

Project code: PN-II-ID-PCE-2011-3-0024
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Project title: *The structure and sensitivity of the solution sets of variational inequalities, optimization and equilibrium problems under generalized monotonicity*
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SCIENTIFIC RESEARCH REPORT No. 2 **covering the period of 16.12.2011 – 15.12.2012**

I. Research Team

Prof. Dr. **Gabor KASSAY** (project leader)
Assoc. Prof. Dr. **Nicolae POPOVICI**
Assoc. Prof. Dr. **Cornel PINTEA**
Assist. Prof. Dr. **Szilard LASZLO**
Dr. Boglarka **BURJAN-MOSONI** (member of the team till October 31, 2012)
PhD Student **Mihaela MIHOLCA (BERCHESAN)** (member of the team since November 1, 2012)
PhD Student **Erika NAGY**

II. Project summary and objectives

(a) The project is mainly motivated by the growing literature in scalar and vector optimization problems, variational inequalities, and equilibrium problems, which neatly shows that these fields are appropriate for applying the modern tools of variational analysis. The following objectives have been proposed in the funding application:

- O1** *Studying condition numbers and metric regularity within parametric variational inequalities and parametric equilibrium problems*
- O2** *Identifying classes of generalized monotone operators for which local and global monotonicity are equivalent and deduce injectivity results*
- O3** *Studying the structure of the solution sets for generalized monotone operators*
- O4** *Characterizing the subdifferential for certain classes of generalized monotone operators*
- O5** *Approaching the sum problem for maximal monotone operators*
- O6** *Constructing algorithms for variational inequalities and equilibrium problems*
- O7** *Extending the proximal point algorithm for equilibrium problems to reflexive Banach spaces*
- O8** *Characterizing generalized convex vector functions by scalarization*
- O9** *Studying the structure of the solution sets of vector variational inequalities and equilibrium problems*

(b) All objectives planned for the period 16.12.2011 – 15.12.2012 have been achieved as follows:

- O1: 1 published paper [A2 in Section III.(a)]
- O2: 1 published paper [A3 in Section III.(a)]
- O5: 1 published paper [A5 in Section III.(a)] and 1 PhD Thesis*
- O6: 3 published papers [A4, A6 and A7 in Section III.(a)]
- O7: 1 published paper [A1 in Section III.(a)]
- O8: 2 papers in preparation**
- O9: 2 published papers [A4 and A6 in Section III.(a)], 1 PhD Thesis* and 1 paper in preparation**

*see Section III.(b)

**see Section III.(c)

III. Scientific results

(a) Published/submitted papers

7 papers have been completed during the period of 16.12.2011 – 15.12.2012. All of them appeared in ISI journals (A1-A7 in the next table).

Ref.	Article	Objectives
A1	Regina Burachik and Gabor Kassay : <i>On a generalized proximal point method for solving equilibrium problems in Banach spaces</i> , Nonlinear Anal. , 75 (2012), 6456-6464, doi:10.1016/j.na.2012.07.020 [JCR Science Edition 2011 IF: 1.536, JCR Science Edition 2013 IF: 1.612]	O7
A2	Monica Bianchi, Gabor Kassay and Rita Pini: <i>An inverse map result and some applications to sensitivity of generalized equations</i> , J. Math. Anal. Appl. , 399 (2013), 279-290, doi:10.1016/j.jmaa.2012.10.023 [JCR Science Edition 2011 IF: 1.001, JCR Science Edition 2013 IF: 1.119]	O1
A3	Gabor Kassay , Cornel Pintea and Szilard Laszlo : <i>Monotone operators and first category sets</i> , Positivity , 16 (2012), 565-577, doi:10.1007/s11117-012-0193-5 [JCR Science Edition 2011 IF: 0.368, JCR Science Edition 2013 IF: 0.682]	O2
A4	Szilard Laszlo : <i>Existence of solutions of inverted variational inequalities</i> , Carpathian J. Math. , 28 (2012), 271-278, [JCR Science Edition 2011 IF: 0.906, JCR Science Edition 2013 IF: 0.642]	O6 and O9
A5	Szilard Laszlo : <i>On the strong representability of the generalized parallel sum</i> , accepted in Bull. Malays. Math. Sci. Soc. , 37 (2014) (4), 1029-1046, [JCR Science Edition 2011 IF: 0.779, JCR Science Edition 2013 IF: 0.854]	O5
A6	Alireza Amini-Harandi and Szilard Laszlo : <i>A coincidence point result via variational inequalities</i> , Fixed Point Theory , 15 (2014) (1), 87-98, [JCR Science Edition 2011 IF: 0.970, JCR Science Edition 2013 IF: 0.951]	O6 and O9
A7	Mihaela Miholca (Berchesan) : <i>On set-valued optimization problems and vector variational-inequalities</i> , Optim. Lett. , .8 (2014) (2), 463-476, doi:10.1007/s11590-012-0591-4 [JCR Science Edition 2011 IF: 0.952, JCR Science Edition 2013 IF: 0.990]	O6

The main results obtained in these seven papers are described below.

