{e}$ *rotates around the origin of the tangent space of the surface at that point.*

Proof. From the Euler's formula, we have

$$k_n(\mathbf{e}) = k_1\cos^2\theta + k_2(1-\cos^2\theta) = k_2 + (k_1-k_2)\cos^2\theta.$$

It is clear now that the maximum value of the normal curvature is reached for $\theta = 0$ (we assumed that $k_1 \geq k_2$!) and, in this case, we get $k_n = k_1$, while the minimal value – for $\theta = \frac{\pi}{2}$, obtaining $k_n = k_2$. □

Definition 4.16.5. The quantities $K_t = k_1 \cdot k_2$ and $K_m = \frac{1}{2}(k_1 + k_2)$ are called, respectively, the *total* (Gaussian) and *mean* curvature of the surface.

The total and mean curvatures of a surface can be computed easily if the matrix of the shape operator in an arbitrary basis is known. Indeed, we have:

Proposition 4.16.2.

$$K_m = -\frac{1}{2}\operatorname{Tr}\mathcal{A} \tag{4.16.2}$$
$$K_t = \det\mathcal{A}. \tag{4.16.3}$$

Proof. As it is well known from linear algebra, the determinant and the trace are invariants of any linear operator, which means that they are the same in any basis, although the matrix of the operator does change, generally, if we modify the basis. In a basis of principal directions, since the eigenvalues of the shape operator are just the opposite of the principal curvature, the matrix of the shape operator will be:

$$\mathcal{A} = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}$$

therefore

$$\begin{aligned} \det \mathcal{A} &= k_1 \cdot k_2 = K_t \\ -\frac{1}{2} \operatorname{Tr} \mathcal{A} &= -\frac{1}{2}(-k_1 - k_2) = \frac{1}{2}(k_1 + k_2) = K_m. \end{aligned}$$

□

Joachimstahl's theorem

We will see below how can one find the lines of curvature of a surface by integrating a differential equation. Anyway, in some special situations it is possible to find these lines using other methods. For instance, sometimes, if we know the curvature lines of a surface it is possible to find such lines on another surface. Such an instance is exemplified by the following theorem, belonging to the German mathematician Joachimstahl.

Theorem. *Let γ be a curve lying at the intersection of two regular oriented surfaces S_1 and S_2 from $\mathbb{R}^3$. Let $\mathbf{n}_i$ be the unit normals of the two surfaces ($i = \overline{1,2}$). Let us assume that S_1 and S_2 intersects under a constant angle, i.e. along the curve γ we have $\mathbf{n}_1 \cdot \mathbf{n}_2 = const.$. Then γ is a curvature line on S_1 iff it is a curvature line on S_2.*

Proof. Let $\mathbf{r} = \mathbf{r}(t)$ be a local parameterization of the curve γ. Then, since $\mathbf{n}_1 \cdot \mathbf{n}_2 =$ const, we have

$$0 = \frac{d}{dt}(\mathbf{n}_1 \cdot \mathbf{n}_2) = \mathbf{n}_1' \cdot \mathbf{n}_2 + \mathbf{n}_1 \cdot \mathbf{n}_2'.$$

If γ is a principal line on S_1, then

$$\mathbf{n}_1' = -k_1 \cdot \mathbf{r}',$$

where k_1 is one of the principal curvatures of the surface S_1. On the other hand, since the curve γ lies also on S_2, we have $\mathbf{r}' \perp \mathbf{n}_2$. From here and from the previous formula we obtain that $\mathbf{n}_1' \cdot \mathbf{n}_2 = 0$, therefore

$$\mathbf{n}_1 \cdot \mathbf{n}_2' = 0.$$

Since $\mathbf{n}_2' \perp \mathbf{n}_2$ (since $\mathbf{n}_2$ has constant length), it follows from here and from the previous equation that $\mathbf{n}_2' \perp \mathbf{r}'$ or, in other words, that there is a $k_2 \in \mathbb{R}$ such that

$$\mathbf{n}_2' = -k_2\mathbf{r}',$$

i.e. γ is a line of curvature on the surface S_2, too. □

Consequence. *The meridians and the parallels on a revolution surface are lines of curvature.*

Proof. Let γ be the rotating curve and S the resulted revolution surface. A meridian is obtained by intersecting a plane Π_m, passing through the rotation axis and S. If $p \in \Pi_m \cap S$, then the unit normal of the surface S, $\mathbf{n}(p)$, lies on the plane Π_m, therefore the normal of the surface and the normal of the plane Π_m make up a constant angle, equal to $\frac{\pi}{2}$. As for the plane the second fundamental form vanishes identically, the same is true for the shape operator, which means that all the plane curves are lines of curvature. It follows, that, in particular, the meridian is, also, a line of curvature in the plane Π_m, which implies, using the Joachimstahl theorem, that it is a line of curvature also for the surface S.

A parallel is an intersection curve between the surface S and a plane Π_p, passing through a point of the curve γ, perpendicular to the rotation axis. It is obvious, from symmetry reasons, that along a parallel we should have $\angle(\mathbf{n}, \Pi_p) = \text{const}$ and we apply again the reasoning we used before. □

4.16.1 The determination of the lines of curvature

As we saw earlier, the lines of curvature are curves on a surfaces whose tangent vectors are eigenvectors of the shape operator. Therefore, before showing how one can find the lines of curvature, we will indicate a way to find the eigenvectors of the shape operator.

Lemma. *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a local parameterization of an oriented surface* S. *A tangent vector* $\mathbf{v} = v_1\mathbf{r}_\mathbf{u}' + v_2\mathbf{r}_\mathbf{v}'$ *has principal direction iff*

$$\begin{vmatrix} v_2^2 & -v_1v_2 & v_1^2 \\ E & F & G \\ D & D' & D'' \end{vmatrix} = 0. \tag{4.16.4}$$

Proof. Since $\mathbf{v}$ has principal direction iff it is an eigenvector of the shape operator A, i.e. $A(\mathbf{v}) = \lambda \cdot \mathbf{v}$, it follows that $\mathbf{v}$ has principal direction iff $A(\mathbf{v}) \times \mathbf{v} = 0$. But, from the definition,

$$A(\mathbf{v}) = \mathcal{A} \cdot \mathbf{v} = \left(\mathcal{G}^{-1} \cdot \mathcal{H}\right) \cdot \mathbf{v} = \frac{1}{H^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$
$$= \frac{1}{H^2}\begin{pmatrix} (GL - FM)v_1 + (GM - FN)v_2 \\ (-FL + EM)v_1 + (-FM + EN)v_2 \end{pmatrix}$$

or

$$A(\mathbf{v}) = \frac{1}{H^2}\overbrace{[(GL - FM)v_1 + (GM - FN)v_2]}^{\alpha_u}\mathbf{r}_\mathbf{u}^{'} + \frac{1}{H^2}\underbrace{[(-FL + EM)v_1 + (-FM + EN)v_2]}_{\alpha_v}\mathbf{r}_\mathbf{v}^{'}.$$

Therefore,

$$A(\mathbf{v})\times\mathbf{v} = 0 \iff (\alpha_u\mathbf{r}_\mathbf{u}^{'} + \alpha_v\mathbf{r}_\mathbf{v}^{'})\times(v_1\mathbf{r}_\mathbf{u}^{'} + v_2\mathbf{r}_\mathbf{v}^{'}) = 0 \quad \iff (\alpha_u\cdot v_2 - \alpha_v\cdot v_1)\cdot(\mathbf{r}_\mathbf{u}^{'}\times\mathbf{r}_\mathbf{v}^{'}) = 0.$$

As $\mathbf{r}_\mathbf{u}^{'} \times \mathbf{r}_\mathbf{v}^{'} \neq 0$, since the surface is regular, it follows that

$$A(\mathbf{v}) \times \mathbf{v} = 0 \iff \alpha_u \cdot v_2 - \alpha_v \cdot v_1 = 0$$

or, having in mind the notations we made,

$$(FL - EM)v_1^2 + (GL - EN)v_1v_2 + (GM - FN)v_2^2 = 0.$$

Using now the coefficients of the second fundamental form instead of the components of the matrix $\mathcal{H}$, the previous relation becomes

$$(ED' - FD)v_1^2 + (ED'' - GD)v_1v_2 + (FD'' - GD')v_2^2 = 0. \tag{4.16.5}$$

which is just another form of the relation (4.16.4) □

Consequence (the differential equation of the lines of curvature)**.** *Let γ be a curve lying in the domain $\mathbf{r}(U)$ of a local parameterization$(\mathbf{r}, U)$ of a surface S, with the local equation $\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t))$. Then γ is a line of curvature on S iff*

$$(ED' - FD)u'^2(t) + (ED'' - GD)u'(t)v'(t) + (FD'' - GD')v'^2(t) = 0 \tag{4.16.6}$$

or

$$(ED' - FD) + (ED'' - GD)\frac{dv}{du} + (FD'' - GD')\left(\frac{dv}{du}\right)^2 = 0. \tag{4.16.7}$$

Proof. The relation (4.16.6) express, clearly, the condition for the vector $\boldsymbol{\rho}' = u'(t)\mathbf{r}_\mathbf{u}^{'} + v'(t)\mathbf{r}_\mathbf{v}^{'}$ to have principal direction, while (4.16.7) follows immediately from (4.16.6), by eliminating the parameter t. □

4.16.2 The computation of the curvatures of a surface

Theorem. *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a local parameterization of an oriented surface* S. *Then the total and the mean curvatures of* S *are given, respectively, by the formulas:*

$$K_t = \frac{DD'' - D'^2}{H^2} \tag{4.16.1}$$

$$K_m = \frac{DG - 2D'F + D''E}{2H^2}. \tag{4.16.2}$$

Proof. As we saw earlier, the matrix of the shape operator of a surface is given by

$$\mathcal{A} = \frac{1}{H^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} \quad \text{or} \quad \mathcal{A} = \frac{1}{H^2}\begin{pmatrix} FD' - GD & FD'' - GD' \\ FD - ED' & FD' - ED'' \end{pmatrix},$$

therefore,

$$K_t = \det \mathcal{A} = \frac{1}{H^4}\left[F^2D'^2 - EFD'D'' - FGDD' + EGDD'' - F^2DD'' + EFD'D'' + \right.$$

$$\left. + FGDD' - EGD'^2\right] = \frac{1}{H^4}\left[\underbrace{(EG - F^2)}_{H^2}\cdot(DD'' - D'^2)\right] = \frac{DD'' - D'^2}{H^2},$$

$$K_m = -\frac{1}{2}\mathrm{Tr}\mathcal{A} = -\frac{1}{2H^2}(2FD' - GD - ED'') = \frac{DG - 2D'F + D''E}{2H^2}.$$

□

Corollary. *The principal curvatures* k_1 *and* k_2 *are the roots of the equation*

$$k^2 - 2K_m \cdot k + K_t = 0, \tag{4.16.3}$$

i.e.

$$k_1 = K_m + \sqrt{K_m^2 - K_t}, \tag{4.16.4}$$

$$k_2 = K_m - \sqrt{K_m^2 - K_t}. \tag{4.16.5}$$

Corollary. *A non-flat point of a surface is*

1. *elliptic iff* $K_t > 0$;
2. *parabolic iff* $K_t = 0$;
3. *hyperbolic iff* $K_t < 0$.

4.17 The fundamental equations of a surface

4.17.1 Introduction

We saw that, in the case of curves, the curvature and the torsion completely determine a space curve (up to a rigid motion of the space). We may ask whether a similar result holds for the case of surfaces. It is not completely obvious what are the entities that should replace the curvature and the torsion, but, clearly, the first two fundamental forms could be good candidates. Thus, we can formulate the question in the following way: If we are given a domain $U \subset \mathbb{R}^2$ and two families of symmetric bilinear forms, with the coefficients depending smoothly on the coordinates on U, such that, at each point of U, the first form is positively defined, is there any regular parameterized surface $\mathbf{r} : U \to \mathbb{R}^3$ such that the two families of bilinear forms be the two first fundamental forms of this surface? The answer is not affirmative, because, as we shall see, the coefficients of the first two fundamental forms of a surface are not independent, therefore, our initial data should satisfy some compatibility conditions (which are, in fact, the integrability conditions for a system of partial differential equation). If, however, this conditions are fulfilled, then the negative answer turns into an affirmative one. It is the aim of this section to establish the compatibility conditions and to formulate the existence and uniqueness theorem for parameterized surfaces.

4.17.2 The differentiation rules. Christoffel's coefficients

If $(U, \mathbf{r})$ is a regular parameterized surface, then, for each $(u, v) \in U$, the vectors $\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{n}$ for a basis of the vector space $\mathbb{R}^3_{\mathbf{r}(u,v)}$. Therefore, in particular, the derivatives of these vectors can be expressed in terms of the vectors themselves. We already saw how to express the derivatives of the normal versor. They define, essentially, the shape operator of the surface. We shall obtain now similar formulae for the second order derivatives of the radius vector. These formulae should be of the form

$$
\begin{aligned}
\mathbf{r}''_{u^2} &= \Gamma^1_{11}\mathbf{r}'_u + \Gamma^2_{11}\mathbf{r}'_v + A\,\mathbf{n}\\
\mathbf{r}''_{uv} &= \Gamma^1_{12}\mathbf{r}'_u + \Gamma^2_{12}\mathbf{r}'_v + B\,\mathbf{n}\\
\mathbf{r}''_{v^2} &= \Gamma^1_{22}\mathbf{r}'_u + \Gamma^2_{22}\mathbf{r}'_v + C\,\mathbf{n}
\end{aligned}
\tag{4.17.1}
$$

The coefficients Γ from these equations are called the *Christoffel's coefficients (of the second kind)*. The coefficients A, B, C are easily identifiable as the coefficients of the second fundamental form. To get the others, first of all, we shall express the scalar products $\mathbf{r}''_{u^2} \cdot \mathbf{r}'_u, \mathbf{r}''_{u_2} \cdot \mathbf{r}'_v$ and the analogues in terms of the coefficients of the first fundamental form and its derivatives. We have, first of all, $\mathbf{r}_u'^2 = E$, whence, differentiating with respect to u, we get

$$\mathbf{r}_{u^2}'' \cdot \mathbf{r}_u' = \frac{1}{2}E_u'. \tag{4.17.2}$$

Differentiating the same equality with respect to v, we get

$$\mathbf{r}_{uv}'' \cdot \mathbf{r}_u' = \frac{1}{2}E_v' \tag{4.17.3}$$

Exactly in the same manner, starting from the definition of the other two coefficients of the first fundamental form, we will obtain

$$\mathbf{r}_{uv}'' \cdot \mathbf{r}_v' = \frac{1}{2}G_u' \tag{4.17.4}$$

$$\mathbf{r}_{u^2}'' \cdot \mathbf{r}_v' = F_u' - \frac{1}{2}E_v' \tag{4.17.5}$$

$$\mathbf{r}_{v^2}'' \cdot \mathbf{r}_u' = F_v' - \frac{1}{2}G_u' \tag{4.17.6}$$

$$\mathbf{r}_{v^2}'' \cdot \mathbf{r}_v' = \frac{1}{2}G_v' \tag{4.17.7}$$

Returning to our problem, we multiply the first equation of (4.17.1) successively by $\mathbf{r}_u'$ and by $\mathbf{r}_v'$. Using the formulae (4.17.2) and (4.17.5), as well as the definitions of the coefficients of the first fundamental form, we get the system

$$\begin{cases} E\Gamma_{11}^1 + F\Gamma_{11}^2 = \frac{1}{2}E_u' \\ F\Gamma_{11}^1 + G\Gamma_{11}^2 = F_u' - \frac{1}{2}E_u' \end{cases}.$$

It is very easy to solve this system and we get

$$\begin{cases} \Gamma_{11}^1 = \dfrac{GE_u' - 2FF_u' + FE_v'}{2(EG - F^2)} \\ \Gamma_{11}^2 = \dfrac{2EF_u' - EE_v' - FE_u'}{2(EG - F^2)} \end{cases} \tag{4.17.8}$$

and, exactly in the manner, starting from the other two equations from (4.17.1), we get

$$\begin{cases} \Gamma_{12}^1 = \dfrac{GE_v' - FG_u'}{2(EG - F^2)} \\ \Gamma_{12}^2 = \dfrac{EG_u' - FE_v'}{2(EG - F^2)} \end{cases} \tag{4.17.9}$$

and

$$\begin{cases} \Gamma^1_{22} = \dfrac{2GF'_v - GG'_u - FG'_v}{2(EG - F^2)} \\ \Gamma^2_{22} = \dfrac{EG'_v - 2FF'_v - FG'_u}{2(EG - F^2)} \end{cases}. \tag{4.17.10}$$

As for the derivatives of the normal versor, looking back at the expression of the shape operator in terms of the coefficients of the first two fundamental forms, we shall get immediately the formulae

$$\begin{cases} \mathbf{n}'_u = \dfrac{FD' - GD}{EG - F^2}\mathbf{r}'_u + \dfrac{FD - ED'}{EG - F^2}\mathbf{r}'_v \\ \mathbf{n}'_v = \dfrac{FD'' - GD'}{EG - F^2}\mathbf{r}'_u + \dfrac{FD' - ED''}{EG - F^2}\mathbf{r}'_v \end{cases} \tag{4.17.11}$$

Remark. The formulae (4.17.11) were obtained by the German mathematician Julius Weingarten, therefore they are called the *Weingartens's formulae*. Clearly, these formulae uniquely define the shape operator. This is the reason why in many books the shape operator (or its opposite) is called the *Weingarten's operator* or *Weingarten's mapping*.

Christoffel's and Weingarten's coefficients in curvature coordinates

Let us assume that in our parameterization the coordinate lines are lines of curvature. Then, as we saw earlier, we should have $F = 0$ and $D' = 0$ on the entire support of the parameterization. Then, as one can convince oneself easy, the Christoffel's coefficients become:

$$\begin{cases} \Gamma^1_{11} = \dfrac{1}{2}\dfrac{E'_u}{E} = \dfrac{\partial}{\partial u}\ln E, \quad \Gamma^2_{11} = -\dfrac{E'_v}{2G}, \\ \Gamma^1_{12} = \dfrac{\partial}{\partial v}\ln E, \quad \Gamma^2_{12} = \dfrac{\partial}{\partial u}\ln G, \\ \Gamma^1_{22} = -\dfrac{G'_u}{E}, \quad \Gamma^2_{22} = \dfrac{\partial}{\partial v}\ln G, \end{cases} \tag{4.17.12}$$

while the Weingarten's equation become:

$$\begin{cases} \mathbf{n}'_u = -\dfrac{D}{E}\mathbf{r}'_u \\ \mathbf{n}'_v = -\dfrac{D''}{G}\mathbf{r}'_v \end{cases} \tag{4.17.13}$$

One should not be surprised that the partial derivatives of the unit normal vector are, each of them, colinear to one of the partial derivative of the radius vector. This is, actually, nothing but the definition of the lines of curvatures.

4.17.3 The Gauss' and Codazzi-Mainardi's equations for a surface

We shall prove now that between the coefficients of the first two fundamental forms of a parameterized surface exists some relations, that we will call the *Gauss' equations* and the *Codazzi-Mainardi's equations*, respectively. We summarize them in the following theorem.

