Characterizations of ε-duality gap statements for composed optimization problems

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Consider two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$ respectively. Let the nonempty closed convex cone $C \subseteq Y$ and its dual cone $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in Y\}$. Let $f : X \to \overline{\mathbb{R}}$ be a proper function, $g : Y \to \overline{\mathbb{R}}$ be a proper function, which is also C-increasing and $h : X \to Y^\bullet$ be a proper vector function fulfilling $dom g \cap (h(dom f) + C) \neq \emptyset$. Unless otherwise stated, these hypotheses remain valid throught the entire chapter. Consider the optimization problem

$$\inf_{x\in X}[f(x) + (g \circ h)(x)]. \qquad (P^C)$$

For $x^* \in X^*$ we also consider the linearly perturbed optimization problem

$$\inf_{x\in X}\left[f(x) + (g \circ h)(x) - \langle x^*, x\rangle\right]. \qquad (P^C_{x^*})$$

To this problem we can attach different dual Fenchel-Lagrange-type problems. If f and (λh) are taken together one gets the following dual to $(P^C_{x^*})$

$$\sup_{\lambda \in C^*} \{-g^*(\lambda) - (f + (\lambda h))^*(x^*)\}. \qquad (D^C_{x^*})$$

When f and (λh) are separated, one gets the following dual problem

$$\sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-g^*(\lambda) - f^*(\beta) - (\lambda h)^*(x^* - \beta)\}. \qquad (\overline{D^C_{x^*}})$$

ε-duality gap statements using epigraphs

Let $\varepsilon \geq 0$. Consider the regularity conditions

$$\Big| \begin{array}{l} \{(x^*, 0, r) : (x^*, r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \bigcup\limits_{\lambda \in C^*} \{(a, -\lambda, r) \\ (a, r) \in epi((f + (\lambda h))^*)\}] \cap (X^* \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon) \end{array} \qquad (RC)$$

and

$$\Big| \begin{array}{l} \{(x^*, 0, r) : (x^*, r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \{(x^*, 0, r) : \\ (x^*, r) \in epi(f^*)\} + \bigcup\limits_{\lambda \in C^*} \{(a, -\lambda, r) : (a, r) \in epi((\lambda h)^*)\}] \cap \\ (X^* \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon) \end{array} \qquad (\overline{RC})$$

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition (RC) is fulfilled if and only if for any $x^* \in X^*$ there exists a $\overline{\lambda} \in C^*$ such that

$$(f + g \circ h)^*(x^*) \geq g^*(\overline{\lambda}) + (f + (\overline{\lambda} h))^*(x^*) - \varepsilon. \tag{1}$$

Remark

In the left-hand side of (1) one can easily recognize $-v(P^C_{x^*})$. The quantity in the right-hand side of (1) is not necessarily $-v(D^C_{x^*}) - \varepsilon$, as the supremum in $(D^C_{x^*})$ is not shown to be attained at $\overline{\lambda}$. Though, (1) implies $v(P^C_{x^*}) \leq v(D^C_{x^*}) + \varepsilon$, which actually means that for $(P^C_{x^*})$ and $(D^C_{x^*})$ there is ε-duality gap. Thus, (RC) yields that there is stable ε-duality gap for (P^C) and (D^C). Note also that $\overline{\lambda} \in C^*$ obtained in the above theorem is an ε-optimal solution of $(D^C_{x^*})$.

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition $(\overline{RC})$ *is fulfilled if and only if for any* $x^* \in X^*$ *there exist some* $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*

$$(f + g \circ h)^*(x^*) \geq g^*(\overline{\lambda}) + f^*(\overline{\beta}) + (\overline{\lambda}h)^*(x^* - \overline{\beta}) - \varepsilon. \qquad (2)$$

