Cesaro means in variable dyadic Hardy spaces

Kristóf Szarvas

Department of Numerical Analysis, Eötvös Loránd University

kristof@inf.elte.hu

Schipp, Wade, Simon and Pál [2] proved that if $f \in L_p$ $(1 < p < \infty)$, then $\lim_{n\to\infty} s_n f = f$ in the L_p-norm, where $s_n f$ denotes the n-th partial sum of the Walsh-Fourier series. Jiao, Zhou, Weisz and Wu [1] generalized this result for $L_{p(\cdot)}$: if $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- := \text{ess inf}_{x\in\Omega} p(x) \le p_+ :=$ ess $\sup_{x\in\Omega} < \infty$ and for all atoms A, the exponent function $p(\cdot)$ satisfies that

$$\mathbb{P}(A)^{p_-(A)-p_+(A)} \le K_{p(\cdot)}, \tag{1}$$

then for all $f \in L_{p(\cdot)}$, $\lim_{n\to\infty} s_n f = f$ in the $L_{p(\cdot)}$-norm. Unfortunately, these results are not true if $p \le 1$ or if $p_- \le 1$. Although, for $p \le 1$, or $p_- \le 1$, we can prove convergence results with the help of Cesaro means. For $\alpha > 0$ and $n \in \mathbb{N}$, the Cesaro means of the martingale f is defined by

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^{\alpha} \widehat{f}(k) w_k,$$

where A_k^α denotes the binomial coefficient $\binom{k+\alpha}{k}$.

We consider three types of maximal operators ($T^{(\alpha)}$, $U^{(\alpha)}$ and $V^{(\alpha)}$) and we prove that (under some conditions) each maximal operator is bounded from the classical dyadic martingale Hardy space H_p to the classical Lebesgue space L_p and these maximal operators are bounded on $L_{p(\cdot)}$.

Theorem 1 *Let $\alpha \in (0,1]$ and $t, r > 0$ such that $\alpha t < r/(r-t) < (1+\alpha)t$.*

1. *If $0 < p \le \infty$, then for all $f \in H_p$,*

$$\left\|T^{(\alpha)} f\right\|_p, \left\|U^{(\alpha)} f\right\|_p, \left\|V^{(\alpha)} f\right\|_p \le C_p \|f\|_{H_p}.$$

2. *If $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \le p_+ < \infty$ and $p(\cdot)$ satisfies the condition (1), then for all $f \in L_{p(\cdot)}$,*

$$\left\|T^{(\alpha)} f\right\|_{p(\cdot)}, \left\|U^{(\alpha)} f\right\|_{p(\cdot)}, \left\|V^{(\alpha)} f\right\|_{p(\cdot)} \le C_{p(\cdot)} \|f\|_{p(\cdot)}.$$

Using this, we can prove the boundedness of the Cesaro maximal operator from $H_{p(\cdot)}$ to $L_{p(\cdot)}$, where the Cesaro maximal operator is defined by $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|$.

Theorem 2 *Let $0 < \alpha \in (0,1]$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1/(\alpha+1) < p_- < \infty$ and suppose that $p(\cdot)$ satisfies condition (1). Then*

$$\|\sigma_*^\alpha f\|_{p(\cdot)} \le C \|f\|_{H_{p(\cdot)}} \qquad \left(f \in H_{p(\cdot)}\right).$$

As a consequence, we can prove theorems about almost everywhere- and norm convergence.

References

[1] Y. Jiao, D. Zhou, F. Weisz, and L. Wu. Variable martingale Hardy spaces and their applications in Fourier analysis. *(preprint).*

[2] F. Schipp, W. R. Wade, P. Simon, and J. Pál. *Walsh Series: An Introduction to Dyadic Harmonic Analysis.* Adam Hilger, Bristol, New York, 1990.