Theorem 4.17.1 (Gauss, Codazzi, Mainardi)**.** *In any local parameterization of a surface, the following systems of equations are fulfilled,*

$$\begin{aligned}
&\frac{\partial \Gamma_{11}^2}{\partial v} - \frac{\partial \Gamma_{12}^2}{\partial u} + \Gamma_{11}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{21}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2 = EK_t \\
&\frac{\partial \Gamma_{12}^1}{\partial u} - \frac{\partial \Gamma_{11}^1}{\partial v} + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1 = FK_t \\
&\frac{\partial \Gamma_{22}^1}{\partial u} - \frac{\partial \Gamma_{12}^1}{\partial v} + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - \Gamma_{12}^1\Gamma_{12}^1 - \Gamma_{12}^2\Gamma_{22}^1 = GK_t \\
&\frac{\partial \Gamma_{12}^2}{\partial v} - \frac{\partial \Gamma_{22}^2}{\partial u} + \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{22}^1\Gamma_{11}^2 = FK_t,
\end{aligned} \tag{4.17.14}$$

called the Gauss's equations*, and the relations*

$$\begin{aligned}
&\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} = D\Gamma_{12}^1 + D'(\Gamma_{12}^2 - \Gamma_{11}^1) - D''\Gamma_{11}^2 \\
&\frac{\partial D'}{\partial v} - \frac{\partial D''}{\partial u} = D\Gamma_{22}^1 + D'(\Gamma_{22}^2 - \Gamma_{12}^1) - D''\Gamma_{12}^2,
\end{aligned} \tag{4.17.15}$$

called the Codazzi-Mainardi's equations*. Here E, F, G are the coefficients of the first fundamental form of the surface, while D, D', D'' are the coefficients of the second fundamental form.*

Proof. To simplify a little bit the notations, let us rewrite the Weingarten's equations as

$$\begin{cases} \mathbf{n}'_u = a_{11}\mathbf{r}'_u + a_{12}\mathbf{r}'_v \\ \mathbf{n}'_v = a_{21}\mathbf{r}'_u + a_{22}\mathbf{r}'_v \end{cases}. \tag{4.17.16}$$

We have, obviously, the relation

$$\mathbf{r}'''_{u^2v} - \mathbf{r}'''_{uvu} = 0.$$

But

$$\mathbf{r}''_{u^2} = \Gamma_{11}^1\mathbf{r}'_u + \Gamma_{11}^2\mathbf{b}'_v + D\,\mathbf{n},$$

hence

$$
\begin{aligned}
\mathbf{r}'''_{u^2v} &= \frac{\partial\Gamma^1_{11}}{\partial v}\mathbf{r}'_u + \Gamma^1_{11}\mathbf{r}''_{uv} + \frac{\partial\Gamma^2_{11}}{\partial v}\mathbf{r}'_v + \Gamma^2_{11}\mathbf{r}''_{v^2} + \frac{\partial D}{\partial v}\,\mathbf{n} + D\,\mathbf{n}'_v = \\
&= \frac{\partial\Gamma^1_{11}}{\partial v}\mathbf{r}'_u + \Gamma^1_{11}(\Gamma^1_{12}\mathbf{r}'_u + \Gamma^2_{12}\mathbf{r}'_v + D'\,\mathbf{n}) + \\
&+ \frac{\partial\Gamma^2_{11}}{\partial v}\mathbf{r}'_v + \Gamma^2_{11}(\Gamma^1_{22}\mathbf{r}'_u + \Gamma^2_{22}\mathbf{r}'_v + D''\,\mathbf{n}) + \frac{\partial D}{\partial v}\,\mathbf{n} + D(a_{12}\mathbf{r}'_u + a_{22}\mathbf{r}'_v) = \\
&= \mathbf{r}'_u\left(\frac{\partial\Gamma^1_{11}}{\partial v} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + a_{12}D\right) + \\
&+ \mathbf{r}'_v\left(\frac{\partial\Gamma^2_{11}}{\partial v} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + a_{22}D\right) + \\
&+ \mathbf{n}\left(\frac{\partial D}{\partial v} + \Gamma^1_{11}D' + \Gamma^2_{11}D''\right).
\end{aligned}
$$

Analogously,

$$
\mathbf{r}''_{uv} = \Gamma^1_{12}\mathbf{r}'_u + \Gamma^2_{12}\mathbf{r}'_v + D'\,\mathbf{n},
$$

hence

$$
\begin{aligned}
\mathbf{r}'''_{uvu} &= \frac{\partial\Gamma^1_{12}}{\partial u}\mathbf{r}'_u + \Gamma^1_{12}\mathbf{r}''_{u^2} + \frac{\partial\Gamma^2_{12}}{\partial u}\mathbf{r}'_v + \Gamma^2_{12}\mathbf{r}''_{uv} + \frac{\partial D'}{\partial u}\,\mathbf{n} + D'\,\mathbf{n}'_u = \\
&= \frac{\partial\Gamma^1_{12}}{\partial u}\mathbf{r}'_u + \Gamma^1_{12}(\Gamma^1_{11}\mathbf{r}'_u + \Gamma^2_{11}\mathbf{r}'_v + D\,\mathbf{n}) + \\
&+ \frac{\partial\Gamma^2_{12}}{\partial u}\mathbf{r}'_v + \Gamma^2_{12}(\Gamma^1_{12}\mathbf{r}'_u + \Gamma^2_{12}\mathbf{r}'_v + D'\,\mathbf{n}) + \frac{\partial D'}{\partial u}\,\mathbf{n} + D'(a_{11}\mathbf{r}'_u + a_{21}\mathbf{r}'_v) = \\
&= \mathbf{r}'_u\left(\frac{\partial\Gamma^1_{12}}{\partial u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + a_{11}D'\right) + \\
&+ \mathbf{r}'_v\left(\frac{\partial\Gamma^2_{12}}{\partial u} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + a_{21}D'\right) + \\
&+ \mathbf{n}\left(\frac{\partial D'}{\partial u} + \Gamma^1_{12}D + \Gamma^2_{12}D'\right).
\end{aligned}
$$

If we subtract the previous relations, we obtain

$$\begin{aligned}0 = \mathbf{r}'''_{u^2v} - \mathbf{r}'''_{uvu} = \mathbf{r}'_u &\left(\frac{\partial \Gamma^1_{11}}{\partial v} - \frac{\partial \Gamma^1_{12}}{\partial u} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{12} + a_{12}D - a_{11}D'\right) + \\ + \mathbf{r}'_v &\left(\frac{\partial \Gamma^2_{11}}{\partial v} - \frac{\partial \Gamma^2_{12}}{\partial u} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + a_{22}D - a_{21}D'\right) + \\ + \mathbf{n} &\left(\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} + \Gamma^1_{12}D - D'(\Gamma^2_{12} - \Gamma^1_{11}) + \Gamma^2_{12}D''\right).\end{aligned}$$

As the vectors $\mathbf{r}'_u, \mathbf{r}'_v$ and $\mathbf{n}$ are linearly independent, a linear combination of them can vanish only if each coefficient vanishes. If we equate to zero the coefficient of $\mathbf{n}$, one sees that one gets the first of the Codazzi-Mainardi's equations. If we equate to zero the coefficient of $\mathbf{r}'_u$ follows, using the expressions of a_{21} and a_{22}, the first of the Gauss' equations, while from the coefficient of $\mathbf{r}'_v$ follow the second of the Gauss's equations. The other three equations can be obtained in the same way, using, this time, the relation

$$\mathbf{r}'''_{uv^2} = \mathbf{r}'''_{vuv}.$$

□

Corollary 4.17.1. *If the coordinate lines are lines of curvature, then the Codazzi-Mainardi equations can be written a lot simpler:*

$$\begin{cases} \dfrac{\partial D}{\partial v} = D\dfrac{\partial \ln E}{\partial v} + \dfrac{D''}{2G}\dfrac{\partial E}{\partial v} \\ \dfrac{\partial D''}{\partial u} = \dfrac{D}{2E}\dfrac{\partial G}{\partial u} + D''\dfrac{\partial \ln G}{\partial u}. \end{cases} \tag{4.17.17}$$

4.17.4 The fundamental theorem of surface theory

The theorem we are going to prove in this section is the analogue of the existence and uniqueness theorem for space curves. It was first established by the French mathematician Ossian Bonnet, in 1860.

To shorten the formulas, in the following we shall denote the coordinates by u^1 and u^2 instead of u and v and we shall use the index notation to denote the components of the matrices of the first two fundamental forms of the surface. More precisely, for the coefficients of the first fundamental form we shall write

$$g_{11} = E, g_{12} = g_{21} = F, g_{22} = G, \tag{4.17.18}$$

while for the coefficients of the second fundamental form we shall write

$$h_{11} = D, h_{12} = h_{21} = D', h_{22} = D''. \tag{4.17.19}$$

Also, we shall denote by g the determinant of the matrix of the first fundamental form. Finally, we shall denote by $\mathbf{r}'_{\mathbf{i}}$ the derivative of r with respect to the coordinate u^i and by $\mathbf{r}''_{\mathbf{ij}}$ – the second order derivative of r with respect to the coordinates u^i and u^j, where i and j can take the values 1 and 2. Clearly, as we always assume that the surfaces are as as smooth as one expects to be (i.e., in this context, at least C^2), the order of differentiation is not relevant, therefore we shall always have $\mathbf{r}''_{\mathbf{ij}} = \mathbf{r}''_{\mathbf{ji}}$.

Theorem (Ossian Bonnet, 1860)**.** *Let $U \subset \mathbb{R}^2$ be an open set. On U there are given the symmetric matrix functions*

$$g_{ij} = g_{ij}(u^1, u^2),\ h_{ij} = h_{ij}(u^1, u^2),\ i, j = 1, 2 \tag{4.17.20}$$

of classes C^2 and C^1, respectively, such that for each $(u^1, u^2) \in U$ the quadratic for associated to the bilinear form whose matrix is (g_{ij}) is positively defined and, moreover, the components of the two functions verify the Gauss and Codazzi-Mainardi compatibility conditions. We choose $u_0 = (u_0^1, u_0^2) \in U$, $p_0 \in \mathbb{R}$ and the vectors

$$\mathbf{r}'^{(0)}_{\mathbf{1}}, \mathbf{r}'^{(0)}_{\mathbf{2}},\ \mathbf{n}^{(0)} \in T_{p_0}\mathbb{R}^3, \tag{4.17.21}$$

such that $\mathbf{r}'^{(0)}_{\mathbf{i}} \cdot \mathbf{r}'^{(0)}_{\mathbf{j}} = g_{ij}(u_0)$, $\mathbf{n}^{(0)} \cdot \mathbf{r}'^{(0)}_{\mathbf{i}} = 0$, $\mathbf{n}^{(0)} \cdot \mathbf{n}^{(0)} = 1$, while the triple $\{\mathbf{r}'^{(0)}_{\mathbf{1}}, \mathbf{r}'^{(0)}_{\mathbf{2}}, \mathbf{n}^{(0)}\}$ is a right-handed basis of the vector space $T_{p_0}\mathbb{R}^3$. Then there exists a single regular parameterized surface of class C^3, $r : V \to \mathbb{R}^3$, with $V \subset U$ – an open set, such that the following conditions are fulfilled:

(i) $r(u_0) = p_0$ *(the surface "passes" through p_0 for $u = u_0$).*

(ii) $\dfrac{\partial r}{\partial u^i}(u_0) = \mathbf{r}'^{(0)}_{\mathbf{i}}$, $i = 1, 2$.

(iii) $\mathbf{n}(u_0) = \mathbf{n}^{(0)}$.

(iv) *g_{ij} and h_{ij} are the coefficients of the first two fundamental forms of the parameterized surface r (with respect to the orientation of r defined by the unit normal vector $\mathbf{n}$).*

Proof. We consider the system of partial differential equations

$$\begin{cases} \dfrac{\partial \mathbf{r_i'}}{\partial u^j} = \sum\limits_{k=1}^{2} \Gamma_{ij}^k \mathbf{r_k'} + h_{ij}\mathbf{n}, \\ \dfrac{\partial \mathbf{n}}{\partial u^i} = -\sum\limits_{j=1}^{2}\sum\limits_{k=1}^{2} h_{ij} g^{jk} \mathbf{r_k'} \end{cases} \tag{4.17.22}$$

with respect to the unknown functions $\mathbf{r_1'}, \mathbf{r_2'}$, $\mathbf{n}$, where the coefficients Γ_{ij}^k are computed with the formula (4.17.8)–(4.17.10). This is a linear and homogeneous system which is completely integrable, because the compatibility conditions

$$\begin{cases} \dfrac{\partial^2 \mathbf{r_i'}}{\partial u^j \partial u^k} = \dfrac{\partial^2 \mathbf{r_i'}}{\partial u^k \partial u^j} \\ \dfrac{\partial^2 \mathbf{n}}{\partial u^j \partial u^k} = \dfrac{\partial^2 \mathbf{n}}{\partial u^k \partial u^j} \end{cases} \tag{4.17.23}$$

are equivalent, as we saw, to the Gauss-Weingarten equations, which are satisfied, by hypothesis. Therefore, from standard results from the theory of partial differential equations of first order, it follows that there exists an open neighborhood $W \subset U$ of the point u_0 and a set of three C^2 vector functions[5] $\mathbf{r_1'}, \mathbf{r_2'}, \mathbf{n} : W \to \mathbb{R}^3$, which are solutions of the system (4.17.22), with the given initial condition at the point u_0.

Let us notice, also, that a set of initial conditions as in the theorem always exists, due, mainly, to the fact that the quadratic form associated to g_{ij} is positively defined. Indeed, we can make, for instance, the following choice:

$$\begin{cases} \mathbf{r_1'}^{(0)} = \left\{ \sqrt{g_{11}(u_0)}, 0, 0 \right\} \\ \mathbf{r_2'}^{(0)} = \left\{ \dfrac{g_{12}(u_0)}{\sqrt{g_{11}(u_0)}}, \dfrac{\sqrt{g_{11}(u_0) g_{22}(u_0) - (g_{12}(u_0))^2}}{\sqrt{g_{11}(u_0)}}, 0 \right\} \\ \mathbf{n}^{(0)} = \{0, 0, 1\} \end{cases} . \tag{4.17.24}$$

We leave to the reader to check that, indeed, these vectors verify the conditions from the hypothesis.

We consider, now, the system of partial differential equations

$$\begin{cases} \dfrac{\partial r}{\partial u^1} = \mathbf{r_1'}, \\ \dfrac{\partial r}{\partial u^2} = \mathbf{r_2'} \end{cases} . \tag{4.17.25}$$

[5] The order of smoothness is with one unit greater then the smallest order of smoothness of the coefficients, as the system is of first order.

This system is, again, completely integrable, because the integrability condition

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = \frac{\partial^2 r}{\partial u^j \partial u^i} \tag{4.17.26}$$

is equivalent to the condition

$$\frac{\partial \mathbf{r}'_{\mathbf{i}}}{\partial u^j} = \frac{\partial \mathbf{r}'_{\mathbf{j}}}{\partial u^i} \tag{4.17.27}$$

which is true, as one can convince oneself looking at the first equation (4.17.22), because of the symmetry of the second matrix ($h_{ji} = h_{ij}$) and because of the symmetry of the Christoffel's coefficients in the lower indices. Therefore, applying again the existence and uniqueness theorem, it follows that there exists an open neighborhood $V \subset W \subset U$ of u_0 and a single C^3 function $r : V \to \mathbb{R}^3$ such that $r(u_0) = p_0$.

We are not done yet, because we still have to show that g_{ij} and h_{ij} are the first two fundamental forms of the parameterized surface defined by r. Apparently, we also have to show that r is regular. But this follows immediately if we prove that g is the first fundamental form, because

$$g_{ij} = \frac{\partial r}{\partial u^i} \cdot \frac{\partial r}{\partial u^j}$$

and then

$$\frac{\partial r}{\partial u^i} \times \frac{\partial r}{\partial u^j} \neq 0,$$

because the square of the norm of this vector is not zero (as it is equal to the determinant of the first fundamental form, which is strictly greater then zero, since the form is positively defined). It is, actually, enough to show that, all over V, the following relations are fulfilled:

$$\begin{cases} \mathbf{r}'_{\mathbf{i}} \cdot \mathbf{r}'_{\mathbf{j}} = g_{ij}, \\ \mathbf{r}'_{\mathbf{i}} \cdot \mathbf{n} = 0, \\ \mathbf{n} \cdot \mathbf{n} = 1 \end{cases} . \tag{4.17.28}$$

To this end, we shall compute the derivatives with respect to the coordinates of the scalar products, taking into account the Gauss-Weingarten equations and we get the system of first order partial differential equations:

$$\begin{cases} \dfrac{\partial(\mathbf{r}'_{\mathbf{i}} \cdot \mathbf{r}'_{\mathbf{j}})}{\partial u^k} = \sum\limits_{l=1}^{2} \Gamma^l_{ik}(\mathbf{r}'_{\mathbf{l}} \cdot \mathbf{r}'_{\mathbf{j}}) + \sum\limits_{l=1}^{2} \Gamma^l_{jk}(\mathbf{r}'_{\mathbf{l}} \cdot \mathbf{r}'_{\mathbf{i}}) + h_{ik}(\mathbf{r}'_{\mathbf{j}} \cdot \mathbf{n}) + h_{jk}(\mathbf{r}'_{\mathbf{i}} \cdot \mathbf{n}) \\ \dfrac{\partial(\mathbf{r}'_{\mathbf{j}} \cdot \mathbf{n})}{\partial u^i} = -\sum\limits_{l,k=1}^{2} h_{il} g^{lk}(\mathbf{r}'_{\mathbf{k}} \cdot \mathbf{r}'_{\mathbf{j}}) + \sum\limits_{l=1}^{2} \Gamma^l_{ij}(\mathbf{r}'_{\mathbf{l}} \cdot \mathbf{n}) + h_{ij}(\mathbf{n} \cdot \mathbf{n}) \\ \dfrac{\partial(\mathbf{n} \cdot \mathbf{n})}{\partial u^i} = -2 \sum\limits_{l,k=1}^{2} h_{il} g^{lk}(\mathbf{r}'_{\mathbf{k}} \cdot \mathbf{n}) \end{cases} . \tag{4.17.29}$$

As probably the reader suspects already (and it is asked to check for himself), this system is, again, completely integrable, which means that it has a single solution for prescribed initial values. We "guess" that this solution is, exactly, (4.17.28) and we shall check this in the following. For the last equation, there is nothing to check: if we substitute (4.17.28), we get, simply, $0 = 0$. For the second equation, after the substitution, the left-hand side is zero, while the right-hand side becomes:

$$-\sum_{l,k=1}^{2} h_{il} g^{lk} g_{kj} + h_{ij} = -\sum_{l=1}^{2} h_{il} \delta^l_j + h_{ij} = -h_{ij} + h_{ij} = 0,$$

so we are done. Finally, the first equation becomes

$$\frac{\partial g_{ij}}{\partial u^k} = \sum_{l=1}^{2} \Gamma^l_{ik} g_{kj} + \sum_{l=1}^{2} \Gamma^l_{jk} g_{li}$$

which is very easy to prove, using the definition of the Christoffel's coefficients. Thus, the functions (4.17.28) give a solution of the system (4.17.29), which, obviously, verify the initial conditions of the theorem. As the solution is unique, for the given initial conditions we have

$$\begin{cases} \mathbf{r}'_\mathbf{i} \cdot \mathbf{r}'_\mathbf{j} = g_{ij}, \\ \mathbf{r}'_\mathbf{i} \cdot \mathbf{n} = 0, \\ \mathbf{n} \cdot \mathbf{n} = 1 \\ (\mathbf{r}'_\mathbf{1}, \mathbf{r}'_\mathbf{2}, \mathbf{n}) > 0 \end{cases}, \tag{4.17.30}$$

which means that we proved already that

- $\mathbf{r}'_\mathbf{i}$ are the derivatives of r;
- $\mathbf{n}$ is the unit normal of the parameterized surface given by r;
- g_{ij} are the coefficients of the first fundamental form of r.

