Remarks on Local Contractions and Isometries

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1 Introduction

This is a brief survey of fixed point results in certain geodesic spaces. We only mention a few specific open questions but the study, especially regarding the local theory, is far from complete.

A *path* in a metric space (X, d) is a continuous image of the unit interval $I = [0, 1] \subset \mathbb{R}$. If $S \equiv f(I)$ is a path then its *length* is defined as

$$\ell(S) = \sup_{(x_i)} \sum_{i=0}^{N-1} d(f(x_i), f(x_{i+1}))$$

where $0 = x_0 < x_1 < \cdots < x_N = 1$ is any partition of $[0, 1]$. If $\ell(S) < \infty$ then the path is said to be *rectifiable.*

Let (X, d) be a metric space and suppose each two points of X are the endpoints of a rectifiable path. We cam associate with (X, d) a metric space (X, ℓ) where the distance $\ell(x, y)$ between each two points x, y of X is the infimum of the lengths of all rectifiable paths joining them. In this case, ℓ is said to be the *length metric* (otherwise known an *inner metric* or *intrinsic metric*) and (X, ℓ) is called a *length space.*

A length space X is called a *geodesic space* if there is a path S joining each two points $x, y \in X$ for which $\ell(S) = d(x, y)$. Such a path is often called a *metric segment* (or *geodesic segment*) with endpoints x and y. There is a simple criterion which assures the existence of metric segments. A metric space (X, d) is said to be *metrically convex* if given any two points $p, q \in M$ there exists a point $z \in X$, $p \neq z \neq q$, such that

$$d(p, z) + d(z, q) = d(p, q).$$

Theorem 1.1 (Menger [21]) *Any two points of a complete and metrically convex metric space are the endpoints of at least one metric segment.*

Menger based the proof of his classical result on transfinite induction. Since then, other proofs have been given – see, e.g., [15] for a proof and citations.

Another criterion for geodesic spaces is given in [23]. (Recall that a Hausdorff space X is said to be *locally compact* if each of its points has a neighborhood that lies in a compact subset of X.)

Theorem 1.2 *If M is a complete metric space, locally compact at all except possibly one of its points, and any pair of points has a path of finite length joining them, then any pair of points has a shortest path joining them.*

There is an analog of Menger's criterion for length spaces. Here we use $B(x;r)$ to denote the closed ball centered at $x \in X$ with radius $r \geq 0$.

Definition 1.3 ([8]) A metric space (X,d) is said to satisfy *property* (A) if given any two points $x,y \in X$, any two numbers $b,c \geq 0$ such that $b+c = d(x,y)$, and any $\varepsilon > 0$,

$$B(x;b+\varepsilon) \cap B(y;c+\varepsilon) \neq \emptyset. \tag{A}$$

The **proof** of Theorem 1 of [8] yields the following fact.

Theorem 1.4 *If a complete metric space (X,d) satisfies property (A) then each two points of X can be joined by a rectifiable path. (Thus X has an intrinsic metric.)*

A metric space is said to be *finitely compact* (or proper) if each of its bounded closed sets is compact. It is a consequence of a result of Mycielski [23] that every finitely compact length space is a geodesic space. (Such a space is necessarily complete.) In fact Mycielski proved the following.

Theorem 1.5 ([23]) *If (X,d) is a complete metric space, locally compact at all except possibly one of its points, and if any pair of points of X has a rectifiable path joining them, then any pair of points has a shortest path joining them. (Thus (X,ℓ) is actually a geodesic space.)*

This result for finitely compact X is found in [7]. However the real line $\mathbb{R}$ with the metric $\rho(x,y) = \min\{|x-y|, 1\}$ is an example of a bounded metric space which is complete and locally compact, but not compact. On the other hand, every metric space which is complete, locally compact, and convex is necessarily finitely compact [20].

Geodesic spaces provide a fruitful setting for a number of results in metric fixed point theory. There is an interesting general problem of the extent to which these theorems lead to 'approximate' fixed point results in length spaces.

2 Local radial contractions

It was also shown by Mycielski in [23] that if (X,d) is a complete metric space which has the property that each two of its points can be joined by a rectifiable path, and if ℓ is the length metric on X, then (X,ℓ) is also complete. However this latter fact is implicit in the proof of Theorem 1 of Hu and Kirk [10], and explicitly stated as a corollary there. (Note that it is not true that if X is compact, then (X,ℓ) is compact – see the example on p. 943 of [7]. Also see Remark 1 in [23].)

