Project code: **PN-II-ID-PCE-2011-3-0024**
Contract: **157/05.10.2011**, financed by UEFISCDI
Project title: ***The structure and sensitivity of the solution sets of variational inequalities, optimization and equilibrium problems under generalized monotonicity***
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SCIENTIFIC RESEARCH REPORT No. 5
covering the period of 01.01.2015 – 31.12.2015

I. Research Team

Prof. **Gabor KASSAY**, PhD (project leader)
Assoc. Prof. **Nicolae POPOVICI**, PhD
Assoc. Prof. **Cornel PINTEA**, PhD
Lect. **Szilard LASZLO**, PhD
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II.Project summary and objectives

(a) The project is mainly motivated by the growing literature in scalar and vector optimization problems, variational inequalities, and equilibrium problems, which neatly shows that these fields are appropriate for applying the modern tools of variational analysis. The following objectives have been proposed in the funding application:

- **O1** *Studying condition numbers and metric regularity within parametric variational inequalities and parametric equilibrium problems*
- **O2** *Identifying classes of generalized monotone operators for which local and global monotonicity are equivalent and deduce injectivity results*
- **O3** *Studying the structure of the solution sets for generalized monotone operators*
- **O4** *Characterizing the subdifferential for certain classes of generalized monotone operators*
- **O5** *Approaching the sum problem for maximal monotone operators*
- **O6** *Constructing algorithms for variational inequalities and equilibrium problems*
- **O7** *Extending the proximal point algorithm for equilibrium problems to reflexive Banach spaces*
- **O8** *Characterizing generalized convex vector functions by scalarization*
- **O9** *Studying the structure of the solution sets of vector variational inequalities and equilibrium problems*

(b) All objectives planned for the period 01.01.2015 – 31.12.2015 have been achieved as follows:

O4: 2 papers under review [A2 and A3 in Section III (a)]
O5: 1 paper under review [A5 in Section III (a)]
O6: 2 papers under review [A4 and A5 in Section III (a)]
O7: 3 papers under review [A2, A3 and A5 in Section III (a)]
O8: 1 accepted paper [A1 in Section III (a)], 2 papers under review [A3 and A8 in Section III (a)] and 2 papers in preparation*
O9: 1 published paper [A1 in Section III (a)], 5 papers under review [A2-A6 in Section III (a)] and 4 papers in preparation*.

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*see Section III (b)

III. Scientific results

(a) Published/submitted papers

7 papers have been completed during the period of 01.01.2015 – 31.12.2015:

- **1** article **has been accepted in an ISI journal** (A1 in the next table);
- **2** articles **are submitted in revised form to ISI journals** (A2 and A3 in the next table);
- **4** papers **have been submitted for publication to ISI journals** (A4-A7 in the next table).

Ref.	Article	Objectives
A1	Ovidiu Bagdasar, **Nicolae Popovici**: *Extremal properties of generalized convex vector functions*, **J. Nonlinear Convex Anal.**, accepted, [JCR Science Edition 2014 IF: 0.655]	O8 and O9
A2	Adela Capata, **Gabor Kassay:** *Existence results for strong vector equilibrium with applications*, 1st revision under review at **J. Nonlinear Convex Anal.**	O4, O7 and O9
A3	**Gabor Kassay**, **Mihaela Miholca**, Nguyen The Vinh: *Vector quasi-equilibrium problems for the sum of two multivalued mappings*, 1st revision under review at **J. Optim. Theory Appl.**	O4, O7, O8 and O9
A4	Shaghaf Alzorba, Christian Guenther, **Nicolae Popovici**, Christiane Tammer: *A new algorithm for solving planar multiobjective location problems involving the Manhattan norm,* under review at **Eur J. Oper. Res.**	O6 and O9
A5	**Szilard Laszlo**: *Vector equilibrium problems on dense sets*, under review at **J. Optim. Theory Appl.**	O5, O6, O7 and O9
A6	Ioan Radu Peter, **Cornel Pintea**: *Necessary conditions for finite critical sets. Maps with infinite critical sets*, under review at **Topol. Methods Nonlinear Anal.**	O9
A7	Juan-Enrique Martinez-Legaz, **Cornel Pintea**: *Closed convex sets of Minkowski type*, under review at **J. Math. Anal. Appl.**	O8

The main results obtained in these seven papers are described below.