We are left with the proof of the fact that h_{ij} are the coefficients of the second fundamental form of r. Let us denote, for the moment, by b_{ij} these coefficients. As we know, they are given by

$$b_{il} = -\mathbf{n}'_\mathbf{i} \cdot \mathbf{r}'_\mathbf{l} = \sum_{j,k=1}^{2} h_{ij} g^{jk} \mathbf{r}'_\mathbf{k} \cdot \mathbf{r}'_\mathbf{l} = \sum_{j,k=1}^{2} h_{ij} g^{jk} g_{kl} = \sum_{j=1}^{2} h_{ij} \delta^j_l = h_{il},$$

which concludes the proof of the Bonnet's theorem. □

4.18 The Gauss' egregium theorem

The theorem that we are going to prove in this section (and which is implicit in the equations of Gauss above) is one of the most important theorem in classical differential geometry. Not by accident Gauss called it, in his famous *Disquisitiones circa superficies curvas* "theorema egregium", i.e. the *remarkable theorem*. As we just saw in the previous section, the total curvature of a surface in $\mathbb{R}^3$ can be expressed in terms of the determinants of the first two fundamental forms of the surface. This would mean, in principle, that the total curvature depends both on the intrinsic data (the first fundamental form) and extrinsic data (i.e. the second fundamental form). It turns out, however, that the situation is different, i.e. we have:

Theorem 4.4 (Gauss, 1827)**.** *The total curvature of a surface of class at least C^3 depends only on the coefficients of the first fundamental form of the surface and their first order derivatives with respect to the coordinates.*

Proof. There exists several proofs of this theorem (which is called, in many books, using the Latin term used by Gauss, namely *theorema egregium*. The original proof, given by Gauss in 1827, is quite complicated. The proof we are going to give here is the first different proof, belonging to the German mathematician Richard Baltzer (1867)[6], although Struik credits the formula which will be established to the Italian mathematician Brioschi. We start with the formula

$$K_t = \frac{DD'' - D'^2}{EG - F^2},$$

which we rewrite in the form

$$K_t(EG - F^2) = DD'' - D'^2, \tag{4.18.1}$$

or, having in mind the expressions of the coefficients of the second fundamental form,

$$K_t(EG - F^2)^2 = (\mathbf{r}''_{\mathbf{u}^2}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) \cdot (\mathbf{r}''_{\mathbf{v}^2}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) - (\mathbf{r}''_{\mathbf{uv}}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}})^2. \tag{4.18.2}$$

The right hand side of the equation (4.18.2) can be written in a more convenient form if we notice that each term is, in fact, the product of two determinants. We shall use, now,

[6]The name of Richard Baltzer (1818–1887) is not well known today, however, in the second half of the nineteenth century he was considered an important geometer. For the history of mathematics, there are important his "Elementhe der Mathematik", published in several editions, where was mentioned, for the first time, the Non-Euclidean geometry.

the following formula for the product of two determinants, known from vector algebra:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) \cdot (\mathbf{d}, \mathbf{e}, \mathbf{f}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{d} & \mathbf{a} \cdot \mathbf{e} & \mathbf{a} \cdot \mathbf{f} \\ \mathbf{b} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{e} & \mathbf{b} \cdot \mathbf{f} \\ \mathbf{c} \cdot \mathbf{d} & \mathbf{c} \cdot \mathbf{e} & \mathbf{c} \cdot \mathbf{f} \end{vmatrix}. \tag{4.18.3}$$

Using the formula (4.18.3), the formula (4.18.2) becomes

$$\begin{aligned} K_t(EG - F^2)^2 = (\mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}''_{\mathbf{v}^2} - {\mathbf{r}''_{\mathbf{uv}}}^2)(EG - F^2) + \\ + \begin{vmatrix} 0 & \mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}'_{\mathbf{u}} & \mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}'_{\mathbf{v}} \\ \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}''_{\mathbf{v}^2} & E & F \\ \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{r}''_{\mathbf{v}^2} & F & G \end{vmatrix} - \\ - \begin{vmatrix} 0 & \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{u}} & \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{v}} \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{u}} & E & F \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{v}} & F & G \end{vmatrix}. \end{aligned} \tag{4.18.4}$$

Thus, already a part of the terms involved in the computation of the total curvature are expressed in terms of the coefficients of the first fundamental form. As one can see immediately, the remaining terms are of two kinds: either a product of a second order derivative of $\mathbf{r}$ and a first order one, either a product of two second order derivatives. The first kind of terms are easier to be taken care of. Indeed, starting from the definition of the coefficients of the first fundamental form, $E = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}'_{\mathbf{u}}$, $F = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}'_{\mathbf{v}}$, $G = \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{r}'_{\mathbf{v}}$ one obtains, differentiating with respect to the coordinates, the following expressions:

$$\begin{cases} \mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}'_{\mathbf{u}} & = \frac{1}{2}E'_u \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{u}} & = \frac{1}{2}E'_v \\ \mathbf{r}''_{\mathbf{v}^2} \cdot \mathbf{r}'_{\mathbf{v}} & = \frac{1}{2}G'_v \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{v}} & = \frac{1}{2}G'_u \\ \mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}'_{\mathbf{v}} & = F'_u - \frac{1}{2}E'_v \\ \mathbf{r}''_{\mathbf{v}^2} \cdot \mathbf{r}'_{\mathbf{u}} & = F'_v - \frac{1}{2}G'_u \end{cases}. \tag{4.18.5}$$

Differentiating once more the fourth equation above with respect to u and the fifth with respect to v and subtracting them side by side, we get also the expression for the products of second order derivatives of $\mathbf{r}$:

$$\mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}''_{\mathbf{v}^2} - {\mathbf{r}''_{\mathbf{uv}}}^2 = -\frac{1}{2}G''_{u^2} + F''_{uv} - \frac{1}{2}E''_{v^2}. \tag{4.18.6}$$

Combining everything we obtained, we get the following expression (due, as we said before, to Baltzer) for the total curvature

$$\begin{aligned} K_t = \frac{1}{(EG-F^2)^2} &\begin{vmatrix} -\frac{1}{2}G''_{u^2} + F''_{uv} - \frac{1}{2}E''_{v^2} & \frac{1}{2}E'_u & F'_u - \frac{1}{2}E'_v \\ F'_v - \frac{1}{2}G'_u & E & F \\ \frac{1}{2}G'_v & F & G \end{vmatrix} - \\ - \frac{1}{(EG-F^2)^2} &\begin{vmatrix} 0 & \frac{1}{2}E'_v & \frac{1}{2}G'_u \\ \frac{1}{2}E'_v & E & F \\ \frac{1}{2}G'_u & F & G \end{vmatrix}, \end{aligned} \tag{4.18.7}$$

which concludes the proof, as we got an expression of K_t which, indeed, depends only on the coefficients of the first fundamental form and their derivatives up to second order. □

Exercise 4.18.1 (Frobenius). Show that the total curvature of a surface can be written, also, in the following form, easier to remember:

$$\begin{aligned} K_t = -\frac{1}{4(EG-F^2)^2} &\begin{vmatrix} E & E'_u & E'_v \\ F & F'_u & F'_v \\ G & G'_u & G'_v \end{vmatrix} + \\ + \frac{1}{2\sqrt{EG-F^2}} &\left\{ \frac{\partial}{\partial u}\left(\frac{F'_v - G'_u}{\sqrt{EG-F^2}} \right) + \frac{\partial}{\partial v}\left(\frac{F'_u - E'_v}{\sqrt{EG-F^2}} \right) \right\}. \end{aligned} \tag{4.18.8}$$

In the particular case of an orthogonal coordinate system ($F \equiv 0$), we get the following nice "divergence" expression for the total curvature:

$$K_t = -\frac{1}{2\sqrt{EG}} \left\{ \frac{\partial}{\partial u}\left(\frac{G'_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v}\left(\frac{E'_v}{\sqrt{EG}} \right) \right\}, \tag{4.18.9}$$

which will be used later for some integral formulae.

Exercise 4.18.2 (Liouville). Prove the following (slightly asymmetric) formula for the total curvature of a surface:

$$K_t = -\frac{1}{2\sqrt{EG-F^2}} \left\{ \frac{\partial}{\partial u}\left(\frac{G'_u + \frac{F}{G}G'_v - 2F'_v}{\sqrt{EG-F^2}} \right) + \frac{\partial}{\partial v}\left(\frac{E'_v - \frac{F}{G}G'_u}{\sqrt{EG-F^2}} \right) \right\}. \tag{4.18.10}$$

4.19 Geodesics

4.19.1 Introduction

The curves we are going to study in this chapter are direct generalizations of the straight lines. More specifically, they are curves that projects on the tangent planes of the surface as straight lines. There are many different approaches to geodesics. We chose here the one we find more elementary.

4.19.2 The Darboux frame. The geodesic curvature and geodesic torsion

Let $(I, \boldsymbol{\rho})$ be a parameterized curve whose support lies on a surface S and let $M = \boldsymbol{\rho}(t_0)$ be a point of the curve, with $t_0 \in I$. As $\boldsymbol{\rho}$ is, in particular, a space curve, we can attach to it the Frenet frame at the point M, $\{M; \boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$. As we saw in the first part of the book, this frame is good enough if we want to investigate the curve $\boldsymbol{\rho}$ as an *independent* object, but it is not very helpful if one wishes to study the connections between the curve and the surface. To this end, we shall introduce another orthonormal frame, which involves both vectors related to the curve and to the surface. The first vector of the new frame will be still the unit tangent vector of the curve, $\boldsymbol{\tau}$. The second, related, this time to the surface, is the unit normal of the surface, $\mathbf{n}$. The third one, let it be denoted by $\mathbf{N}$, will be chosen in such a way that the basis $\{\boldsymbol{\tau}, \mathbf{N}, \mathbf{n}\}$ be direct, or in other words, such that $(\boldsymbol{\tau}, \mathbf{N}, \mathbf{n}) = 1$. This means, of course, that

$$\mathbf{N} = \mathbf{n} \times \boldsymbol{\tau}.$$

Clearly, $\mathbf{N}$ lies in the normal plane of the curve at M, therefore it will be called the *unit tangential normal vector* of the curve. The name *tangential* comes, of course from the fact that $\mathbf{N}$ lies, also, in the tangent plane of the surface at M.

The frame $\{M; \boldsymbol{\tau}, \mathbf{N}, \mathbf{n}\}$ is called the *Darboux frame* or the Ribaucour-Darboux frame of the surface S along the curve $\boldsymbol{\rho}$.

The next step we are going to make is to compute the derivatives of the vectors of the Darboux frame and to obtain a set of linear differential equation which is similar to the Frenet frame and which will play an important role in the following sections. To get them, the intention is exactly to use the Frenet equations. Therefore, we shall start by expressing the vectors $\mathbf{N}$ and $\mathbf{n}$ in terms of the vectors of the Frenet frame. We denote by θ the angle between the vectors $\boldsymbol{\nu}$ and $\mathbf{n}$. Then, as one can see immediately that

$$\begin{cases} \boldsymbol{\nu} = \cos(\mathbf{N}, \boldsymbol{\nu}) \cdot \mathbf{N} + \sin(\mathbf{N}, \boldsymbol{\nu})\mathbf{n} \\ \boldsymbol{\beta} = \cos(\mathbf{N}, \boldsymbol{\beta}) \cdot \mathbf{N} + \sin(\mathbf{N}, \boldsymbol{\beta})\mathbf{n} \end{cases}$$

As $(\mathbf{N}, \boldsymbol{\nu}) = \frac{\pi}{2}$ and $(\mathbf{N}, \boldsymbol{\beta}) = \pi - \theta$, we get

$$\begin{cases} \boldsymbol{\nu} = \sin\theta \mathbf{N} + \cos\theta \mathbf{n} \\ \boldsymbol{\beta} = -\cos\theta \mathbf{N} + \sin\theta \mathbf{n} \end{cases}.$$

Conversely, we get

$$\begin{cases} \mathbf{N} = \sin\theta \cdot \boldsymbol{\nu} - \cos\theta \cdot \boldsymbol{\beta} \\ \mathbf{n} = \cos\theta \cdot \boldsymbol{\nu} + \sin\theta \cdot \boldsymbol{\beta} \end{cases}.$$

Now, the derivatives of the vectors of the Darboux frame with respect to the arclength of the curve can be expressed in terms of these vectors as

$$\begin{cases} \boldsymbol{\tau} = a(s) \cdot \mathbf{N} + b(s)\mathbf{n} \\ \mathbf{N}' = c(s)\boldsymbol{\tau} + d(s) \cdot \mathbf{n} \\ \mathbf{n}' = e(s) \cdot \boldsymbol{\tau} + f(s) \cdot \mathbf{N}, \end{cases} \tag{4.19.1}$$

where a, b, c, d, e, f are smooth functions of the arclength. We remind that the derivative of a vector of the Darboux frame is perpendicular on that vector, because the frame is orthonormal. For the same reason, we deduce immediately that the six coefficients are not independent and, in fact, we have the relations: $c = -a, e = -b, f = -d$, therefore the system (4.19.1) becomes

$$\begin{cases} \boldsymbol{\tau} = a(s) \cdot \mathbf{N} + b(s)\mathbf{n} \\ \mathbf{N}' = -a(s)\boldsymbol{\tau} + d(s) \cdot \mathbf{n} \\ \mathbf{n}' = -b(s) \cdot \boldsymbol{\tau} - d(s) \cdot \mathbf{N}, \end{cases} \tag{4.19.2}$$

We shall express now the quantities a, b, d in terms of the characteristics of the curve and the angle θ. We notice, first of all, that, from the first of the Frenet's formula, we get

$$\boldsymbol{\tau}' = k\boldsymbol{\nu} = k(\sin\theta \cdot \mathbf{N} + \cos\theta \cdot \mathbf{n}),$$

hence, identifying the coefficients with those of the first equation of the system (4.19.2), we obtain

$$a = k \cdot \sin\theta; \quad b = k \cdot \cos\theta. \tag{4.19.3}$$

The quantity $k \cdot \cos\theta$ is already known to us: it is nothing but the *normal curvature* k_n of the surface that was studied previously. The function $k \cdot \sin\theta$, instead, is new. We will denote it by k_g and we will call it *geodesic* or *tangential* curvature of the curve[7]. To find the quantity d, we start from the relation

$$\mathbf{N} = \sin\theta \cdot \boldsymbol{\nu} - \cos\theta \cdot \boldsymbol{\beta}.$$

By differentiating with respect to the arclength of the curve, we get, using the last two of the Frenet's formulae:

$$\begin{aligned}\mathbf{N}' &= \theta' \cdot \sin\theta \cdot \boldsymbol{\nu} - \sin\theta(-k \cdot \boldsymbol{\tau} + \chi \cdot \boldsymbol{\beta}) + \theta' \sin\theta \cdot \boldsymbol{\beta} + \chi \cdot \cos\theta \cdot \boldsymbol{\nu} = \\ &= -k \cdot \sin\theta \cdot \boldsymbol{\tau} + (\theta' + \chi) \cdot (\cos\theta \cdot \boldsymbol{\nu} + \sin\theta \cdot \boldsymbol{\beta}) = \\ &= -k \cdot \sin\theta \cdot \boldsymbol{\tau} + (\theta' + \chi) \cdot \mathbf{n},\end{aligned}$$

and, comparing with the second equation from (4.19.2), we get

$$d = \theta' + \chi.$$

This function is denoted by χ_g and it s called the *geodesic torsion*. His geometric meaning will be made clear later. Clearly, we cannot claim, as we did with the geodesic curvature, that the geodesic torsion is the torsion of the projection of the curve on the tangent plane, as the torsion of that curve is always zero, while the geodesic torsion of the given curve is not, usually.

The geodesic curvature plays a much more important role in the differential geometry of surfaces than the geodesic torsion does, so we start by focusing on it. We notice, to begin with, that

$$k_g = \boldsymbol{\tau}' \cdot \mathbf{N} = -\boldsymbol{\tau} \cdot \mathbf{N}'. \tag{4.19.4}$$

Since, as we saw earlier, $\mathbf{N} = \mathbf{n} \times \boldsymbol{\tau}$, one obtains for k_g the expression

$$k_g = \boldsymbol{\tau}' \cdot \mathbf{N} = \boldsymbol{\tau}' \cdot (\mathbf{n} \times \boldsymbol{\tau}),$$

i.e.

$$k_g = (\boldsymbol{\tau}, \boldsymbol{\tau}', \boldsymbol{n}). \tag{4.19.5}$$

This formula holds for naturally parameterized curves. Let us consider, now, an arbitrary regular parameterized curve on S, given by the local equations $u = u(t), v = v(t)$. We have, therefore, $\mathbf{r} = \mathbf{r}(u(t), v(t))$. If we denote by a dot the differentiation with respect to the parameter t along the curve, we get

$$\boldsymbol{\tau} \equiv \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{1}{\dot{s}} \cdot \dot{\mathbf{r}},$$

[7]Here the term "tangential" refers to the tangent plane of the surface, not to the tangent line of the curve. In fact, it can be shown that the geodesic curvature at a point of a curve lying on a surface is the signed curvature of the projection of the curve on the tangent plane to the surface at that particular point.

therefore

$$\boldsymbol{\tau}' = \frac{d\boldsymbol{\tau}}{ds} = \frac{1}{\dot{s}^3}(\ddot{\mathbf{r}} \cdot \dot{s} - \dot{\mathbf{r}}\ddot{s}).$$

Thus, for the geodesic curvature one gets

$$k_g = (\boldsymbol{\tau}, \boldsymbol{\tau}', \mathbf{n}) = \left(\frac{1}{\dot{s}}\dot{\mathbf{r}}, \frac{1}{\dot{s}^3}(\ddot{\mathbf{r}} \cdot \dot{s} - \dot{\mathbf{r}}\ddot{s}), \mathbf{n}\right) = \frac{1}{\dot{s}^4}(\dot{\mathbf{r}}, \ddot{\mathbf{r}} \cdot \dot{s} - \dot{\mathbf{r}}\ddot{s}, \mathbf{n}) = \frac{1}{\dot{s}^4}(\dot{\mathbf{r}}, \dot{s} \cdot \ddot{\mathbf{r}}, \mathbf{n}),$$

i.e.

$$k_g = \frac{1}{\dot{s}^3}(\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \mathbf{n}). \tag{4.19.6}$$

The formulas we got so far for the geodesic curvature use mixture of information about the curve and about the surface, but they don't use explicitly the things we usually compute when we are given a local parameterization of a surface or, more generally, a regular parameterized surface, namely the coefficients of the first two fundamental forms. So, the next step will be to obtain a formula for the geodesic curvature in terms of the coefficients of the fundamental forms, when the curve is given by its local parametric equations with respect to a local parameterization of the surface. We shall discover, in fact, that the geodesic curvature can be expressed in terms of the coefficients of the first fundamental form and they first order partial derivatives with respect to the coordinates.