Definition 2.1 A mapping g defined on a metric space X is said to be a *local radial contraction* if there exists $k \in (0,1)$ such that $d(g(x), g(u)) \leq kd(x,u)$ for u in some neighborhood N_x of x.(It follows that any local radial contraction is continuous.)

Proposition 2.2 ([9]) *Let (X,d) be a compact metric space and suppose $g: X \to X$ is a local radial contraction. Then there exist numbers $k \in (0,1)$ and $\beta > 0$ such that $d(g(x), g(y)) \leq kd(x,y)$ for all $x, y \in X$ such that $d(x,y) \leq \beta$.*

Theorem 2.3 (Theorem 1 of [10]) *Let (X,d) be a complete metric space and $g : X \to X$ a local radial contraction. Suppose for some $x_0 \in X$ the points x_0 and $g(x_0)$ are joined by a path of finite length. Then the sequence $(g^n(x_0))$ converges to a fixed point of g.*

Rakotch proved the above theorem in [25] under the stronger assumption that g is *locally contractive* in the sense that there exists $k \in (0,1)$ such that each point of $x \in X$ has a neighborhood N_x such that $d(g(u), g(v)) \leq kd(u,v)$ for all $u, v \in N_x$.

Theorem 2.4 ([10]) *Let (X,d) be a complete metric space, and suppose each two points of X can be joined by a rectifiable path. Then (X,ℓ) is also complete, where ℓ is the length metric on X induced by d. Consequently every local radial contraction $g : X \to X$ has a unique fixed point $x_0 \in X$, and moreover $\lim_{n\to\infty} g^n(x) = x_0$ for each $x \in X$.*

An example, which we reproduce below, is given in [10] which shows that Theorem 2.3 is false if x_0 and $g(x_0)$ are merely assumed to be joined by an arbitrary path rather than a rectifiable path. An example in [25] shows that the fixed point in Theorem 2.3 need not be unique, even if the space is connected.

Proposition 2.5 *Each two points of a connected open subset of a Banach space can be joined by a rectifiable path.*

Proof. Let G be a connected open subset of a Banach space and let $x \in G$. Let

$$G_0 = \{y \in G : x \text{ and } y \text{ can be joined by a rectifiable path}\}.$$

If $y \in G$ then some open ball centered at y also lies in G, and any point in this ball is clearly in G_0. So G_0 is an open subset of G. Suppose G_0 is a proper subset of G and let $u \in G \backslash G_0$. Then some open ball centered at U lies in G, and this ball must necessarily lie in $G \backslash G_0$. This would mean that G is the union of two disjoint open sets, which is clearly impossible because G is connected. Hence $G_0 = G$ ■

Theorem 2.6 *Let G be a connected open subset of a Banach space, suppose $g : G \to G$ is a local radial contraction, and suppose g can be extended to a continuous mapping $\overline{g} : \overline{G} \to \overline{G}$. Then $\overline{g}$ has a fixed point in $\overline{G}$.*

Proof. Let ℓ be the path metric on G. In view of proof of Theorem 2.3 g is a contraction mapping on (G,ℓ). Let $x \in G$. By a standard argument $(g^n(x))$ is a Cauchy sequence in (G,ℓ). This in turn implies $(g^n(x))$ is a Cauchy sequence in (G,d). Hence $(g^n(x))$ converges to some point $x_0 \in \overline{G}$. Since $\overline{g}$ is continuous we conclude $\overline{g}(x_0) = x_0$. Moreover if k is the contraction constant for g, and if for some $x \in G$ the segment $(x, g(x))$ lies in G, then we have the estimate

$$d(g^n(x), x_0) \leq \ell(g^n(x), x_0) \leq \frac{k^n}{1-k}\ell(x, g(x)) = \frac{k^n}{1-k}d(x, g(x)).$$

■

Example. In general connected open subsets of a Banach space does not have a path metric. A simple example can be given in $\mathbb{R}^2$. Let $\varepsilon \in (0, 1/2)$ and let H be the open rectangle with vertices $(0,0), (0,1+\varepsilon), (1,1+\varepsilon), (1,0)$. Delete the closed strip centered on the segment joining $(1/2,0)$ to $(1/2,1)$ of width $1/6$. Then delete the closed strip centered on the segment joining $(1/3, 1+\varepsilon)$ to $(1/3,\varepsilon)$ of width $1/12$. In general delete the closed strip centered on the segment joining $(1/2n, 0)$ and $(1/2n, 1)$ of length $1/[2n(2n+1)]$ and delete the closed strip centered on the segment joining $(1/(2n+1), 1+\varepsilon)$ and $(1/(2n+1), \varepsilon)$ of length $1/[(2n+1)(2n+2)]$. Now let G be the points of H remaining after the strips have been deleted. Clearly G is a connected open set in $\mathbb{R}^2$. However the point $(0,1/2)$ is in the closure of G, but no path of finite length can join any point of G to $(0,1/2)$.