Remark

In the left-hand side of (2) *one can easily recognize* $-v(P^C_{x^*})$. *The quantity in the right-hand side of* (2) *is not necessarily* $-v(\overline{D^C_{x^*}}) - \varepsilon$, *as the supremum in* $(\overline{D^C_{x^*}})$ *is not shown to be attained at* $\overline{\lambda}$ *and* $\overline{\beta}$. *Though,* (2) *implies* $v(P^C_{x^*}) \leq v(\overline{D^C_{x^*}}) + \varepsilon$, *which actually means that for* $(P^C_{x^*})$ *and* $(\overline{D^C_{x^*}})$ *there is* ε*-duality gap. Thus* $(\overline{RC})$ *guarantees stable* ε*-duality gap for* (P^C) *and* $(\overline{D^C})$ *and, moreover, also for* (P^C) *and* (D^C). *Note also that the pair* $(\overline{\lambda}, \overline{\beta}) \in C^* \times X^*$ *obtained in the above theorem is an* ε*-optimal solution of* $(\overline{D^C_{x^*}})$.

In order to characterize formulae similar to (1) and (2), where appear actually the optimal values of (D^C) and $(\overline{D^C})$, let us consider the following regularity conditions

$$epi(f + g \circ h)^* \subseteq epi \inf_{\lambda \in C^*} [g^*(\lambda) + (f + (\lambda h))^*(\cdot)] - (0, \varepsilon) \qquad (RCI)$$

and

$$epi(f + g \circ h)^* \subseteq epi \inf_{\substack{\lambda \in C^* \\ \beta \in X^*}} [g^*(\lambda) + f^*(\beta) + (\lambda h)^*(\cdot - \beta)] - (0, \varepsilon). \quad (\overline{RCI})$$

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition (RCI) is fulfilled if and only if for any $x^* \in X^*$ we have

$$(f + g \circ h)^*(x^*) \geq \inf_{\lambda \in C^*} [g^*(\lambda) + (f + (\lambda h))^*(x^*)] - \varepsilon. \quad (3)$$

Remark

Relation (3) means actually $v(P^C_{x^*}) \leq v(D^C_{x^*}) + \varepsilon$, i.e. we have stable ε-duality gap for (P^C) and (D^C).

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition $(\overline{RCI})$ is fulfilled if and only if for any $x^* \in X^*$ we have

$$(f + g \circ h)^*(x^*) \geq \inf_{\substack{\lambda \in C^* \\ \beta \in X^*}} [g^*(\lambda) + f^*(\beta) + (\lambda h)^*(x^* - \beta)] - \varepsilon. \quad (4)$$

ε-duality gap statements using subdifferentials

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) One has

$$\partial(f + g \circ h)(x) \subseteq \bigcap_{\eta > 0} \bigcup_{\substack{\varepsilon_{1,2} \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta \\ \lambda \in C^* \cap \partial_{\varepsilon_2} g(h(x))}} \partial_{\varepsilon_1}(f + (\lambda h))(x) \qquad (RCSC)$$

for all $x \in X$ if and only if (3) holds for all $x^* \in R(\partial(f + g \circ h))$.

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) One has

$$\partial(f + g \circ h)(x) \subseteq \bigcup_{\substack{\varepsilon_{1,2} \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon \\ \lambda \in C^* \cap \partial_{\varepsilon_2} g(h(x))}} \partial_{\varepsilon_1}(f + (\lambda h))(x) \qquad (RCLC)$$

for all $x \in X$ if and only if for all $x^* \in R(\partial(f + g \circ h))$, there exists $\overline{\lambda} \in C^*$ such that (1) holds.

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) One has

$$\partial(f + g \circ h)(x) \subseteq \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_{1,2}\geq 0 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta \\ \lambda\in C^*\cap\partial_{\varepsilon_3} g(h(x))}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) \qquad (\overline{RCSC})$$

for all $x \in X$ if and only if for all $x^* \in R(\partial(f + g \circ h))$, (4) holds.

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) One has

$$\partial(f + g \circ h)(x) \subseteq \bigcup_{\substack{\varepsilon_{1,2}\geq 0 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta \\ \lambda\in C^*\cap\partial_{\varepsilon_3} g(h(x))}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) \qquad (\overline{RCLC})$$

for all $x \in X$ if and only if for all $x^* \in R(\partial(f + g \circ h))$, there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that (2) holds.