To simplify the notations, we shall use, for a while, the index notations. In other words, we shall denote the coordinates with u^1 and u^2, instead of u and v, while the first partial derivatives of the radius vector $\mathbf{r}$ with respect to the coordinates will be denoted by $\mathbf{r}'_i$ and those of second order by $\mathbf{r}''_{ij}$, $i, j = 1, 2$. Moreover, we shall use the *Einstein's summation convention*: every time an index enter twice in a monomial, once in an inferior position and once in a superior one, then one makes summation after all the allowed values of the index (in our case, 1 and 2). Thus, an expression of the form $a_i u^i$ should be read as

$$a_i u^i = a_1 u^1 + a_2 u^2.$$

Going back to our problem, we have, with the newly introduced notations:

$$\begin{cases} \dot{\mathbf{r}} & \equiv \frac{d\mathbf{r}}{dt} = \mathbf{r}'_i \dot{u}^i \\ \ddot{\mathbf{r}} & = \mathbf{r}''_{ij}\dot{u}^i\dot{u}^j + \mathbf{r}'_k \ddot{u}^k \end{cases}.$$

The decomposition of the second order partial derivatives of the radius vectors in terms of the basis $\{\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{n}\}$ is already known to us: it can be described using the Christoffel's coefficients and the second fundamental form as

$$\mathbf{r}''_{ij} = \Gamma^k_{ij}\mathbf{r}'_k + h_{ij} \cdot \mathbf{n},$$

where, as we know, h_{ij} are the coefficients of the second fundamental form of the surface. Thus, we have

$$\ddot{\mathbf{r}} = \left(\Gamma^k_{ij}\mathbf{r}'_k + h_{ij}\cdot\mathbf{n}\right)\dot{u}^i\dot{u}^j + \mathbf{r}'_k\ddot{u}^k = \left(\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j\right)\mathbf{r}'_k + h_{ij}\dot{u}^i\dot{u}^j\mathbf{n}$$

or

$$\ddot{\mathbf{r}} = \left(\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j\right)\mathbf{r}'_k + \varphi_2(\dot{\mathbf{r}},\dot{\mathbf{r}})\cdot\mathbf{n}.$$

With these in hand, we can write now

$$\begin{aligned} k_g =& \frac{1}{\dot{s}^3}\left(\dot{\mathbf{r}},\ddot{\mathbf{r}},\mathbf{n}\right) = \frac{1}{\dot{s}^3}\left(\dot{u}^m\mathbf{r}'_m,\left(\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j\right)\mathbf{r}'_k + \varphi_2(\dot{\mathbf{r}},\dot{\mathbf{r}}),\mathbf{n}\right) = \\ =& \frac{1}{\dot{s}^3}\left[\dot{u}^1\left(\ddot{u}^2 + \Gamma^2_{ij}\dot{u}^i\dot{u}^j\right) - \dot{u}^2\left(\ddot{u}^1 + \Gamma^1_{ij}\dot{u}^i\dot{u}^j\right)\right]\left(\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{n}\right). \end{aligned}$$

But

$$\left(\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{n}\right) = \left(\mathbf{r}'_1\times\mathbf{r}'_2\right)\cdot\mathbf{n} = \left(\mathbf{r}'_1\times\mathbf{r}'_2\right)\cdot\frac{\mathbf{r}'_1\times\mathbf{r}'_2}{\|\mathbf{r}'_1\times\mathbf{r}'_2\|} = \|\mathbf{r}'_1\times\mathbf{r}'_2\| = \sqrt{g},$$

where g is the determinant of the first fundamental form, whence

$$k_g = \frac{\sqrt{g}}{\dot{s}^3}\left[\dot{u}^1\left(\ddot{u}^2 + \Gamma^2_{ij}\dot{u}^i\dot{u}^j\right) - \dot{u}^2\left(\ddot{u}^1 + \Gamma^1_{ij}\dot{u}^i\dot{u}^j\right)\right]. \tag{4.19.7}$$

If, in particular, the curve is naturally parameterized, we get

$$k_g = \sqrt{g}\left[(u^1)'\left((u^2)'' + \Gamma^2_{ij}(u^i)'(u^j)'\right) - (u^2)'\left((u^1)' + \Gamma^1_{ij}(u^i)'(u^j)'\right)\right]. \tag{4.19.8}$$

In the traditional notations, the formula for the geodesic curvature of a naturally parameterized curve is

$$\begin{aligned} k_g = \sqrt{EG-F^2}&\left[\Gamma^2_{11}u'^3 + \left(2\Gamma^2_{12} - \Gamma^1_{11}\right)u'^2v' + \left(\Gamma^2_{22} - 2\Gamma^1_{12}\right)u'v'^2 - \right.\\ &\left. -\Gamma^1_{22}v'^3 + u'v'' - u''v'\right]. \end{aligned} \tag{4.19.9}$$

This equation can also be written in a more elegant form as

$$k_g = \sqrt{EG-F^2}\det\begin{pmatrix} u' & u'' + u'^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + v'^2\Gamma^1_{22} \\ v' & v'' + u'^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + v'^2\Gamma^2_{22}\end{pmatrix}. \tag{4.19.10}$$

If, in particular, the surface S is the plane, with the Cartesian coordinates on it, then the determinant of the first fundamental form is, of course, one, as the matrix of the fundamental form is the unit matrix, while all the Christoffel's coefficients vanish. As a consequence, in this situation the geodesic curvature of the curve (which is, in this case, a plane curve), is nothing but its *signed curvature*:

$$k_g = u'v'' - u''v' \equiv k_\pm.$$

Remark. It can be shown that, in fact, the geodesic curvature of a curve on a surface is nothing but the signed curvature of the projection of the curve on the tangent plane.

4.19.3 Geodesic lines

Definition 4.19.1. Let S be a surface. A parameterized curve $\boldsymbol{\rho} : I \to S$ is called a *geodesic line*, or simply, a *geodesic* if, at each point, its geodesic curvature vanishes.

Remark. According to the formula (4.19.12), the geodesic curvature of a curve can be written as

$$k_g = (\boldsymbol{\tau}, \boldsymbol{\tau}', \boldsymbol{n}).$$

On the other hand, from the first formula of Frenet, $\boldsymbol{\tau} \times \boldsymbol{\tau}' = k\boldsymbol{\beta}$, therefore, k_g vanishes at a point of a curve on the surface if and only if the binormal of the curve is perpendicular on the normal of the surface at that point or, in other words, *the geodesic curvature vanishes if and only if the normal of the surface is contained in the osculating plane*. Thus, the geodesic lines are exactly those lines on the surface for which the osculating plane at each of their points contain the normal of the surface at that point. In fact, in many books, this property is taken as the definition of the geodesics.

Another remark that should be made is that since, as we noticed earlier, the geodesic curvature is the signed curvature of the projection of the curve on the tangent plane, we can say that the geodesics are those curves which project on each tangent plane of the surface after a straight line. In this sense, we might say that the geodesics are the *straightest* lines on the surfaces.

The formula 4.19.9 leads to

Theorem 4.5. *The differential equation of the geodesic lines is*

$$\Gamma_{11}^2 u'^3 + \left(2\Gamma_{12}^2 - \Gamma_{11}^1\right) u'^2 v' + \left(\Gamma_{22}^2 - 2\Gamma_{12}^1\right) u' v'^2 - \Gamma_{22}^1 v'^3 + u' v'' - u'' v' = 0. \tag{4.19.11}$$

Also, from (4.19.10) we can deduce that

Theorem 4.6. *The geodesic lines verify the system of differential equations*

$$\begin{cases} u'' + u'^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + v'^2\Gamma_{22}^1 = 0 \\ v'' + u'^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + v'^2\Gamma_{22}^2 = 0 \end{cases}. \tag{4.19.12}$$

There is, apparently, a contradiction between the previous two theorems, as the first claim that the geodesics are solutions of a single equations, while the second – that they are a solution of a system of different equation. In fact, however, the two equations of the system (4.19.12) are not independent, because we impose the geodesics to be *naturally parameterized*, therefore, between the functions u and v there is an extra relation.

The existence (locally, at least) of geodesics passing through a point of a surface and having at that point a given tangent vector is a consequence of standard results in the theory of ordinary differential equations. Finding geodesics, is, usually, a very delicate business and can be done explicitly only in special situations.

Examples of geodesics

The geodesics of the plane. By the geometrical interpretation of the geodesics, as the curves that project on the tangent plane on straight lines, we deduce immediately that the geodesics of the plane are the straight lines and only them. On the other hand, by using cartesian coordinates, we fined immediately that the Christoffel's coefficients vanish identically, therefore the equations of geodesics become

$$\begin{cases} u'' = 0 \\ v'' = 0 \end{cases},$$

which lead to $u(s) = a_1 s + b_1$, $v(s) = a_2 s + b_2$, i.e., again, the geodesics are straight lines.

The geodesics of the sphere. The geodesics of the sphere can be found very easily from their interpretation as being those curves for which the osculating plane contains the normal to the surface. As we know, in the case of the sphere all the normals are passing through the center of the sphere, therefore the osculating planes should pass, all of them through the center, which leads immediately to the idea that the geodesics are arcs of great circles of the sphere.

Also, using a standard parameterization of the sphere (with spherical coordinates), we can find (do that!) the equations of the geodesics as:

$$\begin{cases} \ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0 \\ \ddot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi} = 0 \end{cases}.$$

We assume that the explicit equation of the geodesic is $\theta = \theta(\varphi)$. Then

$$\dot{\theta} = \frac{d\theta}{ds} = \frac{d\theta}{d\varphi} \cdot \dot{\varphi},$$

$$\ddot{\theta} = \frac{d}{ds}\left(\frac{d\theta}{d\varphi}\right) \cdot \dot{\varphi} + \frac{d\theta}{d\varphi}\ddot{\varphi} = \frac{d\theta}{d\varphi} \cdot \ddot{\varphi} + \frac{d^2\theta}{d\varphi^2} \cdot \dot{\varphi}^2.$$

Therefore, the first equation of the system becomes:

$$\ddot{\varphi} \cdot \frac{d\theta}{d\varphi} + \frac{d^2\theta}{d\varphi^2} \cdot \dot{\varphi}^2 - \sin\theta\cos\theta \cdot \dot{\varphi}^2 = 0.$$

On the other hand, from the first equation,

$$\ddot{\varphi} = -2\cot\theta\dot{\theta} \cdot \dot{\varphi} = -2\cot\theta\frac{d\theta}{d\varphi} \cdot \dot{\varphi}^2,$$

hence:

$$\dot{\varphi}^2\left(\frac{d^2\theta}{d\varphi^2} - 2\cot\theta\cdot\frac{d\theta}{d\varphi} - \sin\theta\cos\theta\right) = 0.$$

We have two possibilities: either $\dot{\varphi} = 0$, i.e. $\varphi = const$, and, in this case, the curve is, obviously, a great circle of the sphere (a meridian), either

$$\frac{d^2\theta}{d\varphi^2} - 2\cot\theta\cdot\frac{d\theta}{d\varphi} - \sin\theta\cos\theta = 0.$$

In this case, we make the substitution $z = \cot\theta$ and, after a straightforward computation, we get

$$\frac{d^2 z}{d\varphi^2} + z = 0,$$

and this equation has the general solution

$$z = \cot\theta = A\cos\varphi + B\sin\varphi$$

or

$$A\sin\theta\cos\varphi + B\sin\theta\sin\varphi - \cos\theta = 0,$$

which is the equation of a great circle, lying in the plane passing through the origin and having as normal vector the vector $(A, B, -1)$.

4.19.4 Liouville surfaces

Definition 4.19.2. A surface is called a *Liouville surface* if it can be parameterized locally in such a way that the first fundamental form can be written as

$$ds^2 = (U(u) + V(v))(du^2 + dv^2), \tag{4.19.13}$$

where U and V are smooth functions of a single variable.

These surfaces have been introduced by the French mathematician Joseph Liouville in the *Note III* of the 5th edition of the book of Gaspard Monge, *Application*. In particular, the surfaces of revolution are examples of Liouville surfaces. The most important characteristic of Liouville surfaces is the fact that their geodesics an be found through quadratures. This is exactly what we are going to prove in the rest of this section.

First of all, it is an easy matter to prove that the equation of the geodesics for the metric (4.19.13) can be written as

$$(U'v' - V'u')(u'^2 + v'^2) + 2(U + V)(u'v'' - v'u'') = 0. \tag{4.19.14}$$

Here, for the functions U and V the prime denotes the derivative with respect to u and v, respectively, while for the coordinate functions u and v the prime denotes the derivative with respect to the parameter t along the geodesic. Of course, U and V are, both, functions of t, through the coordinate functions. Therefore, the equation of geodesics can be rewritten as

$$\frac{u'}{v'}\frac{dU}{dt} - \frac{v'}{u'}\frac{dV}{dt} + 2(U+V)\frac{u'v'' - v'u''}{u'^2 + v'^2} = 0. \tag{4.19.15}$$

We see that this equation actually does not involve the coordinate functions u and v, but only their derivatives. We substitute them with other two functions of t, ρ and α, defined by

$$\begin{cases} u' = \rho\cos\alpha \\ v' = \rho\sin\alpha \end{cases}. \tag{4.19.16}$$

In the sequel, the equation of the geodesics becomes

$$\sin^2\alpha\frac{dU}{dt} - \cos^2\alpha\frac{dV}{dt} + 2(U+V)\sin\alpha\cos\alpha\frac{d\alpha}{dt} = 0.$$

This equation can be also written as

$$\frac{d}{dt}\left(U\sin^2\alpha - V\cos^2\alpha\right) = 0,$$

whence

$$U\sin^2\alpha - V\cos^2\alpha = a,$$

where a is a constant. Returning to the old coordinate functions, we get

$$v'^2U - u'^2V = a(u'^2 + v'^2),$$

whence

$$\int\frac{du}{\sqrt{U-a}} = \pm\int\frac{dv}{\sqrt{V+a}} + b, \tag{4.19.17}$$

where b is another integration constant.

CHAPTER 5

Special classes of surfaces

5.1 Ruled surfaces

Intuitively speaking, the ruled surfaces are the surfaces generated by a moving straight line, which is subject to a further condition, usually this condition being that the moving line remains tangent to a given surface or intersects a given regular space curve. More precisely

5.1.1 General ruled surfaces

After the plane, the ruled surfaces are, no doubt, the simplest surfaces. We recall that a surface is called *ruled* if it is generated by a straight line (the *ruling*, moving in space, lying all the time on a given curve, called the *directrix*. Clearly, once the ruled surface was defined, the directrix can be replaced by another curve , obtained, for instance, by sectioning the surface by a plane or a sphere. In particular, it is, usually, convenient to choose a directrix which is, at each point, orthogonal to the ruling passing through that point (in other words, the directrix is an *orthogonal trajectory* of the rulings. As we shall see shortly, this particular choice of the ruling has as effect the diagonalization of the first fundamental form of the surface.

The simplest ruled surfaces are the *cylindrical surfaces*, for which the rulings are always parallel to a fixed direction and the *conical surfaces*, in the case of which the rulings are passing through a fixed point. The ruled surfaces have been studied in details

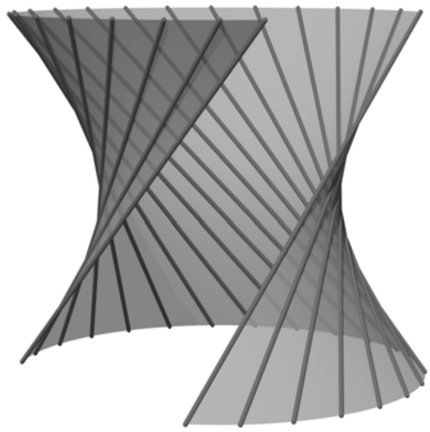


Figure 5.1: The hyperboloid with one sheet and its rulings

for the first time by the French geometer Gaspard Monge, at the end of the XVIIIth century and most of the important results in the field belong to him or to some of his students. In the figure 5.1.1 we represented the hyperboloid of rotation with one sheet, together some of its rulings.

Another way of getting interesting ruled surfaces is to take as rulings the axes of the Frenet frame of a space curve. For instance, in the figure 5.1.1 we represented the surface described by the binormals of the Viviani's temple.

The parameterization of a ruled surface

We assume that the directrix of the surface has a parameterization of the form

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u), \quad u \in I,$$

where I is an interval on the real axis. For each $u \in I$, we denote by $\mathbf{b}(u)$ the versor of the ruling passing through the point $\boldsymbol{\rho}(u)$. Then, if M is a point on this ruling, its coordinates will be determined by the relation

$$\mathbf{r} = \mathbf{r}(u, v) = \boldsymbol{\rho}(u) + v\mathbf{b}(u), \tag{5.1.1}$$

where v is the parameter along the ruling, i.e. if $M_0 = \boldsymbol{\rho}(\boldsymbol{u})$, then we have

$$\overrightarrow{M_0M} = v\mathbf{b}(u).$$

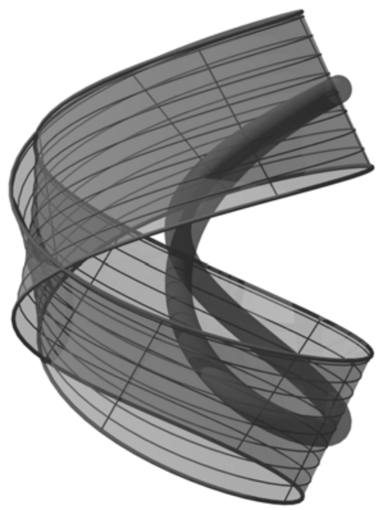


Figure 5.2: The surface of binormals of the Viviani's temple

The relation 5.1.1 provides a parameterization of the ruled surface,

$$\mathbf{r} : I \times \mathbb{R} \to S.$$

This parameterization is *not*, usually, global, as the parameterization of the directrix is not global.

With respect to this parameterization, the rulings will be coordinate lines ($u = const$), and the directrix is, equally, a coordinate line ($v = 0$). Generally, the coordinate lines $v = const$ have the property that they are "parallel" to the directrix, in the sense that all the points from such a coordinate curve lie at the same distance (equal to $|v|$) from the directrix, when we measure the distance along the ruling passing through each point.

The tangent plane and the first fundamental form of a ruled surface

To compute the coefficients of the first fundamental form of a ruled surface we need, first of all, the partial derivatives of the radius vector of a point of the surface. We have, obviously,

$$\mathbf{r}'_u = \boldsymbol{\rho}' + b\mathbf{b}'; \quad \mathbf{r}'_v = \mathbf{b}'_u. \tag{5.1.2}$$

Thus, the coefficients of the first fundamental form of the surface will be

$$\begin{aligned} E &\equiv \mathbf{r}'_u \cdot \mathbf{r}'_u = \boldsymbol{\rho}'^2 + 2v\boldsymbol{\rho}' \cdot \mathbf{b}' + v^2\mathbf{b}'^2; \\ F &\equiv \mathbf{r}'_u \cdot \mathbf{r}'_v = \boldsymbol{\rho}' \cdot \mathbf{b}; \\ G &\equiv \mathbf{r}'_v \cdot \mathbf{r}'_u = 1. \end{aligned} \tag{5.1.3}$$

It follows that the first fundamental form of a ruled surface can be written as:

$$ds^2 = \left(\boldsymbol{\rho}'^2 + 2v\boldsymbol{\rho}' \cdot \mathbf{b}' + v^2\mathbf{b}'^2\right) du^2 + 2(\boldsymbol{\rho}' \cdot \mathbf{b})\, dudv + dv^2. \tag{5.1.4}$$

To find the tangent plane at a point of a ruled surface, we notice, first of all, that the direction of the normal to the plane (and, hence, to the surface) at a given point is given by the vector $\mathbf{r}'_u \times \mathbf{r}'_v$, i.e. by the vector

$$\mathbf{N} \equiv \mathbf{r}'_u \times \mathbf{r}'_v = \boldsymbol{\rho}' \times \mathbf{b} + v\,(\mathbf{b}' \times \mathbf{b}). \tag{5.1.5}$$

Therefore, if $\mathbf{R}$ is the position vector of a point from the tangent plane to the surface at a point corresponding to the pair of parameters(u, v), then the equation of the tangent plane can be written under the form

$$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{N} = 0,$$

i.e.

$$(\mathbf{R} - \boldsymbol{\rho} - v\mathbf{b}) \cdot (\boldsymbol{\rho}' \times \mathbf{b} + v\,(\mathbf{b}' \times \mathbf{b})) = 0$$

or

$$(\mathbf{R}, \boldsymbol{\rho}', \mathbf{b}) + v\,(\mathbf{R}, \mathbf{b}', \mathbf{b}) - (\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{b}) - v\,(\boldsymbol{\rho}, \mathbf{b}', \mathbf{b}) = 0$$

or, also,

$$[\mathbf{R} \times \boldsymbol{\rho}' + v\,(\mathbf{R} \times \mathbf{b}') - \boldsymbol{\rho} \times \boldsymbol{\rho}' - v\,(\boldsymbol{\rho} \times \mathbf{b}')] \cdot \mathbf{b} = 0. \tag{5.1.6}$$

A characteristic property of the ruled surfaces is described by the following proposition:

Proposition 5.1.1. *The tangent planes to a ruled surface in point located along the same ruling, belong to the pencil of planes determined by that ruling, or, to put it another way, the tangent plane at a point of a ruled surface contains the ruling passing through that point.*

Proof. Since the ruling and the tangent plane already have a point in common (the very point of tangency), it is enough to prove that the ruling is parallel to the tangent plane or, which is the same, that it is perpendicular to the normal to the surface at the tangency point. We have, indeed,

$$\mathbf{N} \cdot \mathbf{b} = [\boldsymbol{\rho}' \times \mathbf{b} + v\,(\mathbf{b}' \times \mathbf{b})] \cdot \mathbf{b} = (\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}') + v\,(\mathbf{b}', \mathbf{b}, \mathbf{b}') = 0.$$

□

The proposition we just proved indicates how varies the tangent plane to a ruled surface along a ruling: it just rotates around the ruling. In other words, its normal vector stays all the time parallel to a fixed plane, which is perpendicular to the considered ruling.