- Semistrictly/explicitly quasiconvex real-valued functions play an important role in scalar optimization, since they preserve several properties of convex functions. In particular, semistrictly quasiconvex real-valued functions satisfy the so-called ``local min - global min'' property, i.e., their local minimizers are global minimizers, cf. [J. Ponstein, *Seven kinds of convexity*, SIAM Review 9 (1967), 115—119]. Also, as shown in [O. Bagdasar, **N. Popovici**, *Local maximum points of explicitly quasiconvex functions*, **Optim. Lett**., 9 (2015), 769–777], every explicitly quasiconvex real-valued function satisfies a ``local max - global min'' property, namely: any local maximizer belonging to the intrinsic core of the function's domain is a global minimizer. By continuing their research, in the paper **A1** [*Extremal properties of generalized convex vector functions*, **J. Nonlinear Convex Anal.**, accepted] **N. Popovici** together with O. Bagdasar (University of Derby, UK) have established similar extremal properties for vector functions taking values in a finite-dimensional Euclidean space, with respect to optimality concepts currently used in vector optimization. It is shown that the ``local min - global min'' property can be extended to several classes of semistrictly quasiconvex vector-valued functions, while the ``local max - global min'' property hold for componentwise explicitly quasiconvex vector functions, i.e., any ideal, strong or weak local maximizer, which belongs to the intrinsic core of the domain, is an ideal, strong or weak global minimizer, respectively. These results give new insights on the structure of ideal, strong and weakly optimal solution sets in multicriteria linear fractional programming.
- **G. Kassay** (**project leader)** together with A. Capata (Technical University of Cluj-Napoca) have studied in the paper **A2** [*Existence results for strong vector equilibrium with applications*, under review at **J. Nonlinear Convex Anal.**] the strong vector equilibrium problem in case when the function involved is given by a sum of two bifunctions taking values in a topological vector space ordered by a convex cone. Similar results as in the paper [**G. Kassay** and **M. Miholca**: *Existence results for vector equilibrium problems given by a sum of two functions*, **J. Global Optim.**, 63 (2015), 195-211] are established, but the paper **A2** aims to obtain existence results for solutions in the strong sense. It has to be mentioned that in the literature only few results have been given for strong solutions, and the continuity hypotheses involved are rather strong. By introducing a new type of generalized monotonicity for bifunctions, the authors succeeded to deduce results under usual continuity assumptions thanks to a method based on dilating cones.
- Due to their numerous and useful applications, the vector equilibrium problems have been studied by many authors in last decades. Blum and Oettli [*From optimization and variational inequalities to equilibrium problems*, Math. Student, 63 (1994), 123-145] established existence results for the scalar equilibrium problem given by the sum of two functions, while Tan and Tinh [*On the existence of equilibrium points of vector functions*, Numer. Funct. Anal. Optim., 19 (1998), 141-156] extended the results of Blum and Oettli from the scalar to the vector case. On the other hand, the **project leader** together with **M. Miholca** [*Existence results for vector equilibrium problems given by a sum of two functions*, **J. Global Optim.**, 63 (2015), 195-211] obtained some generalizations of the main results of Tan and Tinh, under a new monotonicity assumption (called C-essential quasimonotonicity). Focusing on multivalued mappings, Fu [*Vector equilibrium problems. Existence theorems and convexity of solution set*, **J. Global Optim.,** 31 (2005), 109-119] obtained some extensions of the results of Blum and Oettli, as well as those of Tan and Tinh. However, his results don't recover **Kassay** and **Miholca**'s results since the monotonicity assumption he considered reduces, in case of single-valued vector bifunctions, to the same monotonicity used by Tan and Tinh, and the latter is stronger than the C-essential quasimonotonicity considered by **Kassay** and **Miholca**. In 2003, Ansari and Yao [*On Vector Quasi-Equilibrium Problems*. In: Daniele, P., Giannessi, F., Maugeri, A. (eds.), *Equilibrium Problems and Variational Models*, vol. 68, pp.1-18, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003] and, approximately at the same time, Ansari and Flores-Bazan [*Generalized vector quasi-*

equilibrium problems with applications, J. Math. Anal. Appl., 277 (2003), 246-256] introduced the so called generalized vector quasi-equilibrium problem, where the feasible set of the problem is provided by a multivalued mapping and the solution should be, in addition, a fixed point of that mapping.