Another example can be visualized by 'enlarging' the graph of $y = \sin \dfrac{1}{x}$.

Theorem 2.7 *Let D be the closure of a connected open set in a Banach space, and suppose D is rectifiably pathwise connected. Then any local radial contraction $g : D \to D$ has a unique fixed point.*

Theorem 2.8 *Let G be a connected open set in a Banach space X, and suppose the intersection of every line in X with G consists of at most finitely many open intervals. Then $\overline{G}$ is rectifiably pathwise connected. Consequently every local radial contraction $g : \overline{G} \to \overline{G}$ has a unique fixed point.*

Proof. Let $x, y \in \overline{G}$. For $z \in G$, the line $\ell(z,x)$ passing through z and x intersects G in a finite number of open intervals. Consequently there is a metric segment $[u,x]$ lying on this line with $[u,x] \subset \overline{G}$ and $u \in G$. Similarly there is a metric segment $[v,y] \subset \overline{G}$ with $v \in G$. By Theorem 2.4 there is a rectifiable path α joining u and v. It follows that $\alpha \cup [u,x] \cup [v,y]$ is a rectifiable path joining x and y. ■

It is not difficult to think of very elaborate examples of open sets in Banach spaces which satisfy the criteria of Theorem 2.8. In fact a more general formulation is true.

Theorem 2.9 *Let G be a connected open set in a Banach space, and suppose for each $x \in \overline{G}$, there exists $z \in G$ such that the interval (x,z) lies in G. Then $\overline{G}$ is rectifiably pathwise connected.*

For $x \in \overline{G}$, we will say x *can see* G if (x, z) lies in G for some $z \in G$. For $x \in G$, let

$$\ell(x) = \left\{ y \in \overline{G} : x \text{ and } y \text{ can be joined by a rectifiable path} \right\}.$$

Theorem 2.10 *Let G be a connected open set in a Banach space. Then G is rectifiably pathwise connected if and only if for each $x \in \overline{G}$ there is a point $y \in \ell(x)$ such that y can see G.*

Here is an example showing that Theorem 2.3 fails if x_0 and $g(x_0)$ are merely joined by an arbitrary path rather than a rectifiable one. We need the following fact.

Proposition 2.11 ([3, p. 130]) *The metric transform of any metric space M by any monotone increasing concave function ϕ which vanishes at the origin is again a metric space.*

Example ([10]). Re-metrize the real line $\mathbb{R}^1$ as follows: Let $(\beta_n)_{n=-\infty}^{\infty}$ be a strictly increasing doubly infinite sequence in $(0, 1)$. For $x, y \in \mathbb{R}^1$, define

$$d(x, y) = \begin{cases} |x - y|^{\beta_n} \text{ if } x, y \in [n, n+1] \\ |x - (n+1)|^{\beta_n} + (p-1) + |(n+p) - y|^{\beta_{n+p}} \\ \text{if } x \in [n, n+1], y \in [n+p, n+p+1], \ p \in \mathbb{N}. \end{cases}$$

It is a straightforward matter to verify that d is a metric and that $(\mathbb{R}^1, d)$ is pathwise connected and complete. Now define $g : \mathbb{R}^1 \to \mathbb{R}^1$ be taking $g(x) = x + 1$. Then g is locally contractive relative to d for any $k \in (0, 1)$. To see this, suppose $x, y \in \left(n, \dfrac{1}{n+1}\right)$. Then

$$d(g(x), g(y)) = |x - y|^{\beta_{n+1}} \leq k|x - y|^{\beta_n} = kd(x, y)$$

provided $|x - y|^{\beta_{n+1} - \beta_n} \leq k$. Since $\beta_{n+1} - \beta_n > 0$ such a choice is always possible; indeed if $d(x, y) < k^{\beta_n/(\beta_{n+1} - \beta_n)}$, then $d(g(x), g(y)) \leq kd(x, y)$. To deal with the case $x = n$, merely take a neighborhood of x in $(\mathbb{R}^1, d)$ with radius less than $\min\left\{k^{\beta_n/(\beta_{n+1} - \beta_n)}, k^{\beta_{n+1}/(\beta_{n+2} - \beta_{n+1})}\right\}$.

