Project code: **PN-II-ID-PCE-2011-3-0024**
Contract: **157/05.10.2011**, financed by UEFISCDI
Project title: ***The structure and sensitivity of the solution sets of variational inequalities, optimization and equilibrium problems under generalized monotonicity***
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SCIENTIFIC RESEARCH REPORT No. 4
covering the period of 16.12.2013 – 15.12.2014

I. Research Team

Prof. Dr. **Gabor KASSAY** (project leader)
Assoc. Prof. Dr. **Nicolae POPOVICI**
Assoc. Prof. Dr. **Cornel PINTEA**
Lect. Dr. **Szilard LASZLO**
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Dr. **Erika NAGY**

II.Project summary and objectives

(a) The project is mainly motivated by the growing literature in scalar and vector optimization problems, variational inequalities, and equilibrium problems, which neatly shows that these fields are appropriate for applying the modern tools of variational analysis. The following objectives have been proposed in the funding application:

- **O1** *Studying condition numbers and metric regularity within parametric variational inequalities and parametric equilibrium problems*
- **O2** *Identifying classes of generalized monotone operators for which local and global monotonicity are equivalent and deduce injectivity results*
- **O3** *Studying the structure of the solution sets for generalized monotone operators*
- **O4** *Characterizing the subdifferential for certain classes of generalized monotone operators*
- **O5** *Approaching the sum problem for maximal monotone operators*
- **O6** *Constructing algorithms for variational inequalities and equilibrium problems*
- **O7** *Extending the proximal point algorithm for equilibrium problems to reflexive Banach spaces*
- **O8** *Characterizing generalized convex vector functions by scalarization*
- **O9** *Studying the structure of the solution sets of vector variational inequalities and equilibrium problems*

(b) All objectives planned for the period 16.12.2013 – 15.12.2014 have been achieved as follows:

O1: 1 published paper [A2 in Section III.(a)] and 1 paper under review [A9 in Section III.(a)]
O2: 1 published paper [A1 in Section III.(a)], 1 paper under review [A10 in Section III.(a)] and 1 paper in preparation**
O3: 1 published paper [A2 in Section III.(a)] and 1 PhD Thesis*
O5: 1 published paper [A1 in Section III.(a)], 1 paper under review [A10 in Section III.(a)] and 1 paper in preparation**
O6: 1 paper under review [A7 in Section III.(a)]
O8: 2 published papers [A3 and A4 in Section III.(a)], 1 paper under review [A6 in Section III.(a)] and 1 paper in preparation**
O9: 1 published paper [A5 in Section III.(a)], 2 papers under review [A7 and A8 in Section III.(a)], 1 PhD Thesis* and 2 papers in preparation**.

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*see Section III.(b)
**see Section III.(c)

III. Scientific results

(a) Published/submitted papers

10 papers have been completed during the period of 16.12.2013 – 15.12.2014:

- **4** articles **appeared in ISI journals** (A1-A4 in the next table);
- **1** article **has been accepted in an ISI journal** (A5 in the next table);
- **4** papers **have been submitted for publication to ISI journals** (A6, A7, A9 and A10 in the next table);
- **1** paper **has been submitted for publication to a journal indexed in international databases** (A8 in the next table).

Ref.	Article	Obiectives
A1	Daniela Marian, Ioan Radu Peter, **Cornel Pintea**: *A class of generalized monotone operators*, **J. Math. Anal. Appl**., 421 (2015) (2), 1827-1843, doi:10.1016/j.jmaa.2014.08.017 [JCR Science Edition 2013 IF: 1,119]	O2 and O5
A2	Monica Bianchi, **Gabor Kassay**, Rita Pini: *Stability results of variational systems under openness with respect to fixed sets*, **J. Optim. Theory Appl.**, doi:10.1007/s10957-014-0560-4 , online [JCR Science Edition 2013 IF: 1,406]	O1 and O3
A3	Daishi Kuroiwa, **Nicolae Popovici**, Matteo Rocca: *A characterization of cone-convexity for set-valued functions by cone-quasiconvexity*, **Set-Valued Var. Anal.**, doi:10.1007/s11228-014-0307-2, online [JCR Science Edition 2013 IF: 0,918]	O8
A4	Ovidiu Bagdasar, **Nicolae Popovici**: *Local maximum points of explicitly quasiconvex functions*, **Optim. Lett**., doi:10.1007/s11590-014-0781-3, online [JCR Science Edition 2013 IF: 0,990]	O8