In the paper **A3** [*Vector quasi-equilibrium problems for the sum of two multivalued mappings*, under review at **J. Optim. Theory Appl.**] the **project leader** together with **M. Miholca** and Nguyen The Vinh (University of Transport and Communications, Hanoi, Vietnam) have studied the vector quasi-equilibrium problem for the sum of two multivalued bifunctions. The authors obtained sufficient conditions for the existence of solutions of such problems in the setting of topological vector spaces. The assumptions are required separately on each of these bifunctions. The results in this paper unify, improve and extend some well-known existence theorems from the literature, like those mentioned before: Tan and Tinh, Fu, **Kassay** and **Miholca** (2015) and Gwinner-Oettli.

- **N. Popovici** together with S. Alzorba, Chr. Guenther and Chr. Tammer (Martin Luther University of Halle-Wittenberg, Germany) developed in the paper **A4** [*A new algorithm for solving planar multiobjective location problems involving the Manhattan norm,* under review at **Eur J. Oper. Res.**] a new effective numerical method for solving planar multiobjective location problems, where the distance-type objective functions are defined by means of the Manhattan norm. The mathematical background of the so-called *Rectangular Decomposition Algorithm* mainly relies on a dual characterization of efficient solutions given by Gerth and Poehler [*Dualitaet und algorithmische Anwendung beim vektoriellen Standortproblem*, Optimization, 19 (1988), 491-512], several structural properties of the efficient solutions'set established by Wendell, Hurter Jr. and Lowe [*Efficient points in location problems*, AIIE Transactions, 9 (1977), 238-246], and a new characterization of this set in terms of minimality with respect to four ordering cones, namely the usual quadrants of the Cartesian system of coordinates. The latter characterization leads to an implementable procedure, embedded in the first phase of the Rectangular Decomposition Algorithm, based on the Jahn-Graef-Younes method for solving discrete vector optimization, cf. [J. Jahn, *Vector Optimization - Theory, Applications, and Extensions* (2nd Ed.), Berlin-Heidelberg, Springer, 2011]. In comparison with other known algorithms for solving location problems, the algorithm developed in the paper **A4** has several special features. First of all it identifies all nonessential objectives, hence the running time needed to generate the set of efficient solutions can be reduced drastically, by eliminating these objectives. In contrast to certain algorithms, which produce one efficient solution corresponding to a particular choice of scalarization parameters, this algorithm provides a well structured representation of the whole set of efficient solutions as a finite union of line segments and rectangles. This representation can be used as input for further applications, as for instance to minimize/maximize an additional cost function over the efficient solution set, cf. [S. Alzorba, Chr. Guenther, **N. Popovici**, *A special class of extended multicriteria location problems*, **Optimization**, 64 (2015), 1305-1320].

- It is interesting, of course, to check whether the conditions imposed for obtaining the existence of an equilibrium point, in the work [**Sz. Laszlo**, A. Viorel, *Densely defined equilibrium problems*, **J. Optim. Theory Appl.**, 166 (2015), 52-75], can also be implemented in the vector and multifunction case. If the answer is affirmative, then these results can further be applied to vector and multifunction variational inequlities, vector and mutifunction optimization or cone saddle points. In the work **A5** [*Vector equilibrium problems on dense sets*, under review at **J. Optim. Theory Appl.] Szilard Laszlo** answered affirmatively to this question for vector equilibrium problems. However, the techniques used in the proofs of the results obtained, that ensure the existence of a vectorial equilibrium point, have been signnificantly different from those used in the above mentioned paper by **Laszlo** and Viorel (2015). The results obtained in the paper **A5** are applied to Minty vector variational inequalities and vector optimization.

- In the paper **A6** [*Necessary conditions for finite critical sets. Maps with infinite critical sets*, under review at **Topol. Methods Nonlinear Anal.**] **Cornel Pintea** and Radu Peter (Technical University of Cluj-Napoca) provide neccessary conditions on a given map, betweentwo compact differential manifolds, for its critical set to be finite. As consequences of these necessary conditions we also provide several examples of pairs of compact differential manifolds such that every map between them has infinitee critical set.