5.1.2 The Gaussian curvature of a ruled surface

We start, of course, from the parameterization (5.1.1). Then we have

$$\mathbf{r}'_u = \boldsymbol{\rho}' + v\mathbf{b}', \quad \mathbf{r}'_v = \mathbf{b}.$$

Therefore, for the first fundamental form, we will get

$$E = \mathbf{r}'_u \cdot \mathbf{r}'_u = (\boldsymbol{\rho}' + v\mathbf{b}') \cdot (\boldsymbol{\rho}' + v\mathbf{b}') = \boldsymbol{\rho}'^2 + 2v\boldsymbol{\rho}' \cdot \mathbf{b}' + v^2\mathbf{b}'^2,$$

$$F = \mathbf{r}'_u \cdot \mathbf{r}'_v = (\boldsymbol{\rho}' + v\mathbf{b}') \cdot \mathbf{b} = \boldsymbol{\rho}' \cdot \mathbf{b},$$

while

$$G = \mathbf{r}'_v \cdot \mathbf{r}'_v = 1,$$

since $\mathbf{b}$ is a versor. Let H be the determinant of the first fundamental form. Then, for the coefficients of the second fundamental form, we will get

$$D = \frac{1}{H}\left(\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{r}''_{u^2}\right) = \frac{1}{H}\left(\boldsymbol{\rho}' + v\mathbf{b}', \mathbf{b}, \boldsymbol{\rho}'' + v\mathbf{b}''\right),$$

$$D' = \frac{1}{H}\left(\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{r}''_{uv}\right) = \frac{1}{H}\left(\boldsymbol{\rho}' + v\mathbf{b}', \mathbf{b}, \mathbf{b}'\right) = \frac{1}{H}\left(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}'\right),$$

$$D'' = \frac{1}{H}\left(\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{r}''_{v^2}\right) = 0.$$

Thus,

Proposition 5.1.2. *The total curvature of a ruled surface given by the local parameterization (5.1.1) is given by*

$$K_t = \frac{DD'' - D'^2}{EG - F^2} = -\frac{(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}')^2}{\|(\boldsymbol{\rho}' + v\mathbf{b}') \times \mathbf{b}\|^2}. \tag{5.1.7}$$

The curvature is always negative and it vanishes if and only if

$$(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}') = 0. \tag{5.1.8}$$

5.1.3 Envelope of a family of surfaces

In this section, a *surface* will always be a *parameterized surface*. The theory of envelopes of surfaces is analogue to the theory of envelopes of plane curves, therefore there will be no proofs, as the proofs are completely analogue to the ones we provided for plane curves.

Definition 5.1.1. Let

$$\mathbf{r} = \mathbf{r}(u, v, \lambda) \tag{5.1.9}$$

be a family of surfaces. The *envelope* of the family (if there is any) is a surface which is tangent to all the surfaces from the family. The contact between the envelope and each surface is made along a curve which is called a *characteristic*. Thus, the envelope is the geometrical locus of the chracteristics of the family of surfaces.

Proposition 5.1.3. *The points of the envelope of the family (5.1.9) verify, beside this equation, the equation*

$$(\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{r}'_\lambda) = 0. \tag{5.1.10}$$

Of course, as in the case of plane curves, these equations are also verified by the singular points of the surfaces and to get the envelope, we have to eliminate them first.

If the family of surfaces is given implicitly, i.e. through the equation

$$F(x, y, z, \lambda) = 0, \tag{5.1.11}$$

then to get the envelope, we have to add to this equation the equation

$$F'_\lambda(x, y, z, \lambda) = 0, \tag{5.1.12}$$

as we did in the case of plane curves. Something that is specific to the theory of envelopes of surfaces is the so-called *regression edge*. Let us assume that we have a family of surfaces, given, say, implicitly. Then we can form the system:

$$\begin{cases} F(x, y, z, \lambda) = 0, \\ F'_{\lambda}(x, y, z, \lambda) = 0 \\ F''_{\lambda^2}(x, y, z, \lambda) = 0 \end{cases} . \tag{5.1.13}$$

The first two equations of this system are the equations of the family of characteristic lines of the family of surfaces. If the system is compatible, then what we find solving it is a curve which, at each point, is tangent to one of the characteristics, in other words, it is the *envelope* of the characteristics. This envelope (when exists), it is called the *regression edge* of the envelope of the family of surfaces.

An example of envelope. We consider the family of spheres of constant radius, with the center of a given circle, verifying the hypothesis that the radius of the circle is greater than the radius of the spheres. Then the envelope of this family is, as one can see immediately, the torus (see the figure 5.1.3).

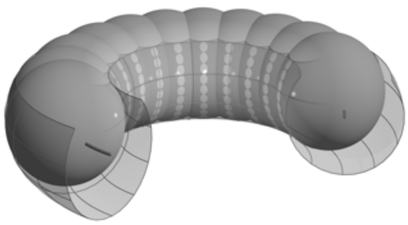


Figure 5.3: The torus as envelope of spheres

5.1.4 Developable surfaces

There is a subclass of ruled playing an important role in differential geometry, the so-called *developable surfaces*:

Definition 5.1.2. A ruled surface is called *developable* if the tangent plane to the surface is the same at all the points of a ruling.

We would like to find conditions for a ruled surface to be developable. We have the following result:

Proposition 5.1.4. *A ruled surface given by the equation*

$$\mathbf{r}(u,v) = \boldsymbol{\rho}(u) + v \cdot \mathbf{b}(u)$$

is developable if and only if

$$(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}') = 0. \tag{5.1.14}$$

Proof. The invariance condition for the tangent plane along a ruling is equivalent to the condition of invariance of the normal line to the surface along the same ruling. As we saw previously, a normal vector to the surface is the vector

$$\mathbf{N}(u,v) = \boldsymbol{\rho}'(u) \times \mathbf{b}(u) + v\left(\mathbf{b}'(u) \times \mathbf{b}(u)\right).$$

Thus, the condition for the surface to be developable is that the direction $\mathbf{N}$ be independent on the coordinate v. This may happen, obviously,in three situations:

1. The first component of the vector $\mathbf{N}$ vanishes, i.e.

   $$\boldsymbol{\rho}'(u) \times \mathbf{b}(u) = 0.$$

   In this case, the vector $\mathbf{N}$ is constant only as direction, but its norm and orientation might vary. As $\mathbf{b}$ is a versor, it cannot vanish, therefore the condition mentioned above can be fulfilled in two situations:

   (a) $\boldsymbol{\rho}' = 0$. In this case, the directrix degenerates to a point, therefore the surface is conical.

   (b) $\boldsymbol{\rho}' \parallel \mathbf{b}$. In this case, the rulings are nothing but the tangents to the directrix, i.e. the surface is generated by the tangents to a space curve.

2. The second component of the normal vector vanishes, i.e.

   $$\mathbf{b}'(u) \times \mathbf{b}(u) = 0.$$

   From this relation follows, in fact, that the direction of $\mathbf{b}$ is fixed, i.e. the surface is cylindrical.

3. The two components of the normal vector are parallel, i.e. we have

   $$\boldsymbol{\rho}'(u) \times \mathbf{b}(u) \parallel \mathbf{b}'(u) \times \mathbf{b}(u).$$

The three conditions we just listed are equivalent to the unique condition

$$(\boldsymbol{\rho}'(u) \times \mathbf{b}(u)) \times (\mathbf{b}'(u) \times \mathbf{b}(u)) = 0.$$

On the other hand,

$$(\boldsymbol{\rho}' \times \mathbf{b}) \times (\mathbf{b}' \times \mathbf{b}) = (\boldsymbol{\rho}', \mathbf{b}', \mathbf{b})\, \boldsymbol{b} - \underbrace{(\mathbf{b}, \mathbf{b}', \mathbf{b})}_{=0} \boldsymbol{\rho}' = -(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}')\, \boldsymbol{b},$$

which concludes the proof of the proposition. □

Corollary 5.1.1. *A ruled surface is developable if and only if its Gaussian curvature vanishes identically.*

Remark. The ruled surfaces which are not developable are also called *scrolls*.

Developable surfaces as envelopes of a one-parameter family of planes. The regression edge of a developable surface

We saw previously that that there are three classes of developable surfaces, namely:

1. cylindrical surfaces;
2. conical surfaces;
3. surfaces generated by the tangents to a space curve.

We shall see, in this paragraph, that, in fact, these three classes exhaust all the developable surfaces.

In the figure 5.1.4 we give an example of developable surface with a regression edge (the tangent developable of the Viviani's temple). One can notice immediately, from the very definition of the developable ruled surfaces, that a ruled surface is developable if and only if it is the envelope of a family of planes, depending on a single parameter[1].

We consider a one-parameter family of planes

$$\mathbf{N}(\lambda) \cdot \mathbf{r} + D(\lambda) = 0. \tag{5.1.15}$$

[1]These planes are, of course, the tangent planes of the surface. Clearly, any surface is the envelope of a family of planes, namely the family of tangent planes to the surface. Nevertheless, this family depends, usually, on two parameters. In the case of developable surfaces, it depends on a single parameter, namely the parameter along the directrix of the surface.

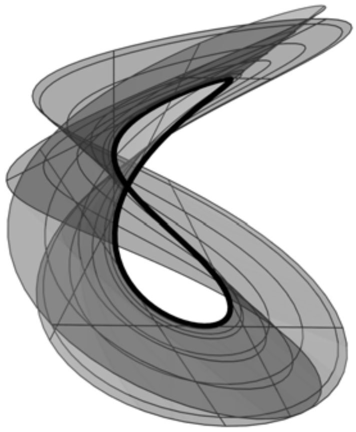


Figure 5.4: The tangent developable of Viviani's temple

We shall assume, without restricting the generality, that the normal vector $\mathbf{N}$ is a versor. By differentiating the equation (5.1.15) with respect to λ, we obtain

$$\mathbf{N}'(\lambda) \cdot \mathbf{r} + D'(\lambda) = 0. \tag{5.1.16}$$

According to the theory of envelopes of the families of surfaces, the equations (5.1.15) and (5.1.16) determine the envelope of the family of planes, if such an envelope exists[2].

The characteristic curves of the family of planes are straight lines, obtained as intersections between the planes of equations (5.1.15) and (5.1.16), respectively. Clearly, these characteristics exist only if the planes of the family are not parallel, which we admit to be true in our case.

The points of the envelope have to verify, also, the equation obtained from (5.1.16), by differentiating once more with respect to λ, i.e.

$$\mathbf{N}''(\lambda) \cdot \mathbf{r} + D''(\lambda) = 0. \tag{5.1.17}$$

We have, thus, the following system of equations:

$$\begin{cases} \mathbf{N}(\lambda) \cdot \mathbf{r} + D(\lambda) = 0 \\ \mathbf{N}'(\lambda) \cdot \mathbf{r} + D'(\lambda) = 0 \\ \mathbf{N}''(\lambda) \cdot \mathbf{r} + D''(\lambda) = 0. \end{cases} \tag{5.1.18}$$

[2]As a matter of fact, these equations describe the *discriminant set*, including, also, the singular points of the surfaces from the family. However, in this particular case, the surfaces are planes and they have no singular points whatsoever.

We can regard this system as being a system of three equations with three unknowns (the components of the radius vector $\mathbf{r}$).

Let us assume, to begin with, that the system is compatible and determined, i.e. the three planes intersect at a single point. We admit, at the first stage, that the solution of the system depends in a nontrivial manner, on the parameter λ, in other words, if

$$\mathbf{r} = \tilde{\mathbf{r}}(\lambda) \tag{5.1.19}$$

is the solution of the system, then $\tilde{\mathbf{r}}(\lambda) \neq 0$. This means, in fact, that the equation (5.1.19) describes a space curve, called the *edge of regression* of the developable surface. This is, of course, the edge of regression of the envelope of the family of planes, in the sense of the theory of envelopes of surfaces. We know from this theory that all the characteristic curves have to be tangent to the edge of regression. Therefore, we must have

$$\mathbf{N} \cdot \tilde{\mathbf{r}}' = 0 \tag{5.1.20}$$

ži

$$\mathbf{N}' \cdot \tilde{\mathbf{r}}' = 0, \tag{5.1.21}$$

because the characteristic is obtained by the intersections of the planes of normal vectors $\mathbf{N}$ and $\mathbf{N}'$, respectively.

By differentiating the relation (5.1.20), we obtain

$$\mathbf{N}' \cdot \tilde{\mathbf{r}}' + \mathbf{N} \cdot \tilde{\mathbf{r}}'' = 0 \tag{5.1.22}$$

whence, using (5.1.21),

$$\mathbf{N} \cdot \tilde{\mathbf{r}}'' = 0. \tag{5.1.23}$$

The relations (5.1.20) and (5.1.23) tell us that the plane of the normal vector $\mathbf{N}$ contains both the vector $\tilde{\mathbf{r}}'$, and the vector $\tilde{\mathbf{r}}''$, i.e. is nothing but the *osculating plane* of the regression edge at the point associated to the value λ of the parameter..

As such, a developable surface admitting an edge of regression can be described in two different ways:

a) as the surface generated by the tangents to the edge of regression;

b) as the envelope of the family of osculating planes of the edge of regression.

In the degenerate case, when the edge of regression reduces to a point, all the characteristic curves (i.e. the generators of the surface) are passing through that point, hence the surface is a conical surface.

We shall focus now on the case when the system (5.1.18) has a vanishing determinant. In principle, two situations are possible: either the system is compatible and undetermined, either it is incompatible. We shall see that the first possibility is, in practice, excluded.

We notice, first of all, that the determinant of the linear system (5.1.18) is nothing but $(\mathbf{N}, \mathbf{N}', \mathbf{N}'')$, therefore, the condition $\Delta = 0$ is equivalent to the condition

$$(\mathbf{N}, \mathbf{N}', \mathbf{N}'') = 0. \tag{5.1.24}$$

This means, of course, that the vectors $\mathbf{N}, \mathbf{N}', \mathbf{N}''$ lie in the same plane. We denote by $\mathbf{n} = \mathbf{n}(\lambda)$ the normal versor of this plane, in other words, we have

$$\mathbf{n} \cdot \mathbf{N} = \mathbf{n} \cdot \mathbf{N}' = \mathbf{n} \cdot \mathbf{N}'' = 0. \tag{5.1.25}$$

By differentiating the first two equations from the previous system, we obtain

$$\mathbf{n}' \cdot \mathbf{N} + \mathbf{n} \cdot \mathbf{N}' = \mathbf{n}' \cdot \mathbf{N} + \mathbf{n} \cdot \mathbf{N}'' = 0,$$

whence, using (5.1.25),

$$\mathbf{n}' \cdot \mathbf{N} = \mathbf{n}' \cdot \mathbf{N}' = 0,$$

i.e. either the vector $\mathbf{n}'$ vanishes, either it is perpendicular on the vectors $\mathbf{N}$ ži $\mathbf{N}'$.

Assuming that we would be in the second situation, as the vectors $\mathbf{N}$ and $\mathbf{N}'$ are not colinear, we would obtain, as one can see easily, that $\mathbf{n}'$ is parallel to $\mathbf{n}$, which is not possible since, as, $\mathbf{n}$ is a versor, we have $\mathbf{n} \cdot \mathbf{n}' = 0$.

We must have, therefore, $\mathbf{n}' = 0$, i.e. the vector $\mathbf{n}$ is constant. It follows from here that for any λ the vectors $\mathbf{N}(\lambda)$ and $\mathbf{N}'(\lambda)$ are perpendiculars on the constant vector $\mathbf{n}$, i.e. all the rulings of the family are parallel to a constant direction, and the surface is a cylindrical surface. It is easy to see that in this case the system of equations (5.1.18) is incompatible (the third plane is parallel to the straight line determined by the first two).

The considerations from this paragraph prove that, in fact, we have the following result::

Proposition 5.1.5. *If S is a developable ruled surface, then it is either a cylindrical surface, either a conical one, either a surface generated by the tangents to a space curve.*

Thus, the classes of surfaces mentioned in the previous paragraph do exhaust all the types of developable ruled surfaces. We mention that the plane can be considered as a limit case for any of the three classes of surfaces listed in the proposition.

5.1.5 Developable surfaces associated to the Frenet frame of a space curve

In all of this section $\mathbf{r} = \mathbf{r}(s)$ will be a smooth naturally parameterized biregular space curve. We assume, moreover, that the curve is *skew*, in other words, its torsion never vanishes.

The envelope of the family of osculating planes

The equation of osculating planes at a point of the curve is

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\beta}(s) = 0. \tag{5.1.26}$$

The parameter along the family of planes will be exactly the natural parameter s along the curve.

By differentiating the relation (5.1.26) with respect to s, we get

$$-\mathbf{r}'(s) \cdot \boldsymbol{\beta}(s) + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\beta}'(s) = 0$$

or, having in mind that $\mathbf{r}' \cdot \boldsymbol{\beta} = 0$ and using the third formula of Frenet,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}(s) = 0, \tag{5.1.27}$$

where we used the fact that, according to the hypotheses we made about the curve, the torsion of the curve never vanishes.

From (5.1.26) and (5.1.27) one obtains that the characteristic straight line is perpendicular both on $\boldsymbol{\beta}$ and on $\boldsymbol{\nu}$, in other words, it is nothing but the tangent to the curve
.

To get the edge of regression, we differentiate once more (5.1.27) and we get

$$-\mathbf{r}'(s) \cdot \boldsymbol{\nu}(s) + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}'(s) = 0$$

or, using the fact that $\mathbf{r}'(s) \cdot \boldsymbol{\nu}(s) = 0$,a s well as the second Frenet formula,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot [\chi(s)\boldsymbol{\beta}(s) - k\boldsymbol{\nu}(s)] = 0.$$

As $\chi(s)\boldsymbol{\beta}(s) - k\boldsymbol{\nu}(s) \neq 0$ if χ and k don't vanish, we obtain that

$$\mathbf{R} = \mathbf{r}(s), \tag{5.1.28}$$

i.e. the edge of regression is nothing but the given curve.

The envelope of the family of normal planes (the polar surface)

The equation of the normal plane at an arbitrary point of a space curve is, as we know,

$$(\mathbf{R}-\mathbf{r}(s))\cdot\boldsymbol{\tau}(s)=0. \tag{5.1.29}$$

We differentiate this relation once, to complete the system of equations for the characteristics and we get

$$-\mathbf{r}'(s)\cdot\boldsymbol{\tau}(s)+(\mathbf{R}-\mathbf{r}(s))\cdot\boldsymbol{\tau}'(s)=0$$

or, having in mind that we have a curve parameterized by the arc length and using the first of the Frenet formulae,

$$-1+(\mathbf{R}-\mathbf{r}(s))\cdot k(s)\cdot\boldsymbol{\nu}(s)=0. \tag{5.1.30}$$

It follows, then, that

$$(\mathbf{R}-\mathbf{r}(s))\cdot\boldsymbol{\nu}(s)=\frac{1}{k(s)}. \tag{5.1.31}$$

The equation (5.1.31) is the equation of a plane which is parallel to the rectifying plane (because it has as normal vector the versor of the principal normal). Thus, the characteristic of the envelope of the normal planes passing through the point of position vector $\mathbf{r}(s)$ is obtained by intersecting the normal plane to the curve at this point with a plane which is parallel to the rectifying plane of the curve at the same point, i.e. it is a straight line which is parallel to the binormal of the curve at the point $\mathbf{r}(s)$.

On the other hand, it is clear that this characteristic passes through the point of position vector $\mathbf{r}(s)+\frac{1}{k(s)}\cdot\boldsymbol{\nu}(s)$, which is exactly the center of curvature of the curve. Thus, the characteristics of the family of normal planes are nothing but the axes of curvature (or polar axes) of the curve. This is the reason why the envelope of the family of normal planes is called, sometimes, the *polar surface* of the curve.

