Finite Functions in $C(Q)$ and Finite Elements in Vector Lattices

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Abstract

Some detailed analysis of continuous functions with compact support (i.e. finite functions) leads to the notion of finite, totally finite and self-majorizing elements in a vector lattice E, e.g. an element $\varphi \in E$ is called *finite* if there exists an element $z \in E$ (a majorant of φ) satisfying the property: for each $x \in E$ there is a number $c_x > 0$ such that the inequality

$$\boldsymbol{|x| \wedge n|\varphi| \leq c_x\, z}$$

holds for any $n \in \mathbb{N}$.

For a vector sublattice $H \subset E$ the relations between finite elements in H and in E are studied. In general, even a finite element in H may not be finite in E.

If the set $\mathfrak{M}(E)$ of all maximal ideals of an Archimedean vector lattice E is equipped with the hull-kernel topology τ_{hk} then the topological space $\big(\mathfrak{M}(E), \tau_{hk}\big) =: \mathfrak{M}(E)$ turns out to be a Hausdorff space but, in general, does not satisfy any stronger separation axiom. The space $\mathfrak{M}(E)$ carries many information on the vector lattice E.

The element φ is called *totally finite*, if it possesses a majorant which itself is a finite element. A finite element φ is called *selfmajorizing*, if $|\varphi|$ is a majorant of φ.

Finite, totally finite and selfmajorizing elements of a vector lattice E can be characterized by means of their *abstract supports*, i.e. the τ_{hk}-closure of subsets $G_x := \{M \in \mathfrak{M}(E) : x \notin M\}$ which are defined in $\mathfrak{M}(E)$ for any $x \in E$.

The τ_{hk}-closure $\mathrm{supp}_{\mathfrak{M}}(x)$ of G_x is the so-called *abstract support* of the element x. In radical-free vector lattices a finite element φ is characterized by the compactness of its abstract support and the majorants z of φ by the inclusion $\mathrm{supp}_{\mathfrak{M}}(\varphi) \subset G_z$. Totally finite and selfmajorizing elements can be similarly characterized.

The subspace $\mathfrak{M}_\Phi(E) = \bigcup\{G_\varphi \colon \varphi \text{ is a finite element in } E\}$ plays an important role for the representation of vector lattices by means of continuous functions, namely in the situation if one asks for representations, where the isomorphic image of each finite element is a functions with compact support.

Literature:

M. R. Weber, *Finite Elements in Vector Lattices.* W. de Gruyter, Berlin/Boston, 2014.