A5	**Gabor Kassay**, **Mihaela Miholca**: *Existence results for vector equilibrium problems given by a sum of two functions*, **J. Global Optim.**, doi:10.1007/s10898-014-0264-1 [JCR Science Edition 2013 IF: 1,355]	O9
A6	Daishi Kuroiwa, **Nicolae Popovici**, Matteo Rocca: *Characterizations of cone-convex vector-valued functions*, under review	O8
A7	**Nicolae Popovici**: *A decomposition approach to vector equilibrium problems*, under review	O6 and O9
A8	Adela Capata, **Gabor Kassay:** *Optimality conditions for vector equilibrium and saddle point problems with constraints*, under review	O9
A9	Monica Bianchi, **Gabor Kassay**, Rita Pini: *Linear openness of the composition of set-valued maps and applications to variational systems*, under review	O1
A10	**Cornel Pintea**, Tiberiu Trif: *The monotoncity of perturbed gradients of convex functions*, under review	O2 and O5

The main results obtained in these ten papers are mentioned below.

- In the paper **A1** [*A class of generalized monotone operators*, **J. Math. Anal. Appl.**, 421 (2015) (2), 1827–1843] **C. Pintea** studied, jointly with D. Marian and I. R. Peter (Technical University, Cluj-Napoca, Romania), the monotonicity of the Minty-Browder operator perturbed by the opposite of the identity operator. Despite the infinite dimensional contexts in which the Minty–Browder monotonicity is generally used, the motivation behind the present work is rather geometric and the results proved here concern mostly the finite dimensional context. The example the authors have in mind, to play the role of the Minty-Browder monotone operator is, in dimension two, that of the gradient of a C^1-smooth convex function perturbed by a rotation. This operator is perturbed, in its turn, by the opposite of the identity operator and the resulting operator is shown to be h-monotone. In other words it is shown the h-monotonicity of such a gradient of convex function perturbed by the difference between a rotation and the identity operator. The h-monotonicity of such a perturbed gradient combined with its local injectivity leads us to its global injectivity. It is also studied the generalized monotonicity of a C^1-smooth function and the generalized monotonicity of their Frechet differentials as well as the relation between the generalized monotonicity of the gradient of a real valued function of one real variable and convexity.

- **G. Kassay** (the project leader) has continued the investigations done in the previous stages of the project jointly with M. Bianchi (University "Cattolica del Sacro Cuore" of Milan, Italy) and R. Pini (University Bicocca of Milan, Italy) in the paper **A2** [*Stability results of variational systems under openness with respect to fixed sets*, **J. Optim. Theory Appl.**, doi:10.1007/s10957-014-0560-4]. It was introduced here the concepts of linear openness and metric regularity for set-valued mappings related to fixed sets and the equivalence between them has been shown. The topic has origins in the early days of functional analysis starting with the Banach open mapping theorem, which concerns continuous linear mappings between two Banach spaces. This fundamental result states that such a mapping is surjective if and only if it is open, i.e., the image of an open set remains open. There is a strong connection between openness, metric regularity, and interiority conditions on domains and ranges, as seen in the Robinson-Ursescu Theorem. Along the years, infinite-dimensional inverse function theorems and implicit function theorems, due to Lyusternik, Graves, and Bartle and Graves, have been derived and extended. Banach spaces could also be replaced by more general metric spaces. The statements of this paper can be included in the stream of results that, starting from the celebrated Lyusternik and Graves

theorems, try to argue under what conditions the regularity properties of a set-valued map are preserved once another set-valued map is added. Two different approaches according to the setting are given; in the case of a Banach space the Ekeland Variational Principle proves to be the main tool. In metric spaces, the result is proved via a fixed point approach. These approaches enable the authors to deal with the sensitivity analysis of a variational system, when a perturbation of the set-valued map, associated to the problem, is considered.