We note that the space $(\mathbb{R}^1, d)$ is topologically equivalent to $\mathbb{R}^1$. In particular $(\mathbb{R}^1, d)$ is complete, connected, and locally connected. (A space X is said to be *locally connected* if given any $x \in X$, each neighborhood U of x contains a connected neighborhood V of x).

The following result is Theorem 1 in [9].

Theorem 2.12 *Let (X, d) be a connected and locally connected metric space and let g be a homeomorphism of X onto X which is a local radial contraction. Then there is a metric δ on X, topologically equivalent to d, such that g is a contraction on (X, δ).*

Problem 2.13 *Holmes also asserts in a corollary that completeness of* (X, δ) *follows from completeness of* (X, d). *In view of the example just given, either the theorem is false or the assertion of the corollary is false. The space* $\left(\mathbb{R}^1, d\right)$ *of the example above satisfies all of the assumption of the above theorem, and it is also complete. Moreover* g *is a homeomorphism of* $\left(\mathbb{R}^1, d\right)$ *onto* $\left(\mathbb{R}^1, d\right)$.

Holmes bases his result on the following lemma.

Lemma 2.14 ([9]) *If* g^n *is a contraction on* (X, d) *and if* g *is continuous, then for each* α, $0 < \alpha < 1$, *there exists a metric* δ *on* X, *equivalent to* d, *such that* g *is an* α*-contraction on* (X, δ).

Holmes neglected to include the continuity assumption on g, but as noted in the proof below this assumption is necessary. The lemma follows from the following result of P. R. Meyers [22]. This theorem appears to be correct, but Meyers asserts in a corollary that if ξ has a compact neighborhood then (i) and (ii) are sufficient. In [13] Janos and Solomon give a counter example to this corollary.

Theorem 2.15 *Let* (X, d) *be a metric space. Suppose* $g : X \to X$ *is continuous and satisfies:*

(i) $\exists\ \xi \in X$ *such that* $g(\xi) = \xi$;

(ii) $g^n(x) \to \xi$ *as* $n \to \infty$;

(iii) *there is an open neighborhood* U *of* ξ *such that* $g^n(U) \to \xi$ *as* $n \to \infty$.

Then for each $\alpha \in (0, 1)$ *there is a metric* δ *on* X *such that* g *is an* α*-contraction on* (X, δ). *Moreover if* (X, d) *is complete, then so is* (X, δ).

Proof of Lemma 2.14. The idea is to show that (i), (ii), (iii) hold under the assumptions of Lemma 2.14.

If the contraction mapping g^n does not have a fixed point then by the Banach Contraction Mapping Theorem we may adjoin a point ξ to X which will be the unique fixed point of g^n. In either case $g^i(x) \to \xi$ as $i \to \infty$ for each $x \in X$. To see that (i) is true observe that $g^n(g(\xi)) = g(g^n(\xi)) = g(\xi)$. Thus $g(\xi)$ is a fixed point of g^n. Since the fixed point of g^n is unique, we must have $g(\xi) = \xi$. So (i) is true. To see that (ii) is true observe that $i \in \mathbb{N} \Rightarrow i = nk + t$ for some $0 \le t \le n-1$, so for $x \in X$

$$g^i(x) = g^{nk+t}(x) = g^{nk}\left(g^t(x)\right) \to \xi \text{ as } i \to \infty.$$

(Note that g^{nk} converges to ξ uniformly on the finite set $S := \left\{x, g(x), \cdots, g^{n-1}(x)\right\}$.)