To get the edge of regression of the surface, we differentiate once more the relation (5.1.30) and we get

$$-\underbrace{\mathbf{r}'(s)\cdot k(s)\cdot\boldsymbol{\nu}(s)}_{=0}+(\mathbf{R}-\mathbf{r}(s))\cdot k'(s)\cdot\boldsymbol{\nu}(s)+(\mathbf{R}-\mathbf{r}(s))\cdot k(s)\cdot\boldsymbol{\nu}'(s)=0$$

or, using (5.1.31) and the second Frenet formula,

$$\frac{k'(s)}{k(s)}+(\mathbf{R}-\mathbf{r}(s))\cdot k(s)\cdot[\chi(s)\cdot\boldsymbol{\beta}(s)-k(s)\boldsymbol{\tau}(s)]=0$$

or, also, from (5.1.29),

$$(\mathbf{R}-\mathbf{r}(s))\cdot\boldsymbol{\beta}(s)=\frac{k'(s)}{k^2(s)\cdot\chi(s)}. \tag{5.1.32}$$

Comparing (5.1.29) to (5.1.32), we find out the following things:

- The straight line of intersection between the planes (5.1.29) and (5.1.32) is parallel to the osculating plane to the rectifying plane and the normal plane, i.e. is parallel to the binormal.
- This straight line passes through the point

$$\mathbf{r}(s) + \frac{k'(s)}{k^2(s) \cdot \chi(s)} \cdot \boldsymbol{\beta}(s).$$

Thus, the current point from the edge of regression is obtained by intersecting the lines

$$\mathbf{R}(\lambda) = \mathbf{r}(s) + \frac{1}{k(s)} \cdot \boldsymbol{\nu}(s) + \lambda\boldsymbol{\beta}(s) \tag{5.1.33}$$

and

$$\mathbf{R}(\mu) = \mathbf{r}(s) + \frac{k'(s)}{k^2(s)\chi(s)} \cdot \boldsymbol{\beta}(s) + \mu\boldsymbol{\nu}(s). \tag{5.1.34}$$

This point will have, thus, the position vector

$$\mathbf{R}(s) = \mathbf{r}(s) + \frac{1}{k(s)} \cdot \boldsymbol{\nu}(s) + \frac{k'(s)}{k^2(s)\chi(s)} \cdot \boldsymbol{\beta}(s). \tag{5.1.35}$$

But the right hand side of the relation (5.1.35) is nothing but the position vector of the osculating sphere of the given curve. Thus, *the edge of regression of the polar surface of a space curve is the locus of the centers of the osculating spheres of the given curve*.

The envelope of the family of rectifying planes of a space curves

The equation of the rectifying plane is

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}(s) = 0. \tag{5.1.36}$$

By differentiation, we get

$$-\underbrace{\mathbf{r}'(s) \cdot \boldsymbol{\nu}(s)}_{=0} + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}'(s) = 0$$

or, using the second Frenet equation,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot [\chi(s)\boldsymbol{\beta}(s) - k(s)\boldsymbol{\tau}(s)] = 0. \tag{5.1.37}$$

The points of the curve verify the equations (5.1.36) and (5.1.37), hence the curve lies on the rectifying surface and can be takes as the directrix of the surface, regarded as a ruled surface. The relations (5.1.36) and (5.1.37) indicate that the rulings (i.e. the characteristic lines of the envelope) are perpendicular both on $\boldsymbol{\nu}$, and on $\boldsymbol{\nu}'$, hence their direction is given by the vector

$$\boldsymbol{\nu} \times \boldsymbol{\nu}' = \boldsymbol{\nu} \times (\chi\boldsymbol{\beta} - k\boldsymbol{\tau}) = \chi(\boldsymbol{\nu} \times \boldsymbol{\beta}) - k(\boldsymbol{\nu} \times \boldsymbol{\tau}) = \chi\boldsymbol{\tau} + k\boldsymbol{\beta} = \boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is the Darboux vector of the curve. Thus, *the characteristics of the rectifying surface of a space curve are the Darboux axes of the curve or the instantaneous axes of rotation of the Frenet frame*.

The tangent plane to the rectifying surface at a point lying on the given curve contain both the tangent vector of the curve, $\mathbf{r}' = \boldsymbol{\tau}$, and the Darboux vector of the curve, $\boldsymbol{\delta}$. Therefore, the normal to the surface at such a point is colinear to the vector

$$\boldsymbol{\tau} \times \boldsymbol{\delta} = \boldsymbol{\tau} \times (\chi\boldsymbol{\tau} + k\boldsymbol{\beta}) = k(\boldsymbol{\tau} \times \boldsymbol{\beta}) = -k\boldsymbol{\nu}.$$

Thus, the normal to the rectifying surface of a space curve at a point of the curve coincides to the principal normal of the curve. This actually means that the *osculating plane of the curve is perpendicular on the tangent space of the surface*, i.e. *the curve is a geodesic of its rectifying surface*. This is, actually, the origin of the name of the rectifying plane and, implicitly, of the rectifying surface: the curve, when regarded as lying on its rectifying surface, is a geodesic, which is the surface analogue of a straight line. In other words, the curve is *rectified* (straightened) by its rectifying planes.

As we saw, the total of a developable surface is always zero. In many books of differential geometry it is claimed that the converse is, also, true, in other words, any surface of zero Gaussian curvature is a developable surface or a part of a developable surface. However, this is true only if we assume that the surface doesn't have any planar points, otherwise, first of all, it might not be not even a ruled surface. An example of such surface is given in the book of Klingenberg [31]. The surface is not, actually, described by using a parameterization or an implicit equation, but rather by prescribing the first two fundamental forms and showing that they fulfill the compatibility conditions.

5.2 Minimal surfaces

5.2.1 Definition and general properties

Definition 5.2.1. A *minimal* surface is, by definition, a surface whose mean curvature vanishes identically.

This is not the original definition of a minimal surface. As a matter of fact, Lagrange extended to the two-dimensional case the method of Euler for finding the extrema of a functional and he proposed the following problem:

Given a closed curve and a connected surface patch S bounded by the curve, to finde the surface such that the inclosed area shall be minimum.

We consider, thus, the area functional,

$$A = \iint\limits_U H du dv.$$

As it is known from the variational calculus, a necessary condition for the minimum of this functional is furnished by the Lagrange equations, which, in this case, are:

$$\begin{cases} \frac{\partial H}{\partial x} - \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x'_u}\right) - \frac{\partial}{\partial v}\left(\frac{\partial H}{\partial x'_v}\right) = 0 \\ \frac{\partial H}{\partial y} - \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial y'_u}\right) - \frac{\partial}{\partial v}\left(\frac{\partial H}{\partial y'_v}\right) = 0 \\ \frac{\partial H}{\partial z} - \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial z'_u}\right) - \frac{\partial}{\partial v}\left(\frac{\partial H}{\partial z'_v}\right) = 0 \end{cases}.$$

After long, but otherwise straightforward computations, the Lagrange equations become

$$\begin{cases} \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix} K_m = 0 \\ \begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix} K_m = 0 \\ \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} K_m = 0 \end{cases},$$

and, thus, they are satisfied iff $K_m = 0$.

Let us mention, also, that, initially, Lagrange used the explicit representation $z = f(x, y)$ for the surface, in which case the area functional is

$$\iint\limits_U \sqrt{1 + p^2 + q^2} dx dy,$$

while the stationarity condition finally becomes

$$\frac{\partial}{\partial x}\frac{p}{\sqrt{1 + p^2 + q^2}} + \frac{\partial}{\partial y}\frac{p}{\sqrt{1 + p^2 + q^2}} = 0$$

or

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0.$$

It was Meusnier who showed, sixteen years later, that the minimality condition induces a relation among the coefficients of the first two fundamental forms of a parameterized surface. Eisenhart claims that Meusnier actually showed that for minimal surfaces the mean curvature vanishes. This is not quite correct: Meusnier was not aware of the true meaning of the the relation he found. In fact, the notion of mean curvature was introduced only fifty years later, by Sophie Germain.

An alternative proof of the fact that the minimal surfaces are the solution of the variational problem was found by Darboux. A particularly clear exposition of this proof can be found in the book of Hsiung.

For further convenience, we shall need the following definition:

Definition 5.2.2. A local parameterization $(U, \mathbf{r})$ of a surface S is called *isothermic* if, with respect to this parameterization, the coordinate lines are orthogonal (i.e. $F = 0$) and $E = G$. This means, actually, that the matrix of the first fundamental form is just the unit matrix multiplied by a positive function.

On any surface which is at least C^2 there exist isothermic parameterizations. The proof of this claim is based on the theory of partial differential equations and it is not interesting for our exposition.

Proposition 5.2.1. *A surface is minimal iff its asymptotic directions are orthogonal.*

Proof. A surface is minimal iff the coefficients of the first two fundamental forms verify the equation

$$ED'' - 2FD' + GD = 0.$$

Let us choose, for the surface, an isothermic parameterization, with respect to which we have

$$E = G = \lambda^2(u, v), \quad F = 0.$$

Then the minimality condition for the surface becomes

$$E(D'' + D) = 0$$

and, as $E \neq 0$, the surface being regular, we have

$$D' + D'' = 0.$$

On the other hand, the equation for the slopes of the asymptotic directions is

$$Dt^2 + 2D't + D'' = 0$$

and one can see immediately that the asymptotic directions are orthogonal iff the roots of this equation verify $t_1 t_2 = -1$, which is equivalent to $D = -D''$. □

The following technical lemma will allow us to obtain a very nice characterization of minimal surfaces

Lemma 1. *Let* $\mathbf{r} = \mathbf{r}(u, v)$ *be an isothermic parameterization of a surface* S*. Then*

$$\mathbf{r}''_{\mathbf{u^2}} + \mathbf{r}''_{\mathbf{v^2}} = 2\lambda^2 K_m \,\mathbf{n}, \tag{5.2.1}$$

where $\lambda^2 = E = G$*.*

Proof. Since $\mathbf{r}$ is isothermic, we have

$$\mathbf{r}'^{\,2}_{\mathbf{u}} = \mathbf{r}'^{\,2}_{\mathbf{v}} = \lambda^2, \quad \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}'_{\mathbf{v}} = 0, \tag{5.2.2}$$

therefore the mean curvature is just

$$K_m = \frac{D + D''}{\lambda^2}. \tag{5.2.3}$$

By differentiating (5.2.2), we get

$$\mathbf{r}''_{\mathbf{u^2}} \cdot \mathbf{r}'_{\mathbf{u}} = \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{v}} = -\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}''_{\mathbf{v^2}},$$

whence

$$(\mathbf{r}''_{\mathbf{u^2}} + \mathbf{r}''_{\mathbf{v^2}}) \cdot \mathbf{r}'_{\mathbf{u}} = 0,$$

and, similarly,

$$(\mathbf{r}''_{\mathbf{u^2}} + \mathbf{r}''_{\mathbf{v^2}}) \cdot \mathbf{r}'_{\mathbf{v}} = 0.$$

Thus, the vector $\mathbf{r}''_{\mathbf{u^2}} + \mathbf{r}''_{\mathbf{v^2}}$ is perpendicular both to $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$, i.e. is colinear with the unit normal:

$$\mathbf{r}''_{\mathbf{u^2}} + \mathbf{r}''_{\mathbf{v^2}} = a\,\mathbf{n}. \tag{5.2.4}$$

If we multiply (5.2.4) by $\mathbf{n}$ and use the equation (5.2.2), as well as the definition of the second fundamental form, we obtain

$$a = D + D'' = 2\lambda^2 K_m, \tag{5.2.5}$$

which proves the claim. □

Corollary 1. *if* $\mathbf{r} = \mathbf{r}(u, v)$ *is an isothermic parameterization of a surface* S*, then* S *is minimal iff the components of* $\mathbf{r}$ *are harmonical functions, in other words iff we have*

$$\Delta x = \Delta y = \Delta z = 0.$$

It is a classical (but nontrivial) result that on any smooth (at least C^2) surface there exists isothermic parameterization. On a minimal surface we have a very special isothermic parameterization:

Proposition 5.2.2. *The lines of curvature on a minimal surface form an isothermic system.*

Proof. We choose as parametric lines the lines of curvature of the surface. Then, as we already know, we should have $F = D' = 0$ (both first fundamental forms are diagonal in such a kind of parameterization). Thus, the minimality condition for the surface becomes

$$ED'' + GD = 0$$

In this parameterization the Codazzi-Mainardi equations become

$$\frac{\partial D}{\partial v} - \frac{1}{2}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0,$$
$$\frac{\partial D''}{\partial u} - \frac{1}{2}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0,$$

which, for a minimal surface, reduce to

$$\frac{\partial D}{\partial v} = 0,$$
$$\frac{\partial D''}{\partial u} = 0,$$

i.e. D depends only on u and D'' – only on v. Since

$$\frac{E}{D} = -\frac{G}{D''},$$

we have

$$\frac{\partial^2 \log E}{\partial u \partial v} = \frac{\partial^2 \log G}{\partial u \partial v},$$

which is exactly the condition that the parametric lines are isothermic. □

Proposition 5.2.3. *The spherical image of a minimal surface is conformal (i.e. it preserves the angles).*

Proof. Let e, f, g be the coefficients of the first fundamental form of the spherical image. We have, then,

$$e = -EK_t + DK_m, \quad f = -FK_t + D'K_m, \quad g = -GK_t + D''K_m.$$

Thus, for a minimal surface, we have

$$e = -EK_t, \quad f = -FK_t, \quad g = -GK_t,$$

i.e. the two surfaces are, indeed, conformally equivalent. □

The reciprocal of this proposition is almost true, i.e. we have

Proposition 5.2.4. *If the spherical map of a surface is conformal, then the surface is either minimal, either a sphere.*

Proof. If the spherical map is conformal, then we have

$$-EK_t + DK_m = \mu E, \quad -FK_t + D'K_m = \mu F, \quad -GK_t + D''K_m = \mu G. \tag{5.2.6}$$

Thus,

$$DK_m = E(K_t + \mu), \quad D'K_m = F(K_t + \mu), \quad D''K_m = G(K_t + \mu). \tag{5.2.7}$$

We multiply the three relations by $D'', -2D', D$, respectively and we add the results, we get

$$2K_tK_m = K_m(K_t + \mu)$$

or

$$K_m(\mu - K_t) = 0. \tag{5.2.8}$$

On the other hand, if we multiply the relations (5.2.7) by $G, -2F, E$, respectively, we get

$$K_m^2 = 2(K_t + \mu). \tag{5.2.9}$$

If we put K_m, the (5.2.8) is satisfied, while (5.2.9) is satisfied iff $\mu = -K_t$ and we rediscover the result from the proof of the previous proposition, i.e. the spherical image of a minimal surface is conformal and the conformal factor is $\sqrt{-K_t}$. The second possibility to satisfy (5.2.8) is $K_t = \mu$. Notice that, as $\mu > 0$, we cannot have at the same time $K_t = \mu$ and $K_m = 0$, because then we would get, also, $K_t = -\mu$, i.e. $K_t = \mu = 0$.

If $K_t = \mu$, then, from (5.2.9) we get

$$K_m^2 = 4\mu = 4K_t,$$

i.e. the principal curvatures are equal at each point (all the points are umbilical). Or, as we know, the only surface with only umbilical points is the sphere (and its limit case, the plane). □

Proposition 5.2.5. *The spherical image of isothermic lines on a minimal surface are isothermic lines on the sphere.*

Proof. This is an immediate consequence of the fact that the spherical map of a minimal surface is conformal. □

The first two minimal surfaces (apart from the plane) were discovered in the eighteenth century by the French mathematician Meusnier. These surface were the catenoid and the right helicoid. As we know, the first surface is a revolution surface, while the second is a ruled surface. For a couple of decades, nobody was able to find other examples. The surfaces found by Meusnier are very simple and they belong classes of surfaces which are relatively easy to study, but, as we shall see in the following, these classes do not contain other minimal surfaces.

5.2.2 Minimal surfaces of revolution

Proposition 5.2.6. *The catenoid is the only revolution surface which is, also, a minimal surface.*

Proof. An arbitrary surface of revolution (around the z-axis, in our case), can be parameterized as

$$\begin{cases} x &= u\cos v \\ y &= u\sin v \\ z &= f(u) \end{cases}.$$

Therefore, the coefficients of the first two fundamental forms of the surface can be written as

$$\begin{aligned} &E = 1 + f'^2, \quad F = 0, \quad G = u^2,\\ &D = \frac{f''}{\sqrt{1+f'^2}}, \quad , D' = 0, \quad D'' = \frac{uf'}{\sqrt{1+f'^2}}. \end{aligned}$$

Then the minimality condition for the surface reads

$$(1 + f'^2)uf' + f''u^2 = 0$$

which can be integrated and gives for f an expression of the form

$$f(u) = a\cosh^{-1}\frac{u}{a} + c,$$

where a and c are constants, which corresponds to the catenoid. □

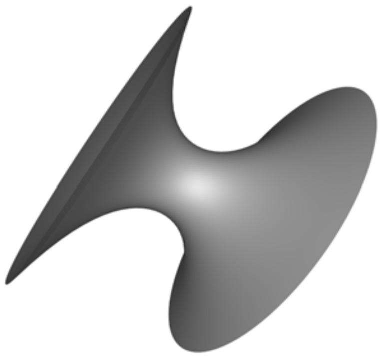


Figure 5.5: The catenoid

5.2.3 Ruled minimal surfaces

We recall that the right conoid is the surface generated by s straight line moving parallel to a plane ($z = 0$ in our case) and intersecting a line which is perpendicular to this plane (in our case – the z-axis). As we saw, the equations of a right conoid can be written as

$$\begin{cases} x &= u\cos v \\ y &= u\sin v \\ z &= f(v) \end{cases}.$$

Then the coefficients of the first two fundamental forms of the surface will be

$$\begin{aligned} &E = 1, \quad F = 0, \quad G = u^2 + f'^2, \\ &D = 0, \quad D' = -\frac{f'}{\sqrt{u^2 + f'^2}}, \quad D'' = \frac{uf''}{\sqrt{u^2 + f'^2}}. \end{aligned}$$

We are interested to see which right conoids are minimal surfaces. Well, as we will show, there is only one.

Proposition 5.2.7. *The only right conoid which is a minimal surface is the right helicoid.*

Proof. As we know, the condition for a surface to be minimal is that the coefficients of the first two fundamental forms verify the equation

$$ED'' - 2FD' + GD = 0,$$

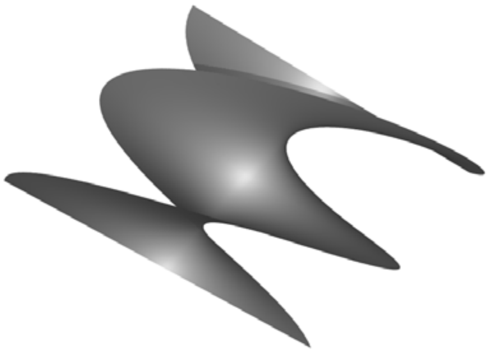


Figure 5.6: The minimal helicoid

which, in our case, reduces to

$$\frac{uf''}{\sqrt{u^2 + f'^2}} = 0,$$

or $f'' = 0$. But this leads to $f(v) = av + b$, where a and b are constants, i.e. the surface is a right helicoid. □

One might think that there are other ruled minimal surfaces, which are not right conoids. For such more general ruled surfaces we do not have such a nice parameterization, therefore this direct approach cannot be applied. Still, as we shall see next, there is another method to prove that, in fact, there are no other ruled minimal surfaces.