- An interesting characterization of convex real-valued functions by means of quasiconvexity has been established by J.-P. Crouzeix in his thesis [*Contribution a l'etude des fonctions quasi-convexes*, These de Doctorat, Universite de Clermont-Ferrand II, 1977]. It states that a real-valued function f is convex if and only if the sum f+x* is quasiconvex for every linear functional x*. In the paper **A3** [*A characterization of cone-convexity for set-valued functions by cone-quasiconvexity*, **Set-Valued Var. Anal.**, doi:10.1007/s11228-014-0307-2] **N. Popovici,** jointly with M. Rocca (University of Insubria, Varese, Italy) and D. Kuroiwa (Shimane University, Matsue, Japan), have extended the result of J.-P. Crouzeix to set-valued functions taking values in topological linear spaces, partially ordered by a convex cone C. It is shown that, under suitable assumptions (i.e., the hypotheses of the Radstrom-Urbanski cancellation rule) a set-valued function F is C-convex if and only if F+G is C-quasiconvex for every set-valued function G, which is affine in the sense of V. V. Gorokhovik [*Representations of affine multifunctions by affine selections*, Set-Valued Anal., 16 (2008), 185–198].

- It is known that explicitly quasiconvex real-valued functions, i.e., functions which are both quasiconvex and quasiconcave (as for instance, the ratios of affine functions), play an important role in nonlinear optimization. These functions preserve several fundamental properties of convex functions. C. Zalinescu has shown in his monograph [*Convex Analysis in General Vector Spaces*. World Scientific, River Edge, 2002] that any local maximum point of a convex function belonging to the interior of the function's domain is actually a global minimum point, hence the function is constant on a neighborhood of that point. In the paper [P. Phohomsiri: *On the extrema of linear least-squares problems*, J. Optim. Theory Appl., 127 (2005), 665–669] the author asserted that any local extremum point of the residual function associated to the least squares problem is actually a local minimum point. In the paper **A4** [*Local maximum points of explicitly quasiconvex functions*, **Optim. Lett**., doi:10.1007/s11590-014-0781-3] **N. Popovici** and O. Bagdasar (collaborator from Derby, UK) extended the aforementioned result of C. Zalinescu to explicitly quasiconvex functions, under the mild hypothesis that a local maximum point exsts in the intrinsic core of function's domain. Moreover, the authors shown that whenever the residual function $f(x) = \|A(x)-b\|^2$ admits a local maximum point, then A should be the null operator.

- **The project leader** together with **M. Miholca** (**Berchesan**) obtained in the paper **A5** [*Existence results for vector equilibrium problems given by a sum of two functions*, **J. Global Optim.**, doi:10.1007/s10898-014-0264-1] new existence results for weak solutions for equilibrium problem, where the function involved can be represented as a sum of two functions. These functions take values in a topological vector space ordered by a convex cone with nonempty interior and the hypotheses are separately imposed on each of the functions. In this way it was possible to obtain, in particular, either variants of Ky Fan's theorem, or variants of Minty-Browder theorem concerning variational inequalities. Following the idea of E. Blum and W. Oettli [*From optimization and variational inequalities to equilibrium problems*, Math. Student, 63 (1994), 123-145], K. R. Kazmi considered the same problem for the vector case [*On vector equilibrium problem*, Proc. Indian Acad. Sci. Math. Sci., 110 (2000), 213-223], but his hypotheses are too strong in order to recover the results of Blum-Oettli, in particular. The aim of this paper is to weaken the assumptions of Kazmi in such a way to be able to recover Blum-Oettli's results on one hand, and

by assuming alternative conditions on the vector functions, to deduce new existence theorems, on the other hand. The special case of reflexive Banach spaces endowed with the weak topology is separately treated; in that case, mild sufficient conditions for guaranteeing coercivity are presented.