For (iii) set $V = B(\xi; 1)$ and let λ be the contraction constant of g^n. Then if $v \in V$ and $k \in \mathbb{N}$,

$$d\left(g^{nk}(v), g^{nk}(\xi)\right) \le \lambda^k d(v, \xi)$$

so $g^{nk}(V) \subset B\left(\xi; \lambda^k\right)$. Set $U = \cap_{i=0}^{n-1} g^{-i}(V)$. **Since g is continuous**, U is a neighborhood of ξ, and, if $0 \le t < n$,

$$g^{nk+t}(U) \subset g^{nk}(V) \subset B\left(\xi; \lambda^k\right).$$

Thus if $i \in \mathbb{N}$, then $i = nk + t$ for some $0 \leq t < n - 1$. Let $u \in U$. Then $u \in g^{-t}(V)$, i.e., $g^t(u) \in V$. Hence $g^i(u) = g^{nk+t}(u) \in g^{nk}(V) \subset B(\xi; \lambda^k)$. Since $i \to \infty \Rightarrow k \to \infty$ (iii) follows. ∎

Remark. In [26] it is shown that if (X, d) is a metric space and if $g : X \to X$ is a contraction with constant K, then for any λ such that $K^{1/n} < \lambda < 1$ there is a metric δ on X such that g is a λ-contraction on (X, δ). Moreover if g is uniformly continuous on (X, d), and if d is complete, then so is δ.

3 Locally nonexpansive mappings

A mapping f of a metric space (M, ρ) into itself is said to be *locally uniformly β-lipschitzian* for $\beta > 0$ if each point $x \in M$ has a neighborhood U such that for all $u, v \in U$ and all $n \in \mathbb{N}$, $\rho(f^n(u), f^n(v)) \leq \beta\rho(u, v)$. If $\beta = 1$ f is said to be *locally uniformly nonexpansive.* The following is Theorem 1 of [11]. A mapping $f : M \to M$ is said to be *locally nonexpansive* if each point $x \in M$ has a neighborhood U such that for all $u, v \in U$, $\rho(f(u), f(v)) \leq \rho(u, v)$.

Theorem 3.1 *Let (M, ρ) be a compact metric space and let $f : M \to M$ locally uniformly β-lipschitzian and surjective. Then f is a homeomorphism and f^{-1} is also locally uniformly β-lipschitzian on M.*

A classical result of Freudenthal and Hurewicz [6] asserts that a surjective nonexpansive mapping of a compact metric space is always an isometry. We use Theorem 3.1 to prove the following local version of this fact. This is Corollary 1 of [11].

Theorem 3.2 *Suppose (X, ρ) is compact and suppose $\phi : X \to X$ is locally nonexpansive. Then ϕ is locally uniformly nonexpansive. If ϕ is also surjective, then ϕ is a local isometry.*

Proof. We first show that each point $x \in X$ has a neighborhood U with the property that for all $u, v \in U$ and $n \in \mathbb{N}$, $\rho(\phi^n(u), \phi^n(v)) \leq \rho(u, v)$. By assumption for each $x \in X$ there exists $r_x > 0$ such that for each $u, v \in B(x; r_x)$, $\rho(\phi(x), \phi(y)) \leq \rho(u, v)$. Since X is compact there exists a finite set $\{x_1, \cdots, x_n\} \subset X$ such that $X \subset \cup_{i=1}^n B(x_i; r_{x_i}/2)$. Let

$$r = \min\{r_{x_i} : i = 1, \cdots, n\},$$

and suppose $\rho(u, v) \leq r/2$. There exists $i \in \{1, \cdots, n\}$ such that $\rho(u, x_i) \leq r_{x_i}/2$. Thus

$$\rho(v, x_i) \leq \rho(u, v) + \rho(u, x_i) \leq r/2 + r_{x_i}/2 \leq r_{x_i}.$$

Therefore $u, v \in B(x; r_{x_i})$, so $\rho(\phi(x), \phi(y)) \leq \rho(u, v) \leq r/2$. We can now conclude that $\rho(\phi^n(u), \phi^n(v)) \leq \rho(u, v)$ for all $u, v \in B(x; r/4)$ and all $n \in \mathbb{N}$.

If ϕ is surjective it now follows from Theorem 3.1 (taking $\beta = 1$) that ϕ is a local isometry. ■

There is some interesting history connected to the above result. Following A. Edrei [4], if (X, d) is a metric space and $f : X \to X$, then f is called a *local contraction* provided for each $x \in X$ there is a positive number $\mu(x)$ such that for $y \in X$,

$$d(x, y) < \mu(x) \Rightarrow d(f(x), f(y)) \leq d(x, y).$$

Edrei conjectured that a surjective local contraction on a compact metric space is actually a local isometry. In [**?**] R. Williams gives an elaborate counterexample to this conjecture.