Results concerning ε-optimality conditions, ε-Farkas statements and (ε, η)-saddle points

From the results presented in the previous sections one can derive other useful statements concerning ε-optimality conditions, ε-Farkas assertions and characterizations for (ε, η)-saddle points. Let us consider the following regularity conditions:

$$\left| \begin{array}{c} (epi(f+g\circ h)^*)\cap(\{0\}\times\mathbb{R})\subseteq(epi\inf_{\lambda\in C^*}[g^*(\lambda)+(f+(\lambda h))^*(\cdot)])\cap \\ (\{0\}\times\mathbb{R})-(0,\varepsilon) \end{array} \right. \qquad (RCI^0)$$

and

$$\left| \begin{array}{l} (epi(f+g\circ h)^*)\cap(\{0\}\times\mathbb{R})\subseteq(epi\inf_{\substack{\lambda\in C^*\\ \beta\in X^*}}[g^*(\lambda)+f^*(\beta)+(\lambda h)^*(\cdot-\beta)] \\ \cap(\{0\}\times\mathbb{R})-(0,\varepsilon). \end{array} \right. \qquad (\overline{RCI}^0)$$

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) (a) Let $\varepsilon, \eta \geq 0$. *Suppose that the condition* (RCI^0) *is fulfilled. If* $\overline{x}$ *is an* ε*-optimal solution of the problem* (P^C)*, then there exist* $\varepsilon_1, \varepsilon_2 \geq 0$*, and* $\overline{\lambda} \in C^*$ *such that*

(i) $g^*(\overline{\lambda}) + g(h(\overline{x})) \leq (\overline{\lambda}h)(\overline{x}) + \varepsilon_2$,

(ii) $(f + (\overline{\lambda}h))^*(0) + (f + (\overline{\lambda}h))(\overline{x}) \leq \varepsilon_1$,

(iii) $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta$.

Moreover, $\overline{\lambda}$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* (D^C).

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$ *and* $\overline{\lambda} \in C^*$ *such that the relations (i)-(iii) hold for* $\overline{x} \in X$ *and* $\overline{\lambda} \in C^*$ *then* $\overline{x}$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* (P^C)*. Moreover,* $\overline{\lambda}$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* (D^C).

The similar statement for $(\overline{D^C})$ can be proven analogously.

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) (a) Let $\varepsilon, \eta \geq 0$. *Suppose that the condition* $(\overline{RCI}^0)$ *is fulfilled. If* $\overline{x}$ *is an* ε*-optimal solution of the problem* (P^C)*, then there exist* $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*
(i) $g^*(\overline{\lambda}) + g(h(\overline{x})) \leq (\overline{\lambda}h)(\overline{x}) + \varepsilon_3$,
(ii) $f^*(\overline{\beta}) + f(\overline{x}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1$,
(iii) $(\overline{\lambda}h)^*(-\overline{\beta}) + (\overline{\lambda}h)(\overline{x}) \leq \langle -\overline{\beta}, \overline{x} \rangle + \varepsilon_2$,
(iv) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \eta$.
Moreover, $(\overline{\lambda}, \overline{\beta})$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* $(\overline{D^C})$.
(b) If there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that the relations (i)-(iv) hold for* $\overline{x} \in X$, $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *then* $\overline{x}$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* (P^C). *Moreover,* $(\overline{\lambda}, \overline{\beta})$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* $(\overline{D^C})$.

In the following we give ε-Farkas-type results for (P^C) and its duals, too. Consider the following conditions:

$$\{(0,0,r) : (0,r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \bigcup_{\lambda \in C^*} \{(a, -\lambda, r) : (a,r) \in epi((f + (\lambda h))^*)\}] \cap (\{0\} \times \{0\} \times \mathbb{R}) - (0,0,\varepsilon) \qquad (RC^0)$$

and

$$\{(0,0,r) : (0,r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \{(0,0,r) : (0,r) \in epi(f^*)\} + \bigcup_{\lambda \in C^*} \{(a, -\lambda, r) : (a,r) \in epi((\lambda h)^*)\}] \cap (\{0\} \times \{0\} \times \mathbb{R}) - (0,0,\varepsilon) \qquad (\overline{RC}^0)$$

Theorem

(i) Suppose that (RC^0) holds. If $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$ then there exists $\bar{\lambda} \in C^*$ such that $g^*(\bar{\lambda}) + (f + \bar{\lambda} h)^*(0) \leq \varepsilon/2$.
(ii) If there exists $\bar{\lambda} \in C^*$ such that $g^*(\bar{\lambda}) + (f + \bar{\lambda} h)^*(0) \leq -\varepsilon/2$, then $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$.