- In the paper **A6** [*Characterizations of cone-convex vector-valued functions*, under review] **N. Popovici**, M. Rocca (University of Insubria, Varese, Italy) and D. Kuroiwa (Shimane University, Matsue, Japan) extend the result of J.-P. Crouzeix mentioned in the description of the paper A3 (namely, the characterization of convex real-valued functions by means of quasiconvexity of all linear perturbations) to the framework of vector functions, defined on a nonempty convex subset of a real linear space X with values in a real linear space Y, partially ordered by a convex cone C. More precisely, it is shown that a vector-valued function f is C-convex if and only if the sum function f+A is C-quasiconvex for any linear operator A acting between X and Y. The proof of the main result is based on single-valued selections of set-valued linear processes, whose existence was proven by A. Szaz in the paper [*Linear extensions of relations between vector spaces*, Comment. Math. Univ. Carolinae, 44 (2003), 367–385].

- Vector equilibrium problems have been introduced by Q. H. Ansari in his paper [*Vector equilibrium problems and vector variational inequalities*, The Second World Congress of Nonlinear Analysts, Athens, Greece, 1996; In: F. Giannessi (Ed.), Vector Variational Inequalities and Vector Equilibria. Mathematical Theories, Kluwer, 2000, pp. 1-16]. In the paper **A7** [*A decomposition approach to vector equilibrium problems*, under review] **N. Popovici** considered a particular class of vector equilibrium problems, defined by a bifunction taking values in a finite-dimensional Euclidean space, partially ordered by the usual ordering cone (the positive orthant). It is shown that such a vector equilibrium problem can be naturally decomposed into a family of equilibrium subproblems, obtained by selecting certain scalar components of the initial vector-valued bifunction. Under suitable generalized convexity assumptions imposed on the bifunction, the author shows that every weak solution of the initial vector equilibrium problem is actually a proper solution (hence a strong one) for at least one equilibrium subproblem. By considering the particular case when the vector-valued bifunction is componentwise convex, two known results are recovered from [T. J. Lowe, J.-F. Thisse, J. E. Ward and R. E. Wendell: *On efficient solutions to multiple objective mathematical programs*, Management Science, 30 (1984) (11), 1346-1349] and [**N. Popovici**: *Pareto reducible multicriteria problems*, Optimization, 54 (2005) (3), 253-263].

- In the paper **A8** [*Optimality conditions for vector equilibrium and saddle point problems with constraints*, under review], the project leader together with A. Capata (Technical University of Cluj-Napoca)studied the weak vector equilibrium problem. It is well-known that the equilibrium problem contains, in particular, the minimax (saddlepoint) problem and the latter represents a mathematical model for two-person, zero-sum noncooperative games. For a long time, only real-valued payoff functions have been considered in game theory. Motivated by the necessity to describe real-world situations, in the last decades much attention has been attracted to multicriteria games, i.e., games with vector-valued payoff functions. Moreover, for a better description, one needs to consider further restrictions on the strategy sets. In this way the feasible set of the game becomes a proper subset of the product of the strategies, leading in this way to a minimax problem with constraints, and, in the more general case, to an equilibrium problem with constraints. In this paper two independent results are given concerning the existence of weak solutions for vector equilibrium problems. Their proofs are based on separations theorems of convex sets with closed hyperplanes: the first uses a recent result of F. Flores-Bazan and G. Mastroeni, while the second is based on the classical separation theorem of M. Eidelheit. As a

byproduct, one obtains sufficient conditions for the existence of saddlepoint for constrained noncooperative games.

- **The project leader** together with M. Bianchi (University "Cattolica del Sacro Cuore"of Milan, Italy) and R. Pini (University Bicocca of Milan, Italy) have studied in the recent paper **A9** [*Linear openness of the composition of set-valued maps and applications to variational systems*, under review] sufficient conditions under which the composition of two set-valued mappings becomes open with linear rate. The paper can be seen as a continuation of the investigations done in A2, but in a more general framework, since the paper A2 dealt with the sum of two mappings. The method consists in constructing a convenient set-valued map whose fixed point set is precisely the inverse image of the composed mapping. Then the results can be obtained through Nadler's fixed point theorem and Lim's lemma. One works with a new concept introduced here, namely a kind of upper semicontinuity for set-valued mappings. The paper contains also two sections with possible applications. In the first a generalized equation depending on a parameter was considered and the Lipschitz stability of the solution set was established with respect to the parameter. In the second, the Lipschitz stability of the solution set of parametric variational inequalities was established.