4 Surjective isometries

A metric space (M, d) is said to be *hyperconvex* if given any family $\{B(x_i; r_i)\}_{i \in I}$ of closed balls in M satisfying $d(x_i, x_j) \leq r_i + r_j$ it is the case that

$$\bigcap_{i \in I} B(x_i; r_i) \neq \emptyset.$$

Also M is said to be *injective* if it has the following extension property: If Y is a subspace of a metric space X, and if $f : Y \to M$ is nonexpansive, then f has a nonexpansive extension $\tilde{f} : X \to M$. Since a metric space is injective if and only if it is hyperconvex [1], it follows that a hyperconvex metric space is a nonexpansive retract of any space in which it is isometrically embedded. Such spaces are also complete and metrically convex; hence they for a special class of geodesic spaces with interesting properties, especially in connection with nonexpansive mappings.

Indeed the following is true. The proof is similar to one given in [1]. Also see, e.g., [5].

Lemma 4.1 *A metric space is hyperconvex if and only if it is a nonexpansive retract of any superspace in which it is metrically embedded.*

Lemma 4.1 motivates the following definition.

Definition 4.2 A metric space M is said to be *hyperuniversal* (for nonexpansive mappings) if whenever M is a subspace of a metric space N, there is a universal nonexpansive map of N onto M.

One fact is immediate.

Theorem 4.3 *If M is bounded and hyperconvex, then M is hyperuniversal.*

Proof. Since M is a nonexpansive retract of any space in which it is embedded, the conclusion follows from the fact that M has the fixed point property for nonexpansive mappings; hence by Proposition **??** any nonexpansive retraction $R : N \to M$ is a universal nonexpansive mapping. ■

This raises a fundamental question.

Question 1. Is the converse of Theorem 4.3 true? Specifically, is any hyperuniversal space hyperconvex?

The remainder of this section is devoted to showing that the answer to Question 1 is 'yes' if M is compact. These results are taken from [16].

Definition 4.4 ([12]) *A metric space εM is said to be an injective hull of a metric space M if (i) εM is injective; and (ii) no proper subspace of εM which contains M is injective.*

Isbell showed in [12] that every metric space M has an injective hull εM, that M is isometric with a subspace of εM, and that any two injective hulls of M are isometric. Consequently (in view of the result of [1]) εM is a minimal hyperconvex space containing (the isometric copy of) M. Moreover if M is compact, εM is compact (also [12]).

For our next result we need two lemmas.

Lemma 4.5 *If M is hyperuniversal, then M is pathwise connected.*

Proof. By a standard technique M can be embedded as metric subset of a Banach X. By assumption there exists a universal nonexpansive mapping f from X onto M. Since X is pathwise connected and f is continuous, M must be pathwise connected. ■

We also need a slight modification of Lemma 4.1. This result is likely known as well, but we include a proof for the sake of completeness.

Lemma 4.6 *A compact metric space M is hyperconvex if and only if it is a nonexpansive retract of any compact superspace in which it is metrically embedded.*

Proof. Suppose M is a nonexpansive retract of any compact superspace in which it is metrically embedded. Let N be an arbitrary metric space with $M \subset N$. We show that M is a nonexpansive retract of N. Let εN denote the injective hull of N and identify M and N with their isometric copies in εN (under the same embedding). Baillon [2] has shown that the intersection of any descending chain of hyperconvex metric spaces is hyperconvex. This fact with Zorn's Lemma assures the existence of a minimal hyperconvex metric space $\tilde{M}$ which is contained in εN and which contains M. It follows that $\tilde{M}$ and εM (the injective hull of M) are isometric. Since εM is compact, $\tilde{M}$ is compact and by assumption there exists a nonexpansive retraction R_1 of $\tilde{M}$ onto M. Also, since $\tilde{M}$ is hyperconvex there exists a nonexpansive retraction R_2 of εN onto $\tilde{M}$. Thus $R := R_1 \circ R_2$ is a nonexpansive retraction of εN onto M, and since $M \subset N \subset \varepsilon N$ the restriction of R to N is a nonexpansive retraction of N onto M. The conclusion now follows from Lemma 4.1. ■

Theorem 4.7 *If M is compact, then M is hyper-universal if and only if M is hyperconvex.*