Proposition 5.2.8 (Catalan, 1842)**.** *The only minimal ruled surface is the right helicoid.*

Proof. Let S be a ruled surface which is not a plane. Then, as we saw previously, the surface is minimal iff the asymptotic lines are orthogonal. Through each point of the surface pass exactly two asymptotic lines. On the other hand, through each point passes one straight line (the generator). To prove that the surface is a right helicoid, it is enough to prove that it has as directrix a circular cylindrical helix. Since the surface is minimal, the second asymptotic line through each point is orthogonal to the generator passing through that point. The asymptotic lines have the property that their principal normals are contained into the tangent plane. This means that for an asymptotic line from the second family, the generators that intersect it are principal normals. But they are principal normals also for any other asymptotic line that intersects them. Thus, any asymptotic line from the second family has an infinity of Bertrand mates. But then, as we know, they have to be cylindrical helices. □

We shall give now, following Barbosa and Colares, a more direct prove of the Catalan's theorem, using a local parameterization of the ruled surface.

Another proof of the Catalan's theorem. Let $S \subset \mathbb{R}^3$ be a ruled surface. Then from the definition, S has local parameterizations of the form

$$\mathbf{r}(u, v) = \alpha(u) + v\boldsymbol{\gamma}.$$

We may choose the directrix α of the ruled surface to be an orthogonal trajectory of the family of rulings, and $\boldsymbol{\gamma}$, the directing vector of the rulings, to be a unit vector, at each point of the directrix. We shall assume, moreover, that u is the natural parameter of the directrix. A straightforward computation provides, for the coefficients of the first two fundamental forms of the surface, the expressions:

$$\begin{aligned}
E &= 1 + 2v\boldsymbol{\alpha}' \cdot \boldsymbol{\gamma}' + v^2\boldsymbol{\gamma}'^2,\\
F &= 0,\\
G &= 1,\\
D &= \frac{1}{\sqrt{1 + 2v\boldsymbol{\alpha}' \cdot \boldsymbol{\gamma}' + v^2\boldsymbol{\gamma}'^2}}(\boldsymbol{\alpha}' + v\boldsymbol{\gamma}', \boldsymbol{\gamma}, \boldsymbol{\alpha}'' + v\boldsymbol{\gamma}''),\\
D' &= \frac{1}{\sqrt{1 + 2v\boldsymbol{\alpha}' \cdot \boldsymbol{\gamma}' + v^2\boldsymbol{\gamma}'^2}}(\boldsymbol{\alpha}' + v\boldsymbol{\gamma}', \boldsymbol{\gamma}, \boldsymbol{\gamma}'),\\
D'' &= 0.
\end{aligned}$$

Thus, the minimality condition reads

$$(\boldsymbol{\alpha}' + v\boldsymbol{\gamma}', \boldsymbol{\gamma}, \boldsymbol{\alpha}'' + v\boldsymbol{\gamma}'') = 0,$$

or, which is the same,

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}'' + v\left[(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' + (\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}''\right] + v^2(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' = 0.$$

As the left hand side of the previous equation is a polynomial function in v, the equality holds for any v iff the coefficients of the polynomial function vanish simultaneously:

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}'' = 0 \tag{a}$$

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' + (\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}'' = 0 \tag{b}$$

$$(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' = 0. \tag{c}$$

From the relation (a) it follows that the vector $\boldsymbol{\alpha}''$ must be contained in the plane generated by the vectors $\boldsymbol{\alpha}'$ and $\boldsymbol{\gamma}$. But, as the curve α is naturally parameterized, the vector $\boldsymbol{\alpha}'$ has unit length, which means that $\boldsymbol{\alpha}''$ is perpendicular to $\boldsymbol{\alpha}'$. As, from construction, $\boldsymbol{\gamma}$ is also perpendicular to $\boldsymbol{\alpha}'$, it follows that $\boldsymbol{\alpha}''$ is colinear to $\boldsymbol{\gamma}$. But the, the relation (b) reduces to

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' = 0. \tag{b'}$$

Now, from (b') and (c) follows that the vector $\boldsymbol{\gamma}''$ should belong to the intersection of the planes determined by the vectors $\boldsymbol{\alpha}', \boldsymbol{\gamma}$ and $\boldsymbol{\gamma}', \boldsymbol{\gamma}$, respectively. Clearly, this two planes are not parallel, since they contain at least the vector $\boldsymbol{\gamma}$. Therefore, only two situations are possible: either their intersection is a straight line, either they coincide. Suppose, first, that there exists a point where $\boldsymbol{\gamma}''$ is not parallel to $\boldsymbol{\gamma}$. Then, at this point (and, by continuity, in an entire neighborhood of it), these planes have to coincide. Indeed, otherwise $\boldsymbol{\gamma}''$ cannot belong simultaneously to both planes. Moreover, in this situation we have, also $\boldsymbol{\alpha}' \parallel \boldsymbol{\gamma}'$. Indeed, both $\boldsymbol{\gamma}'$ and $\boldsymbol{\alpha}'$ are perpendicular on $\boldsymbol{\gamma}$. As the three vectors are coplanar, it follows the parallelism just mentioned. Now, as we assumed that α and γ are not just smooth, but analytic functions, the two vectors are parallel for any value of the parameter, and, also, the two planes will also coincide. The plane that contains all our vectors could, in principle, vary from point to point. In reality, instead, this not happens. Indeed, we have

$$(\boldsymbol{\gamma} \times \boldsymbol{\alpha}')' = \boldsymbol{\gamma}' \times \boldsymbol{\alpha}' + \boldsymbol{\gamma} \times \boldsymbol{\alpha}'' = 0.$$

But this means that the plane of the vectors is invariant, therefore the curve α is a plane curve and the surface is a plane.

Suppose, now, that $\boldsymbol{\gamma}''$ is always parallel to $\boldsymbol{\gamma}$. If $\boldsymbol{\alpha}'$ is always parallel to $\boldsymbol{\gamma}'$, this means, actually, that the two planes always coincide and we are, in fact, in the previous situation. Let us assume, therefore, that there exists a point (hence an entire open set) where the two vectors are not parallel. We are going to show that, in this case, α is a circular helix. As we know, to this end it is enough to prove that both the curvature and the torsion of the curve are constants.

Let $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$ the Frenet frame of the curve α. We intend to use the Frenet equation to find the curvature and the torsion, instead of using the explicit formulas we obtained when we studied the space curves. First of all, as α is naturally parameterized, we have

$$\boldsymbol{\alpha}''(u) = k(u)\boldsymbol{\nu}(u).$$

On the other hand, again, because α is naturally parameterized, we have $\boldsymbol{\alpha}''(u) \perp \boldsymbol{\alpha}'(u)$. But $\boldsymbol{\alpha}'$ is also perpendicular on $\boldsymbol{\gamma}$, and the three vectors are coplanar, which means that, as $\boldsymbol{\gamma}$ is a unit vector, we have $\boldsymbol{\gamma}'(u) = \pm\boldsymbol{\nu}$, hence, using the previous equation,

$$\pm k(u) = \boldsymbol{\alpha}''(u) \cdot \boldsymbol{\gamma}(u).$$

But, since $\boldsymbol{\alpha}' \perp \boldsymbol{\gamma}$, we have

$$0 = (\boldsymbol{\alpha}' \times \boldsymbol{\gamma})' = \boldsymbol{\alpha}' \times \boldsymbol{\gamma}' + \boldsymbol{\alpha}'' \times \boldsymbol{\gamma},$$

therefore, we have for the curvature the expression

$$\pm k(u) = -\boldsymbol{\alpha}' \times \boldsymbol{\gamma}'.$$

Thus, the derivative of the curvature will be given by

$$\pm\frac{dk}{du} = -(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}')' = -\boldsymbol{\alpha}'' \times \boldsymbol{\gamma}' - \boldsymbol{\alpha}' \times \boldsymbol{\gamma}'' = 0,$$

i.e. the curvature of α is constant. The torsion, on the other hand, can be found using the third Frenet's formula. Indeed, this formula is

$$\boldsymbol{\beta}' = -\chi\boldsymbol{\nu}.$$

As we saw, $\boldsymbol{\nu} = \pm\boldsymbol{\gamma}$. On the other hand, we have $\boldsymbol{\tau} = \boldsymbol{\alpha}'$ and, thus,

$$\boldsymbol{\beta} \equiv \boldsymbol{\tau} \times \boldsymbol{\nu} = \pm\boldsymbol{\alpha}' \times \boldsymbol{\gamma},$$

whence

$$\boldsymbol{\beta}' = \pm(\boldsymbol{\alpha}' \times \boldsymbol{\gamma})' = \pm(\boldsymbol{\alpha}'' \times \boldsymbol{\gamma} + \boldsymbol{\alpha}' \times \boldsymbol{\gamma}') = \pm\boldsymbol{\alpha}' \times \boldsymbol{\gamma}'.$$

Hence the torsion will be given by

$$\chi = -\boldsymbol{\beta} \cdot \boldsymbol{\nu} = \pm(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}$$

and it is easy to see that the derivative of the torsion is, also, identically zero, i.e. the torsion is a constant function.

Now, from the existence and uniqueness theorem for space curves, we know that there exists a curve which has the prescribed curvature and torsion and this curve is unique, up to a motion in space. But, on the other hand, we know a curve which has constant curvature and torsion, namely the circular helix. Thus, α should be a circular helix and has a parameterization of the form

$$\alpha(u) = (A\cos au, A\sin au, bu),$$

where, since α is supposed to be naturally parameterized, the constants A, a, b are subject to $A^2a^2 + b^2 = 1$. On the other hand, as $\boldsymbol{\gamma}$ is parallel to $\boldsymbol{\alpha}''$, we have, also,

$$\boldsymbol{\gamma} = \pm\{\cos au, \sin au, 0\}.$$

If, now, we put $s = A \pm u$, $t = v$, the equation of the surface, with α and $\boldsymbol{\gamma}$ just found, becomes

$$\mathbf{r}(s,t) = (s\cos at, s\sin at, bt),$$

which means that the surface is a part of a right helicoid, as claimed. □

Consequence. *The only developable minimal surface is the plane.*

As we already mentioned, Meusnier was the one who discovered that the right helicoid is a minimal surface. It is both interested and instructive to see how Meusnier got to this result. He searched a minimal surface in an explicit representation, $z = f(x, y)$. As we saw, in this case, the minimality condition can be written as

$$(1 + f'^2_y)f''_{x^2} - 2f'_x f' y f''_{xy} + (1 + f'^2_x)f''_{y^2} = 0.$$

Meusnier was interested in solutions of the previous equation for which the level curves $f(x, y) = \text{const}$ are straight lines in the plane. Obviously, such surfaces would be ruled. He started by noticing that, for a plane curve given in the implicit form $f(x, y) = c$, the signed curvature can be written as

$$k_\pm = \frac{-f''_{x^2} f'^2_y + 2f'_x f'_y f''_{xy} - f''_{y^2} f'^2_x}{\|\operatorname{grad} f\|^3}.$$

Therefore, the minimality condition becomes

$$\Delta f = k_\pm \|\operatorname{grad} f\|^3.$$

If the level curves are straight lines, then their signed curvature vanishes, which means that f is a harmonical function, i.e.

$$\Delta f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Now, it can be shown that the only solution of this equation for which the level curves are straight lines is

$$f(x, y) = a \operatorname{arctg} \frac{y - y_0}{z - z_0} + b,$$

with a, b, x_0, y_0 constants.

The graph of the function $z = f(x, y)$ is, then, either a a plane (if $a = 0$), either a portion of the right helicoid

$$\begin{cases} x = x_0 + u\cos v \\ y = y_0 + u\sin v \\ z = av + b. \end{cases}$$

As one can see, even the original deduction of Meusnier could make one suspect that the only ruled minimal surface is the right helicoid.

5.3 Surfaces of constant curvature

Definition 5.3.1. We say that a surface S has *constant curvature* if its Gaussian curvature is independent of the point, i.e. is a constant.

We already met surfaces which have zero curvature, namely the plane and the developable surfaces. We also know an example of a surface that has constant positive curvature (the sphere). We shall give, now, an example of a (parameterized) surface that has constant *negative* curvature, namely the *pseudosphere*, given by the parametric equations

$$\begin{cases} x = a\sin u\cos v \\ y = a\sin u\sin v \\ z = a\left(\ln\tan\frac{u}{2} + \cos u\right) \end{cases}. \tag{5.3.1}$$

A straightforward computation shows that the pseudosphere has the Gaussian curvature $-1/a^2$. Loosely, we can define a surface to be *complete* if it is unextendible (like the sphere, or the plane). It turns out that any surface of constant negative curvature, if extended enough, becomes singular. In other words, there are no non-singular complete surfaces of constant negative curvature in $\mathbb{R}^3$. The pseudosphere (see the figure 5.3) is no exception. A classical theorem of Minding (1839) claims that all the surfaces that have the same constant curvature are locally isometric. This means that they have local parameterizations, defined on the same domain in the plane, in such a way that, for a given pair of parameters, they have the same coefficients for the first fundamental form. This result does not hold, however, if the surfaces don't have constant curvature, as the following counterexample shows.

Counterexample. Let $\mathbf{r}_1 : D_1 \equiv (0, \infty) \times (0, 2\pi) \to \mathbb{R}^3$, given by

$$\mathbf{r}_1(u_1, v_1) = (u_1\cos v_1, u_1\sin v_1, \ln u_1)$$

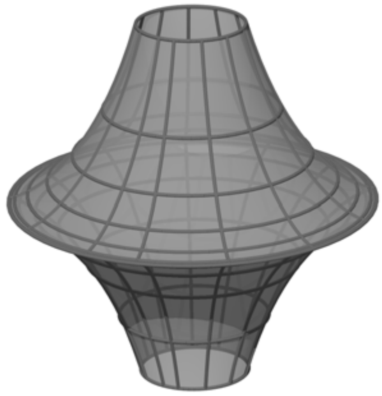


Figure 5.7: The pseudosphere

and $\mathbf{r}_2 : D_2 \equiv (0,\infty)\times(0,2\pi) \to \mathbb{R}^3$, given by

$$\mathbf{r}_2(u_2, v_2) = (u_2\cos v_2, u_2\sin v_2, v_2)\,.$$

As one can see, the first surface is a surface of revolution, generated by the graph of the logarithmic function, while the second one is part of the right helicoid. We consider the map $\Lambda : D_1 \to D_2$, $\Lambda(u_1, v_1) = (u_2, v_2)$. As $D_1 = D_2$, clearly Λ is, in fact, the identity and it is, therefore, a diffeomorphism. An easy computation shows that the first fundamental forms for the two parameterized surfaces are, respectively:

$$E_1 = 1 + \frac{1}{u_1^2}, \quad F_1 = 0, \quad G_1 = u_1^2$$

and

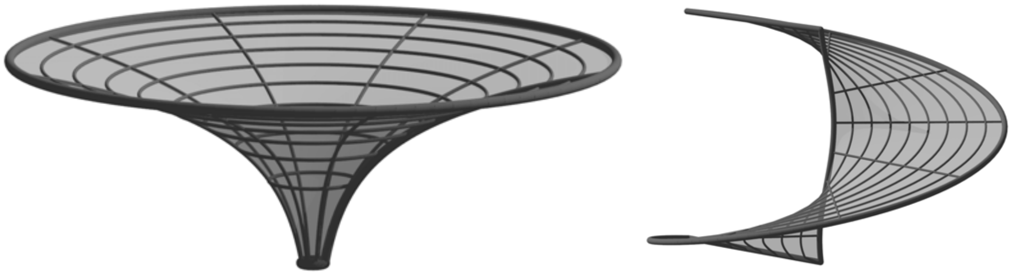
$$E_2 = 1, \quad F_2 = 0, \quad G_2 = 1 + u_2^2,$$

therefore they are *not* isometric (because, for instance, $E_1 \neq E_2$). However, as one can convince oneself, for their Gaussian curvatures we get

$$K_1 = -\frac{1}{(1+u_1^2)^2}, \quad K_2 = -\frac{1}{(1+u_2^2)^2},$$

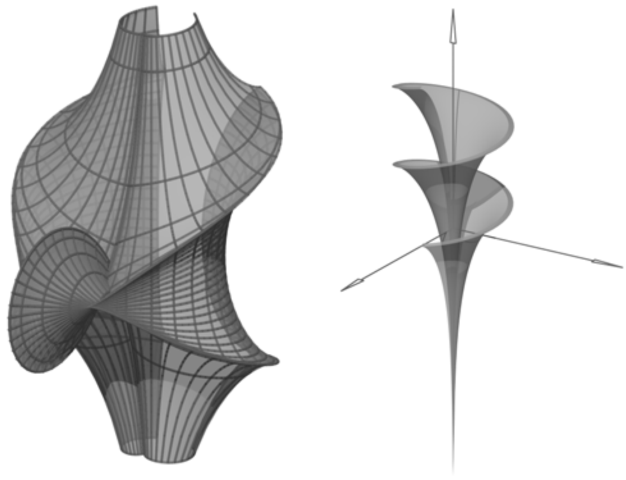
i.e. they are equal.

On the other hand, even if, for instance, the surfaces of the same constant negative curvature are locally isometric with the pseudosphere, they might look very different, as the surfaces from the figure 5.3, belonging to Kuen and Dini show.



(a) Surface of revolution (b) Helicoid

Figure 5.8: Non-isometric parameterized surfaces



(a) Kuen's surface (b) Dini's surface

Figure 5.9: Surfaces of constant negative curvature

Problems

1. We consider, in the coordinate plane xOz, the curve given by

$$\begin{cases} x = f(u), \\ z = \varphi(u). \end{cases}$$

Find the parametrical equation of the surface obtained rotating this curve around the z-axis.

2. Find the equations of the *torus*, obtained by rotating the circle

$$\begin{cases} x = a + \cos u, \\ y = 0, \\ z = b \sin u, \quad b < a, \end{cases}$$

around the z-axis.

3. Find the equations of the *catenoid*, obtained by rotating the catenary

$$\begin{cases} x = a \cosh \dfrac{u}{a}, \\ y = 0, \\ z = u \end{cases}$$

around the z-axis.

4. Find the equations of the *pseudosphere*, obtained by rotating the tractrix

$$\begin{cases} x = a \sin u, \\ y = 0, \\ z = a\left(\ln \tan \frac{u}{2} + \cos u\right) \end{cases}$$

around the z-axis.

5. Find the equation of the tangent plane at an arbitrary point of the right helicoid

$$\mathbf{r} = \{u \cos v, u \sin v, hv\}.$$

6. Show that if a smooth surface Φ and a plane α have a single common point P, then the plane α is the tangent plane to the surface at the point P.

7. Write the equation of the tangent plane to the sphere of radius a, centred at the origin, at the point of coordinates $(0, 0, a)$ ("north pole").

8. Write the equations of the normal to the pseudosphere

$$\begin{cases} x = a \sin u \cos v, \\ y = a \sin u \sin v, \\ z = a \ln \tan \frac{u}{2} + a \cos u \end{cases}$$

at an arbitrary point and find the unit normal vector.

9. Find the unit normal vector to the right helicoid

$$\begin{cases} x = u \cos v \\ y = u \sin v \\ z = hv \end{cases} .$$

10. Determine the equation of the tangent plane to the surface $xy^2 + z^2 = 8$ at the point $(1, 2, 2)$ and find the unit normal vector at this point.

11. Show that the surfaces

$$x^2 + y^2 + z^2 = \alpha x,$$
$$x^2 + y^2 + z^2 = \beta y$$

and

$$x^2 + y^2 + z^2 = \gamma z$$

are pairwise orthogonal to each other.

12. Show that the normal to a surface of rotation at an arbitrary point passes through the rotation axis.

13. Show that the tangent plane at an arbitrary point of the surface $z = x\varphi\left(\frac{y}{x}\right)$, where φ is a smooth function of a single real variable, passes through the origin of coordinates.

14. Show that the tangent plane to a surface generated by the tangents to a space curve is the same at all the points of a given generator.

15. Write the equation of the tangent plane and the equations of the normal at the surface generated by the binormals of a curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$.

16. The surface Φ is generated by the binormals of a curve γ. Show that at the points of the curve γ the tangent plane to Φ coincide with the osculating plane to γ, while the normal to the surface is the principal normal to the curve γ.

17. A surface is generated by the principal normals of a curve γ. Establish the equation of the surface. Show that at the points of the curve γ the tangent plane to the surface coincide with the osculating plane to the curve γ.