- In the paper **A10** [*The Monotoncity of Perturbed Gradients of Convex Functions*, under review], **C. Pintea** and T. Trif (Babes-Bolyai University of Cluj-Napoca) study the monotonicty of the perturbed gradients of some convex functions. More precisely, the authors provide sufficient conditions on the convex function f and choose suitable linear transformations A which assure the η-monotonicity of the perturbed gradient grad(f)+A, under the hypothesis $-1<\eta<0$. These conditions are given in terms of the magnitude of the Hessian operators associated with f and the spectra of the operators $(A+A^*)/2$ and A^*A and they make the Minty-Browder monotonicity of the perturbation grad(f)+A to fail. While the symmetric perturbations are only considered as instructive examples in the Minty-Browder monotone case, we mostly rely, in the even dimensional context, on linear transformations which can be represented as linear combinations between the identity operator and skew-symmetric isometries. In dimension two the rotations do admit representations as such linear combinations and the perturbations of the gradients with rotations has been already studied by D. Marian, I. R. Peter and **C. Pintea** in the paper A1. In this present work **C. Pintea** and T. Trif also prove, in higher dimension, some global injectivity results for the gradients of some strictly convex functions perturbed by such linear combinations. For this purpose, the authors rely on the η-monotonicity of the perturbed gradients with such linear combinations and their local injectivity.

(b) Ph.D. Thesis related to the project objectives **O3** and **O9**:

Mihaela Miholca (**Berchesan**): *Contributions to the Theory of Equilibrium Problems and Variational Inequalities* (Ph.D. Supervisor: Professor Gabor Kassay), Faculty of Mathematics and Computer Science, Babes-Bolyai University, Cluj-Napoca, Romania, Ph.D. Dissertation defended on June 6, 2014, Summary available at http://www.cs.ubbcluj.ro/~grupanopt/PN-II-ID-PCE-2011-3-0024/thesis-summary-mihaela-miholca-berchesan.pdf

(c) Work in progress

Preliminary results have been obtained in the period of 16.12.2013 – 15.12.2014 by

- **G. Kassay** with A. Capata (Technical University of Cluj-Napoca, Romania) studied strong solutions of vector equilibrium problems (within objective **O9**);

- **N. Popovici** with M. Rocca (University of Insubria, Varese, Italy) and D. Kuroiwa (Shimane University, Matsue, Japan) obtained some preliminary results which could be used to characterize con-convex set-valued functions by means of affine perturbations (within objective **O8**);
- **N. Popovici** with S. Alzorba, C. Guenther and C. Tammer (Martin-Luther University of Halle-Wittenberg, Germany) developed a new algorithm for solving location problems with respect to the Minkowski norm (within objective **O9**);
- **C. Pintea** with D. Marian and I. R. Peter (Technical University of Cluj-Napoca, Romania) studied the monotonicity parameters of certain operators (within objectives **O2** and **O5**).

Also, preliminary investigations of certain classes of generalized convex vector functions have been initiated by **N. Popovici** with Ovidiu Bagdasar (University of Derby, UK, who visited the project team in Cluj-Napoca during May 15 - June 5, 2014)

IV. Dissemination of research results

The scientific results mentioned within Section III of this report have been presented by the authors (members of the project research team) at **18** conferences, workshops and research seminars in Romania or abroad, namely:

- **6 international conferences and workshops,**
- **2 research seminars abroad;**
- **10 research seminars in Romania.**

The detailed list of talks is available on the project webpage at

http://www.cs.ubbcluj.ro/~grupanopt/PN-II-ID-PCE-2011-3-0024/index_eng.htm

Proiect leader,
Prof. Dr. Gabor Kassay