Proof. In view of Theorem 4.3 we need only show that if M is hyper-universal then M is hyperconvex. Suppose M is a subspace of a compact metric space N, and let S be a countable dense subset of $N \backslash M$. By assumption there is a universal nonexpansive mapping

$$f : M \cup S \to M.$$

Suppose $f(M) \neq M$. Since M is compact and f is continuous this implies $\overline{f(M)} \neq M$. Since M is pathwise connected it must be the case that $M \backslash f(M)$ is uncountable. But $M \backslash f(M) \subseteq f(S)$ and the latter set is at most countable. Therefore $f(M) = M$ and by Theorem **??** there exists a nonexpansive retraction r of $M \cup S$ onto M. Since $\overline{S} \supset N \backslash M$ it is possible to extend r continuously to all of $N \backslash M$ and obtain a nonexpansive retraction R of N onto M. The conclusion now follows from Lemma 4.6. ■

REMARKS

The following definition is due to M. A. Khamsi [14].

Definition 4.8 *A metric space M is said to be a* 1*-local retract of $N \supset M$ if for each family $\{B_i\}_{i \in I}$ of closed balls centered at points of M for which*

$$\cap_{i \in I} B_i \neq \emptyset$$

it is the case that $M \cap (\cap_{i \in I} B_i) \neq \emptyset$.

Question 2. If M is hyperuniversal, then is M a 1-local retract of N for any $N \supset M$?

Question 3. Suppose M is hyperuniversal and suppose $H \subseteq M$ is the range of a universal nonexpansive $f : M \to H$. Is H also hyperuniversal? If H is a nonexpansive retract of M is H hyperuniversal?

Notice that if $N \supset M$ then since M is hyperuniversal there exists $g : N \to M$ such that g is a universal nonexpansive map. Since $f \circ g : N \to H \subseteq M$ is nonexpansive and since g is universal nonexpansive there exists $x \in N$ such that $g(x) = f \circ g(x)$.

Let M be bounded and hyper-universal and let εM be the injective hull of M. Then there exists $z \in \varepsilon M$ such that

$$z \in \cap_{x \in \varepsilon M} B(x; d/2)$$

where $d = diam(\varepsilon M) = diam(M)$. By assumption there exists a universal nonexpansive mapping $f : M \cup \{z\} \to M$, and since f is surjective this implies

$$f(z) \in \cap_{x \in M} B(x; d/2).$$

Thus we observe that the Chebyshev center of M is nonempty, and the Chebyshev radius of M is $(1/2)\mathrm{diam}(M)$.

5 The Freudenthal-Huewicz Property

The result of Freudenthal and Hurewicz [6] states that a surjective nonexpansive self-mapping of a compact metric space is necessarily an isometry. It is interesting to note that there are *noncompact* spaces for which this assertion also holds.

Example 5.1 Let $\{e_n\}$ be the standard unit basis in ℓ_2, and for each $n \geq 1$ let $L_n = \left\{te_n : 0 \leq t \leq 1 - \frac{1}{n}\right\}$. Take $X = \cup L_n$. Then the only surjective nonexpansive mapping of X onto X is the identity.

Example 5.2 Consider $\mathbb{R}^2$ with the radial metric ρ. Let $\{x_n\}$ be a sequence of distinct points on the unit sphere of $\mathbb{R}^2$. Let $y_n = x_n$ for $1 \leq n \leq 10$, and for $n > 10$ let y_n be the point on the segment $[0, x_n]$ such that $\rho(0, y_n) = 1 - \frac{1}{n}$. Now let $L_n = [0, y_n]$ and take $X = \cup L_n$. In this case every surjective nonexpansive mapping of X onto X is an isometry, but there exist nontrivial surjective nonexpansive mappings.

We raised the following question in [18].

Question 4. *Is it possible to classify metric spaces for which surjective nonexpansive self-mappings are always isometries? Do such spaces share any additional properties with compactness?*

The answer to Question 4 is likely 'No'. Indeed, it is possible to construct a bounded closed convex subset K of ℓ_2 which has the property that every surjective isometry of $K \to K$ is necessarily the identity mapping. Just take K to be the closed convex hull of the set X in Example 5.1.

We say that a metric space M has the *Freudenthal-Hurewicz property* if every surjective nonexpansive mapping $T : M \to M$ is necessarily an isometry.

Question 5. *If a metric space M has the Freudenthal-Hurewicz property, does its injective hull, εM, also have the Freudenthal-Hurewicz property?*

Question 6. *If the answer to Question 5 is affirmative, can compactness of M in Theorem 4.7 be replaced with the assumption that M has the Freudenthal-Hurewicz property?*

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