18. On the normals to a surface Φ, in the same direction, there are taken segments of the same length, measured from the surface. The ends of these segments describe another surface, Φ^*, "parallel" to the surface Φ. Show that the surfaces Φ and Φ^* have, at corresponding points, common normals and parallel tangent planes.

19. Study the variation of the sign of the function $\mathbf{r} \cdot \mathbf{n}$ for the torus, where $\mathbf{r}$ is the radius vector of a point of the torus, while $\mathbf{n}$ is the normal to the torus at that point. Find the geometrical locus of the points from the torus at which $\mathbf{r} \cdot \mathbf{n} = 0$ (the origin of the coordinates is taken at the centre of the torus).

20. Find a close surface on which the function $\mathbf{r} \cdot \mathbf{n}$ is a constant.

21. Find the points of the torus

$$\boldsymbol{\rho} = \{(R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\}$$

at which its normal is parallel to the vector $\mathbf{a} = \{l, m, n\}$

22. Write the equation of the tangent plane to the torus

$$\begin{cases} x = (3 + 2\cos u)\cos v, \\ y = x = (3 + 2\cos u)\sin v, \\ z = 2\sin u, \end{cases}$$

parallel to the plane $x + y + \sqrt{2}z + 5 = 0$.

23. Find the tangent plane to the surface $z = x^4 - 2xy^3$, perpendicular to the vector $\mathbf{a} = \{-2, 6, 1\}$. Find, also, the tangency point.

24. Find the envelope and the regression edge of the family of spheres of constant radius, equal to a, with the centers on the circle

$$\begin{cases} x^2 + y^2 = b^2, \\ z = 0. \end{cases}$$

25. Find the envelope of the family of spheres

$$x^2 + y^2 + (z - C)^2 = 1.$$

26. Find the envelope, the characteristics and the regression edge for a family of spheres of constant radius a, with centers on a given curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ (tubular surface).

27. Find the envelopes and the regression edge for a family of spheres passing through the origin of the coordinates and with the centers on the curve

$$\begin{cases} x = t^3, \\ y = t^2, \\ z = t. \end{cases}$$

28. Find the envelope of a family of planes forming with the coordinate solid angle $x > 0, y > 0, z > 0$ a triangular pyramid of constant volume V.

29. Find the envelope and the regression edge for the family of normal planes to the helix

$$\begin{cases} x = a \cos t, \\ y = a \sin t, \\ z = bt. \end{cases}$$

30. Find the envelopes, the characteristics and the regression edge for the family of normal planes to a space curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ (*the polar surface of the curve*).

31. Find the envelope, the characteristics and the regression edge for the family of rectifying planes of a space curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$.

32. Find the envelope, the characteristics and the regression edge of the family of osculating planes of a space curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$.

33. Find the envelope, the characteristics and the regression edge of the family of planes

$$(\mathbf{r}, \mathbf{n}(\alpha)) + D(\alpha) = 0,$$

where $||\mathbf{n}|| = 1$, $||\mathbf{n}'(\alpha)|| \neq 0$, $||\mathbf{n}''(0)|| \neq 0$.

34. Find the envelope and the characteristics of the family of circular cylinders of constant radius r:

a) $(x - C)^2 + y^2 = r^2$;

b) $x^2 + (y - C)^2 = r^2$;

c) $(x - C)^2 + (y - C)^2 = r^2$.

35. Write the equation of the tangent plane at an arbitrary point of the surface generated by the tangents to the space curve $\mathbf{r} = \mathbf{r}(s)$. Study its position when the tangency point moves along a generators of the surface.

36. Show that the osculating plane to the regression edge of a developable surface coincides with the tangent plane to the surface at that point.

37. Show that the intersection curve between a developable surface and the osculating plane at its regression edge has, for arbitrary point of this edge, the curvature equal to $\frac{3}{4}$ from the curvature of the regression edge at that point.

38. A *Catalan surface* is a skew ruled surface whose generators are parallel to a given plane, the *directing plane*. Show that a ruled surface

$$\mathbf{r} = \mathbf{r}_1(u) + v\mathbf{b}(u)$$

is a Catalan surface iff

$$(\mathbf{b}, \mathbf{b}', \mathbf{b}'') = 0, \quad \mathbf{b}'' \neq 0.$$

39. Find the first fundamental form of the second degree surfaces

1. $x^2 + y^2 + z^2 = r^2$ (the sphere);
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (the ellipsoid);
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (the hyperboloid with one sheet);
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ (the hyperboloid with two sheets);

5. $\frac{x^2}{p} + \frac{y^2}{q} = 2z$ (the elliptical paraboloid);
6. $\frac{x^2}{p} - \frac{y^2}{q} = 2z$ (the hyperbolical paraboloid);
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ (the cone);
8. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (the elliptical cylinder);
9. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (the hyperbolical cylinder);
10. $y^2 = 2px$ (the parabolical cylinder).

40. Specify which of the following quadratic forms can play the role of the first fundamental form for a surface:

1. $ds^2 = du^2 + 4dudv + dv^2$;
2. $ds^2 = du^2 + 4dudv + 4dv^2$;
3. $ds^2 = du^2 - 4dudv + 6dv^2$;
4. $ds^2 = du^2 + 4dudv - 2dv^2$.

41. Find the transformation formulas for the coefficients of the first fundamental form and of the quantity

$$H = \sqrt{EG - F^2}$$

for a parameters' change.

42. Show that any revolution surface has a parameterization for which the first fundamental form can be written under the form

$$ds^2 = du^2 + G(u)dv^2.$$

43. A coordinate net on a surface is called a *Tchebysheff net* if the segments of coordinate lines from a family intercepted between two coordinate lines from the other family have constant length. Show that a coordinate net on a surface is a Tchebysheff net iff $E'_v = 0, G'_u = 0$.

44. Find a coordinate transformation such that the first fundamental form of the pseudosphere,

$$ds^2 = a^2 \operatorname{ctg}^2 u du^2 + a^2 \sin^2 u dv^2$$

becomes

$$ds^2 = d\overline{u}^2 + G(\overline{u})d\overline{v}^2.$$

45. Find the equation of a curve which intersect the meridians of a revolution surface under a constant angle α (the *loxodrom*).

46. Find the loxodroms of the sphere.

47. Find the orthogonal trajectories of the rectilinear generators of a conical surface.

48. Find the differential equation of the curve which intersects the rectilinear generators of a tangent developable under a constant angle α.

49. Find the orthogonal trajectories of the family of curves

$$u + v = \text{const},$$

lying on the sphere

$$\begin{cases} x = R\cos u \cos v, \\ y = R\cos u \sin v, \\ z = R\sin u. \end{cases}$$

50. On the circular cone

$$\begin{cases} x = u\cos v, \\ y = u\sin v, \\ z = u, \end{cases}$$

one considers the family of curves

$$v = u^2 + \alpha,$$

where α is a parameter. Find the orthogonal trajectories of this family.

51. Find the equations of the curves from the sphere

$$\begin{cases} x = R\cos u \sin v, \\ y = R\sin u \sin v, \\ z = R\cos v \end{cases}$$

which bissect the angle between the meridians and the parallels.

52. Find the intersection angle between the curves

$$u + v = 0, \quad u - v = 0$$

from the right helicoid

$$\begin{cases} x = u\cos v, \\ y = u\sin v, \\ z = av. \end{cases}$$

53. Find the perimeter and the interior angles of the curvilinear triangle

$$u = \pm\frac{1}{2}av^2, \quad v = 1,$$

from the surface with the first fundamental form

$$ds^2 = du^2 + (u^2 + a^2)dv^2.$$

54. On the surface with the first fundamental form

$$ds^2 = du^2 + \sinh^2 u dv^2$$

find the length of the arc of the curve $u = v$ between the points $M_1(u_1, v_1)$ and $M_2(u_2, v_2)$.

55. Find the angle between the curves

$$v = u + 1 \quad \text{şi} \quad v = 3 - u$$

from the surface

$$\begin{cases} x = u\cos c, \\ y = u\sin v, \\ z = u^2. \end{cases}$$

56. On a sphere there is given a right angle triangle, whose sides are arcs of great circles of the sphere. Find:

1. the relation between the sides of the triangle;
2. the area of the triangle.

57. Find the area of the quadrilater from the right helicoid

$$\begin{cases} x = u\cos v, \\ y = u\sin v, \\ z = av. \end{cases}$$

bounded by the curves

$$u = 0, \quad u = a, \quad v = 0, \quad v = 1.$$

58. Find the second fundamental form of the following surfaces of rotation:

1. $x = f(u)\cos v, \quad , y = f(u)\sin v, \quad z = \varphi(u)$ – the surface of rotation of rotation axis Oz;
2. $x = R\cos u \cos v, \quad y = R\cos u \sin v, \quad z = R\sin u$ – the sphere;
3. $x = a\cos u \cos v, \quad y = a\cos u \sin v, \quad z = c\sin u$ – the ellipsoid of rotation;
4. $x = a\cosh u \cos v, \quad y = a\cosh u \sin v, \quad z = c\sinh u$ – the hyperboloid of rotation with one sheet;
5. $x = a\sinh u \cos v, \quad y = a\sinh u \sin v, \quad z = c\cosh u$ – the hyperboloid of rotation with two sheets;
6. $x = u\cos v, \quad y = u\sin v, \quad z = u^2$ – the paraboloid of rotation;
7. $x = R\cos v, \quad y = R\sin v, \quad z = u$ – the circular cylinder;
8. $x = u\cos v, \quad y = u\sin v, \quad z = ku$ – the circular cone;
9. $x = (a + b\cos u)\cos v, \quad (a + b\cos u)\sin v, \quad z = b\sin u$ – the torus;
10. $x = a\cosh\frac{u}{a}\cos v, \quad y = a\cosh\frac{u}{a}\sin v, \quad z = u$ – the catenoid;
11. $x = a\sin u\cos v, \quad y = a\sin u \sin v, \quad z = a\left(\ln\operatorname{tg}\frac{u}{2} + \cos u\right)$ – the pseudosphere.

59. Find the principal curvatures and the principal directions of the right helicoid

$$x = u\cos v, \quad y = u\sin v, \quad z = av.$$

60. Find the principal curvatures at the vertices of the hyperboloid with two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

61. Compute the principal curvatures of the surface

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z$$

at the point $M(0, 0, 0)$.

62. Show that at each point of the surface

$$x = u\cos v, \quad y = u\sin v, \quad z = \lambda u$$

one of the normal principal sections is a straight line.

63. Find the curvatures of the normal sections of the surface $y = \frac{1}{2}x^2$:

1. at an arbitrary point and in an arbitrary direction;
2. at the points of the lines obtained by sectioning the surface with planes $z = k$ and in directions tangents to this curves;
3. at the point $M(2, 2, 4)$, in the direction of the tangent to the curve
$$y = \frac{1}{2}x^2, \quad z = x^2.$$

64. On the surface

$$x = u^2 + v^2, \quad y = u^2 - v^2, \quad z = uv$$

we take the point $P(u = 1, v = 1)$.

1. Compute the principal curvatures of the surface at P.
2. Find the equations of the tangents PT_1, PT_2 to the principal normal sections at the indicated point.
3. Find the curvature of the normal section passing through the tangent to the curve $v = u^2$.

65. We are given the surface

$$z = 2x^2 + \frac{9}{2}y^2.$$

1. Find the equation of the Dupin's indicatrix at the origin of coordinates.
2. Compute, at the origin of coordinate, the curvature radius of the normal section curve whose angle with the x-axis is 45°.

66. In the tangent plane at the point M to a surface on draws n straight lines passing through the origin and intercepting between them angle equal to $\frac{\pi}{n}$. Show that

$$K_m = \frac{1}{n}\left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}\right),$$

where $\frac{1}{r_i}$ are the normal curvatures of the curves from the surface which are tangent to the given straight lines, while K_m is the mean curvature of the surface at the point M.

67. Through a point M of an ellipsoid of rotation one draws all the possible curves, lying on the ellipsoid. Find the geometrical locus of the curvature centers of these curves at M.

68. Find the surfaces for which the first fundamental form is a perfect square.

69. Through a point M of a surface one considers all the possible plane sections. Find the equation of the surface which contains the centers of the osculating circles of these sections.

70. A parabolical cylinder intersects with a plane which is perpendicular to its generators after a parabola C.Let M be the vertex of this parabola and MT – the tangent to the parabola at the vertex M. On what curve lie the foci of the parabolas obtained sectioning the cylinder with planes passing through the straight line MT?

71. Starting from the fact that an ellipse can be projected onto a circle, find the curvature radius of an ellipse at one of its vertices, using Meusnier's theorem.

Bibliography

[1] Bär, C. – *Elementare Differentialgeometrie*, Walter de Gruyter, 2001

[2] Barbosa, J.L.M., Colares, A.G. – *Minimal Surfaces in* $\mathbb{R}^3$, Springer Verlag, Berlin, 1986 (Lecture Notes in Mathematics, 1195)

[3] Beltrami, E. – *Ricerche di analisi applicata alla geometria*, Giorn. di Mat., **2-3**, 1864–1865

[4] Bianchi, L. – *Lezioni di geometria differenziale*, Spoerri, Pisa, 1894

[5] Blaschke, W. – *Vorlesungen über Differentialgeometrie. I. Elementare Differentialgeometrie*, Springer Verlag, Berlin, 1921

[6] Burali-Forti, C. – *Fondamenti per la geometria differenziale di una superficie col metodo vettoriale generale*, Rend. Palermo, **33**(1912)

[7] do Carmo, M.P. – *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976

[8] Catalan, E. – Journal de Mathém., **7** (1842), p.203

[9] Codazzi, D. – *Sulle coordinate curvilinee d'una superficie e dello spazio*, Ann. di Mat. (2), **1**, 1867–1868; **2**, 1868–1869; **4**, 1870–1871.

[10] Darboux, G. – *Leçons sur la théorie générale des surfaces*, Volumes I to IV, Gauthier-Villars, Paris, 1887-1896

[11] Dupin, Ch. – *Développement de géométrie*, Paris, 1813

[12] Dupin, Ch. – *Applications degéométrie et de m'ecanique*, Paris, 1822

[13] Enneper, A. – Zeitschrift für Mathem. und Physik, **9** (1864), p.108

[14] Euler, L. – *Introduction in analysin infinitorum*, I–II, Lausannae, 1748

[15] Euler, L. – *De projectione geographica superficiei sphaerici*, Acta Ac. Petrop., I, 1777

[16] Eisenhart, L.P. – *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn & Co., Boston, 1909

[17] Eisenhart, L.P. – *An Introduction to Differential Geometry*, Princeton University Press, 1947

[18] Favard, J. – *Cours de géométrie differentielle locale*, Gauthier-Villars, Paris, 1957

[19] Fedenko, A.S. (ed.) – *Differencial'naja geometrija*, Minsk, 1982

[20] Finikov, S.P. – *Kurs differencial'noi geometrii*, Moscow, 1952

[21] Forsyth, A.R. – *Lectures on the Differential Geometry of Curves and Surfaces*, Cambridge University Press, 1912

[22] Frenet, F. – *Sur les courbes à double courbure*, Journ. de Math. **1**, 1852

[23] Gauss, C.F. – *Disquisitiones generales circa superficies curvas*, Comm. Soc. Göttingen, Bd. 6, 1823-1827

[24] Gray, A. – *Modern Differential Geometry of Curves and Surfaces*, CRC Press, 1993

[25] Hilbert, D., Cohn-Vossen, S. – *Geometry and the Imagination*, Chelsea, New York, 1952

[26] Hopf, H. – *Selected Topics in the Differential Geometry in the Large*, Notes by T. Klotz, New York, Institute of Mathematical Sciences, New York University, 1955

[27] Hopf, H. – *Lectures on Differential Geometry in the Large*, Notes by J. W. Gray, Standford, Applied Mathematics and Statistics Laboratory, Standford University, 1955

[28] Hsing, C.C. – *A First Course in Differential Geometry*, John Wiley, 1981

[29] Jellet, J.H. – *On the properties of inextensible surfaces*, Trans. Irish Ac. Dublin, textbf22 [1855], 1856

[30] Joachimstahl, F. – *Demonstrationes theorematum ad superficies curvas spectantum*, Journ. reine angew. Math. **30**, 1846

[31] Klingenberg, W. – *A Course in Differential Geometry*, Springer (Graduate Texts in Mathematics, 51), 1983

[32] Kreyszig, E. – *Differential Geometry*, University of Toronto Press, Toronto, 1959

[33] Lagrange, J.L. – *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies*, Miscellanea Taurinesnea, II, 1760–1761

[34] Lagrange, J.L. – *Sur la construction des cartes géographiques*, Nouv. Mém. Ac. Berlin, 1779

[35] Lamé, G. – *Leçons sur les coordonnées curvilignes et leurs diverses applications*, Paris, 1859

[36] Lelong-Ferrand, J., Arnaudiès, J.M. – *Cours de mathématiques*, Tome 3: *Géométrie et cinématique*, 2^e édition, Dunod, Paris, 1977

[37] Meusnier, J.B. – *Mémoire sur la courbure des surfaces*, Mémoires des savants étrangers, **10** (lu 1776), 1785, 477–510

[38] Mainardi, G. – *Sulla teoria generale delle superficie*, Giorn. Ist. Lombardo, **9**, 1856

[39] Minding, F. – *Ueber die Biegung krummer Flächen*, Journ. reine angew. Math., **18**, 1838

[40] Minding, F. – *Wie sich entscheiden lässt, ob zwei gegebene krumme Flächen auf einander abwickelbar sind oder nicht; näbst Bemerkungen über die Flächen von unveränderlichem Krümmungsmasse*, Journ. reine angew. Math., **19**, 1839

[41] Monge, G. – *Mémoire sur l'intégration de quelques équations aux derivées partielles*, Mém. Ac. sci., 1787

[42] Monge, G. – *Applications de l'analyse à la géométrie*, Paris, 1807

[43] Nitsche, J.C.C. – *Vorlesungen über Minimalflächen*, Springer, 1975

[44] O'Neill, B. – *Elementary Differential Geometry*, Academic Press, New York, 1966

[45] Osserman, R. – *A Survey of Minimal Surfaces*, second edition, Dover, 1986

[46] Pogorelov, A.V. – *Differential Geometry*, Noordhoff, Groningen, 1966

[47] Raševskii, P.K. – *Kurs differencial'noi geometrii*, 4^{th} edition, Moscow, 1956

[48] Riemann, B. – *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, Gött. Abh., **13**, 1868

[49] Rodrigues, O. – *Recherches sur la théorie analytique des lignes et des rayons de courbure des surfaces*, Corrésp. École polyt., textbf3, 1815

[50] Germain, S. – *Mémoire sur la courbure des surfaces*, Journ. reine angew. Math., **7**, 1830

[51] Scheffers, G. – *Anwendung der Differential- und Integralrechnung auf Geometrie. I. Einführung in die Theorie der Kurven in der Ebene und im Raume; II. Einführung in die Theorie der Flächen*, Teubner, Berlin-Leipzig, 1900, 1902

[52] Scherk, P. – Crelle's Journal f. Mathem., **13** (1835)

[53] Serret, A. – *Mémoire sur quelques formules relatives á la théorie des courbes à double courbure*, Journ. de Math., **16**, 1851

[54] Steiner, J. – *Ueber parallele Flächen*, Mon. Ber. Ak. Wiss., Berlin, 1840, pp. 114–118

[55] Stoker, J.J. – *Differential Geometry*, John Wiley and Sons, 1969

[56] Struik, D.J. – *Lectures on Classical Differential Geometry*, second edition, Dover, 1988

[57] Thorpe, J.A. – *Elementary Topics in Differential Geometry*, Springer (Undergraduate Texts in Mathematics), 1979

[58] Vygodskii, M. Ya. – *Differential'naja geometriya*, Moscow-Leningrad, 1949

[59] Willmore, T. – *An Introduction to Differential Geometry*, Clarendon Press, Oxford, 1959

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