Ramanujan expansions of arithmetic functions of several variables

László Tóth

Department of Mathematics, University of Pécs
Ifjúság útja 6, 7624 Pécs, Hungary
`ltoth@gamma.ttk.pte.hu`

Let $c_q(n)$ denote the Ramanujan sum, defined as the sum of n-th powers of the primitive q-th roots of unity. Let $\sigma(n)$ and $\tau(n)$ be the sum and the number of divisors of n, respectively. According to Ramanujan's classical identities, for every fixed $n \in \mathbb{N}$,

$$\frac{\sigma(n)}{n} = \zeta(2) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}, \qquad \tau(n) = -\sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n),$$

where ζ is the Riemann zeta function.

We discuss the expansions of certain arithmetic functions of several variables with respect to the Ramanujan sums $c_q(n)$ and their unitary analogues $c_q^*(n)$. We show, among others, that the series

$$\frac{\sigma(\gcd(n_1,\ldots,n_k))}{\gcd(n_1,\ldots,n_k)} = \zeta(k+1) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{\operatorname{lcm}(q_1,\ldots,q_k)^{k+1}} \qquad (k \geq 1),$$

$$\tau(\gcd(n_1,\ldots,n_k)) = \zeta(k) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{\operatorname{lcm}(q_1,\ldots,q_k)^{k}} \qquad (k \geq 2),$$

are absolutely convergent for every fixed $n_1,\ldots,n_k \in \mathbb{N}$. Our results and proofs generalize and simplify those of Ushiroya [1].

References

[1] N. Ushiroya, Ramanujan-Fourier series of certain arithmetic functions of two variables, *Hardy-Ramanujan J.* **39** (2016), 1–20.