- In their work **A7 [**Closed convex sets of Minkowski type, under review at **J. Math. Anal. Appl.**]**,** J.E. Martínez-Legaz and **C. Pintea** provide several characterizations of the Minkowski sets, i.e. the closed, possibly unbounded, convex sets which are representable as the convex hulls of their sets of extreme points. The equality between the relative boundary of a closed convex set containing no lines and its Pareto-like associated set ensure the Minkowski property of the set. In two dimensions this equality characterizes the Minkowski sets containing no lines (within objective **O8)**.

(b) Work in progress

- The **project leader** (Prof. Gabor Kassay) together with Monica Bianchi (Universita Cattolica del Sacro Cuore di Milano, Italy) and Rita Pini (Universita degli Studi di Milano Bicocca, Italy) have studied the Lipschitz property of the solution mapping defined by a parametric equilibrium problem (within objective **O9**). The latter has three variables (the first where the solutions has to be found, the second a so called free variable, while the third is the parameter). The problem itself can be equivalently reformulated as a problem of finding zeroes of a certain operator, defined by the subdifferential mapping of the initial function, with respect to its second variable. The results are obtained by applying the methods in some recent papers of the authors, as for instance [M. Bianchi, G. Kassay, R. Pini: *Stability results of variational systems under openness with respect to fixed sets*, **J. Optim. Theory Appl.**, 164 (2015), 92-108]–this paper being described within the project's Report nr. 4 (2014);
- **Nicolae Popovici** jointly with Matteo Rocca (University of Insubria, Varese, Italy) and D. Kuroiwa (Shimane University, Matsue, Japan) introduced new classes of generalized convex set-valued maps by an embedding technique, following [D. Kuroiwa, T. Nuriya, *A generalized embedding vector space in set optimization*, In: S.-B. Hsu et al. (Eds.), Nonlinear Analysis and Convex Analysis, 297–303, Yokohama Publ., Yokohama, 2007]. Among other results, they obtained a characterization of embedded cone-convex set-valued functions (within objective **O8**);
- **Nicolae Popovici** and O. Bagdasar (University of Derby, UK) succeeded to extend the main results of their papers [*Local maximum points of explicitly quasiconvex functions*, **Optim. Lett**., 9 (2015) (4), 769–777] and **A1** [*Extremal properties of generalized convex vector functions*, **J. Nonlinear Convex Anal.**, accepted] to a more general framework of (possibly infinite-dimensional) partially ordered spaces spaces (within objectives **O8** and **O9**);
- **Nicolae Popovici** together with Christian Guenther and Christiane Tammer (Martin-Luther University of Halle-Wittenberg, Germany) obtained new results concerning the structure of efficient solutions' set of multicriteria planar location problems with respect to block-type norms and developed a new algorithm for solving such problems (within objective **O9**);
- Following the research direction on self-segment-dense sets, **Szilard Laszlo** obtained some new minimax results on dense sets (within objective **O9**). The techniques used in proofs rely on the powerful tools of Convex Analysis and Functional Analysis, and is shown that the notion of a self-segment-dense set, introduced in the paper [**Sz. Laszlo**, A. Viorel: *Generalized monotone operators on dense sets*, **Numer. Funct. Anal. Optim.**, 36 (2015), 901-929], is essential in all of

these results, in order that they cannot be replaced by usual denseness. As a matter of fact, minimax results on dense sets are absent in the literature. In this paper a motivation is given for this absence, by an example is shown that the general minimax results of Fan and Sion cannot be extended on usual dense sets. The minimax theorems obtained in this paper lead to some results which can be viewed as generalizations or extensions of James Compactness Theorem (one of the fundamental Theorems of Functional Analysis).

IV. Dissemination of research results

The scientific results mentioned within Section III of this report have been presented by the authors (members of the project research team) at **17** conferences, workshops and research seminars in Romania or abroad, namely:

- **6 international conferences and workshops,**
- **3 research seminars abroad;**
- **8 research seminars in Romania.**

The detailed list of talks is available on the project webpage at

http://www.cs.ubbcluj.ro/~grupanopt/PN-II-ID-PCE-2011-3-0024/index_eng.htm

Proiect leader,
Prof. Dr. Gabor Kassay