Analogously, one can prove the following statements for (P^C) and $(\overline{D^C})$, too.

Theorem

(i) Suppose that $(\overline{RC}^0)$ *holds. If* $f(x) + (g \circ h)(x) \geq \varepsilon/2$ *for all* $x \in X$ *then there exist* $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*
$f^*(\overline{\beta}) + g^*(\overline{\lambda}) + (\overline{\lambda}h)^*(-\overline{\beta}) \leq \varepsilon/2.$
(ii) If there exist $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*
$f^*(\overline{\beta}) + g^*(\overline{\lambda}) + (\overline{\lambda}h)^*(-\overline{\beta}) \leq -\varepsilon/2$, *then* $f(x) + (g \circ h)(x) \geq \varepsilon/2$ *for all* $x \in X$.

Nevertheless, one can extend the investigations from this section also towards generalized saddle points.

The Lagrangian function assigned to $(P^C) - (D^C)$ is $L^C : X \times Y^* \to \overline{\mathbb{R}}$, defined by (cf. [5])

$$L^C(x, \lambda) = \begin{cases} f(x) + (\lambda h)(x) - g^*(\lambda), & \text{if } \lambda \in C^* \\ -\infty, & \text{otherwise.} \end{cases}$$

Let $\eta \geq 0$. We say that $(\overline{x}, \overline{\lambda}) \in X \times Y^*$ is (η, ε)-*saddle point* of the Lagrangian L^C if

$$L^C(\overline{x}, \lambda) - \eta \leq L^C(\overline{x}, \overline{\lambda}) \leq L^C(x, \overline{\lambda}) + \varepsilon, \text{ for all } (x, \lambda) \in X \times Y^*.$$

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) Assume that g is a convex and lower semicontinuous function fulfilling $g(y) > -\infty$ for all $y \in Y$. If $(\overline{x}, \overline{\lambda})$ is an (η, ε)-saddle point of L^C then $\overline{x} \in X$ is an $(\varepsilon + \eta)$-optimal solution to (P^C), $\overline{\lambda} \in C^*$ is an $(\varepsilon + \eta)$-optimal solution to (D^C) and there is $(\varepsilon + \eta)$-duality gap for the pair of problems (P^C) and (D^C), i.e. $v(P^C) \leq (D^C) + \varepsilon + \eta$.

An analogous result with the anterior theorem can be formulated for the pair of problems (P^C) and $(\overline{D^C})$ with the corresponding Lagrangian function given by (cf. [5]) $\overline{L^C} : X \times X^* \times Y^* \to \overline{\mathbb{R}}$

$$\overline{L^C}(x, \beta, \lambda) = \begin{cases} \langle \beta, x \rangle + (\lambda h)(x) - f^*(\beta) - g^*(\lambda), & \text{if } \lambda \in C^* \\ -\infty, & \text{otherwise.} \end{cases}$$

Theorem

Assume that g is a convex and lower semicontinuous function fulfilling $g(y) > -\infty$ for all $y \in Y$. If $(\overline{x}, \overline{\lambda})$ is an (η, ε)-saddle point of $\overline{L^C}$ then $\overline{x} \in X$ is an $(\varepsilon + \eta)$-optimal solution to (P^C), $\overline{\lambda} \in C^*$ is an $(\varepsilon + \eta)$-optimal solution to $(\overline{D^C})$ and there is $(\varepsilon + \eta)$-duality gap for the pair of problems (P^C) and $(\overline{D^C})$, i.e. $v(P^C) \leq (\overline{D^C}) + \varepsilon + \eta$.

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