- The proximal point algorithm developed by R. T. Rockafellar in 1976 has shown to be a very efficient method for solving (approximating) generalized equations defined by maximal monotone operators. In the last 35 years lots of generalizations of this algorithm have been published, but only a few dealt with equilibrium problem. Among them we mention the paper of Iusem and Sosa (2010), where the proximal algorithm was established in Hilbert spaces. In the paper **A1** [*On a generalized proximal point method for solving equilibrium problems in Banach spaces*, **Nonlinear Anal.**, 75 (2012), 6456-6464], **the project leader** jointly with R. Burachik proposed an extension of the method of Iusem and Sosa for the equilibrium problem in reflexive Banach spaces. To this end, they introduced a regularization of the problem based on generalized Bergman functions and showed the existence and uniqueness of solution for this problem. The solution for each regularized problem provides the proximal method for solving the (initial) equilibrium problem. It was shown that each weak cluster point of the proximal sequence is a solution, whenever the function defining the problem is weakly upper semicontinuous with respect to its first argument. Under some additional hypotheses the sequence converges weakly to a solution.

- **The project leader** together with M. Bianchi (University "Cattolica del Sacro Cuore" of Milan, Italy) and R. Pini (University Bicocca of Milan, Italy, (external collaborators in the project) studied an important topic in the field of variational analysis in the paper **A2** [*An inverse map result and some applications to sensitivity of generalized equations*, **J. Math. Anal. Appl.**, 399 (2013), 279-290]. In the recent literature, two different aspects, closely related, are faced: the problem of the Lipschitzian properties of the inverse of the sum of two maps, giving rise to "inverse map theorems", and the Lipschitz regularity of the solutions of a parametric generalized equation. It is worthwhile noticing that the first topic has also strong connection with Milyutin's covering mapping theorem. Most of the results on these subjects are given in terms of modulus of regularity and Aubin constant of the maps involved, therefore showing their local point of view. Moreover, they usually concern the sum of a set-valued and a single-valued map. Only recently, more attention has been drawn on the study of the regularity properties of the sum of set-valued maps. The paper deals with the latter situation and sheds some new light on this aspect and on its possible implications in terms of sensitivity of parametric generalized equations. The results are neither global, nor local in the classical sense: their novelty consists in specifying exactly the relationship that must hold between the domains and the range of the maps involved.

- In the paper **A3** [*Monotone operators and first category sets*, **Positivity**, 16 (2012), 565-577] **the project leader** with **C. Pinte**a and **Sz. Laszlo** show that local monotonicity in the sense of Minty and Browder on certain residual sets ensure global monotonicity, and, according to an earlier result, the convexity of inverse images of that operator. A special attention is paid to residual sets that appear as complements of some special Baire sets, such as σ -compact sets or σ -algebraic varieties. This result is achieved gradually, showing first that continuous functions of one real variable that are local nonincreasing on some certain special sets are global nonincreasing. The convexity of the inverse images combined with their property to be discrete, for locally injective operators, ensure the global injectivity for that operators. The paper concludes with a brief subsection on global convexity which is obtained from local convexity certain residual sets.

- Based on one of his previous works, in which he defined the notion of operator of type ql (quasi-linear), in the paper **A4** [*Existence of solutions of inverted variational inequalities*, **Carpathian J. Math.**, 28 (2012), 271-278], **Sz. Laszlo** introduces two generalized variational inequalities and obtains some existence results of solutions to these problems. The notion of operator of type strict ql is introduced and using a famous result of Ky Fan's, respectively the

properties of the weak topology in a Banach space some existence results of solutions are obtained. It is shown by examples that these results fail in absence of the condition that the operators involved belong to the class mentioned above. Moreover, a result that can be viewed as a generalization of Minty's theorem is also obtained, more precisely, it is shown that under certain circumstances, the solutions of these variational inequalities coincide. Also in this case it is shown by examples that the operator involved to be of type strict ql is essential. Moreover, it is shown that even the condition that operator to be of type ql is not sufficient. As application it is shown that Brouwer's fixed point theorem is a consequence of the author's results.

- In the paper **A5** [*On the strong representability of the generalized parallel sum*, **Bull. Malays. Math. Sci. Soc**, accepted] **Sz. Laszlo** obtains a closedness type condition that ensures the maximal monotonicity of the generalized parallel sum $(AS^{-1}A^* + T^{-1})^{-1}$, where the operators S and T are strongly representable, and shows that this condition is weaker than the interior type condition considered by S. Simons. Using the same technique, the author obtains the interior type condition mentioned before. The technique is based on strong representative functions, on Fenchel conjugate and stable strong duality. As particular cases, besides the new results, are obtained some results known ones, analogous to paper **P4** of the list above. Some interior and closedness type conditions are formulated, that can easily be transcribed in relations involving only the domains of the operators S and T . The particular cases obtained from the generalized parallel sum, can easily be obtained by identifying the spaces involved and taking A the identity operator, or taking T identically zero, respectively. Further, the conditions for the reflexive case can easily be deduced from the author's results.

- Using a new class of nonlinear operators (ql type), in the paper **A6** [*A coincidence point result via variational inequalities*, **Fixed Point Theory** 15 (2014) (1), 87-98], **Sz. Laszlo** with A. Amini-Harandi (University of Shahrekord, Iran) obtained some existence results of solutions for a general variational inequality considered already in the literature. Several types of nonlinear operators (operators gql type) are introduced that generalizes the notion of operator of type ql . An operator belonging this class, which is not of type ql , is also presented. Some sufficient conditions, that ensure the existence of solution for the variational inequality mentioned before, are obtained. It is shown by an example, that these results fail in absence of the condition that operators involved are of type gql . As applications, the authors obtain coincidence and fixed point results in Hilbert spaces. It is worth mentioning that the coincidence point result, which can be regarded as an extension of the Brouwer's fixed point theorem, was unknown in the literature.

- In paper **A7** [*On set-valued optimization problems and vector variational-inequalities*, **Optim. Lett.**, 8 (2014) (2), 463-476] **M. Miholca (Berchesan)** introduced some notions of generalized convexity for set-valued mappings and studied the existing relations between them. Furthermore, by means of several examples it was shown that the generalized convexity concepts introduced here are different from those considered by Zeng and Li (2011). The relationship between the solution sets of Minty-type variational inequalities and a set-valued optimization problem has been established. It has to be mentioned that in order to obtain the results, the weak contingent generalized subdifferential was considered.

(b) Ph.D. Thesis related to the project objectives **O5** and **O9**:

Boglarka Burjan-Mosoni: *The equilibrium problem and applications* (Ph.D. Supervisor: Professor Gábor Kassay), Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania, Ph.D. Dissertation defended on March 9, 2012, Summary available at <http://www.cs.ubbcluj.ro/~grupanopt/PN-II-ID-PCE-2011-3-0024/thesis-summary-boglarka-burjan-mosoni>

(c) Work in progress

Preliminary results have been obtained in the period of 16.12.2011 – 15.12.2012, namely:

- **N. Popovici** with R.I. Bot (Technical University of Chemnitz, Germany, on leave to University of Vienna, Austria) studied some special classes of generalized convex vector functions and their characterization in terms of scalar functions (within objective **O8**);
- **C. Pinte**a with J. E. Martinez Legaz (Autonomous University of Barcelona, Spain) studied Krein-Milman-type results (within objective **O8**);
- **N. Popovici** with S. Alzorba and C. Guenther (Martin-Luther University, Halle-Wittenberg, Germany) studied a special class of multicriteria optimization problems obtained from a given location problem by adding new criteria (within objective **O9**).

Also, some investigations have been initiated by the **project leader** with Monica Bianchi (Universita Cattolica del Sacro Cuore di Milano, Italy, who visited the project team in Cluj-Napoca during April 24-28, 2012), Marco Castellani (Universita degli Studi dell'Aquila, Italy, who visited the project team in Cluj-Napoca during October 20-27, 2012) and Rita Pini (Universita degli Studi di Milano Bicocca, Italy, who visited the project team in Cluj-Napoca during April 24-28, 2012).

IV. Dissemination of research results

The scientific results mentioned within Section III of this report have been presented by the authors (members of the project research team) at **13** conferences, workshops and research seminars in Romania or abroad, namely:

- **4 international conferences and workshops;**
- **6 research seminars abroad;**
- **3 research seminars in Romania.**

The list of talks is available on the project webpage at

http://www.cs.ubbcluj.ro/~grupanopt/PN-II-ID-PCE-2011-3-0024/index_eng.htm

Project leader,
Prof. Dr. Gabor Kassay