

# A Sharp Power Inequality and Its $n$ -Variable Extension

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## Abstract

We solve Problem 5167, proposed by Vasile Cîrtoaje, concerning an inequality between reciprocal sums and power sums under the normalization  $ab + bc + ca = 3$ , with at most one variable below 1. The largest exponent is  $3/2$ . We then extend the problem to  $n$  variables with normalized sum of pairwise products and determine the complete range of positive exponents. For every  $n \geq 2$ , the sharp endpoint is  $K_n = 2 - 1/(n - 1)$ . A comparison lemma and a coordinate-monotonicity argument prove the endpoint inequalities; perturbations of the all-one tuple establish sharpness. All equality cases, including the exceptional two-variable endpoint, are described. A further extension replaces reciprocals by arbitrary nonnegative inverse powers and gives the sharp endpoint  $p + (n - 2)/(n - 1)$  for  $n \geq 3$ . We also obtain remainder estimates, homogeneous and integer-multiplicity formulations, extremal applications, and a collection of related problems with solutions. Finally, for the normalization  $e_j = \binom{n}{j}$ , we prove the sharp endpoint  $p + (n - j)/(n - 1)$  for every symmetric order  $1 \leq j \leq n$ . Arithmetic and product normalizations have distinct endpoint equality patterns, which are classified explicitly.

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## 1 Introduction and the original problem

Problem 5167, proposed by Vasile Cîrtoaje, appeared in *Cruz Mathematicorum*, Volume 52, No. 7 (September 2026) [1]. Its question can be stated as follows.

**Problem 5167 (Vasile Cîrtoaje).** Determine the largest  $k > 0$  for which

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^k + b^k + c^k \tag{1}$$

is valid whenever  $a, b, c$  are positive real numbers satisfying

$$ab + bc + ca = 3,$$

and at least two of the three numbers are at least 1.

We first give a complete solution of this problem. We then replace the three variables by  $n$  variables and normalize the sum of their pairwise products to  $\binom{n}{2}$ . The condition that at most one variable is below 1 remains unchanged. This extension admits a sharp answer in every dimension  $n \geq 2$ .

The proof has two complementary parts. A local perturbation around  $(1, \dots, 1)$  identifies the only possible largest exponent. A comparison of derivatives then proves that this exponent works throughout the permitted domain. We formulate the required comparison in a form that can be used for both the original problem and its extension.

Classical background on arithmetic–geometric mean inequalities and convexity is provided by Hardy, Littlewood, and Pólya [2]. For power means and symmetric polynomial means, see Bullen [3]; for a broader treatment of convexity and symmetric inequalities, see Marshall, Olkin, and Arnold [4]. These references supply general background; the exponent classifications under the one-small-variable restriction are proved explicitly below.

## 2 Solution of the original problem

### 2.1 A comparison principle

**Lemma 2.1.** *Let  $m \geq 1$ , and define*

$$r = m + 1, \quad d = \frac{m}{m+1}, \quad \kappa = 1 + d = \frac{2m+1}{m+1},$$

$$h(x) = x^{-2} + \kappa x^d, \quad J(x) = (m+x)h(x) \quad (x > 0).$$

*If  $u \geq 1$ ,  $v > 0$ , and*

$$uv + m(u+v) \leq 2m+1, \tag{2}$$

*then  $J(v) \geq J(u)$ , with equality if and only if  $u = v = 1$ .*

*Proof.* The identities

$$md + \kappa = r, \quad \kappa r = 2m + 1$$

will be used repeatedly. First, differentiating  $J$  yields

$$x^3 J'(x) = -2m - x + \kappa m d x^{d+2} + \kappa^2 x^{d+3}.$$

For  $0 < x < 1$ , both  $x^{d+2}$  and  $x^{d+3}$  are at most  $x$ . Therefore

$$\begin{aligned} x^3 J'(x) &\leq -2m - x + \kappa(md + \kappa)x \\ &= -2m - x + (2m+1)x \\ &= 2m(x-1) < 0. \end{aligned}$$

Hence  $J$  is strictly decreasing on  $(0, 1]$ .

Condition (2) implies

$$0 < v \leq w := \frac{2m+1-mu}{m+u} \leq 1.$$

In particular,  $w > 0$ . If  $u = 1$ , then  $w = 1$ , so strict decrease proves the conclusion, including equality only at  $v = 1$ . Suppose now that  $u > 1$ . Then  $0 < w < 1$ , and it is enough to prove  $J(w) > J(u)$ , because  $v \leq w$  implies  $J(v) \geq J(w)$ .

Set  $p = uw$ . The definition of  $w$  gives

$$uw + m(u + w) = 2m + 1, \quad u + w = 2 + \frac{1 - p}{m}, \quad 1 - p = \frac{m(u - 1)^2}{m + u} > 0. \quad (3)$$

Thus  $0 < p < 1$  and  $u > w$ . We next estimate two power differences. Put

$$z = \frac{1}{2} \log \frac{u}{w} > 0, \quad u = \sqrt{p} e^z, \quad w = \sqrt{p} e^{-z}.$$

Since  $\sinh t = (e^t - e^{-t})/2$  is convex on  $[0, \infty)$  and  $\sinh 0 = 0$ , for  $0 \leq \lambda \leq 1$  and  $t \geq 0$  we have

$$\sinh(\lambda t) \leq \lambda \sinh t.$$

Apply this with  $(\lambda, t) = (d, z)$  and then with  $(\lambda, t) = (\kappa/2, 2z)$ . Both choices are legitimate, since  $0 < d < 1$  and  $1 < \kappa < 2$ . We obtain

$$\frac{u^d - w^d}{u - w} = p^{(d-1)/2} \frac{\sinh(dz)}{\sinh z} \leq d p^{(d-1)/2}, \quad (4)$$

$$\frac{u^\kappa - w^\kappa}{u - w} = p^{(\kappa-1)/2} \frac{\sinh(\kappa z)}{\sinh z} \leq \frac{\kappa}{2} (u + w) p^{(d-1)/2}. \quad (5)$$

For the last step in (5), use

$$\sinh(2z) = 2 \sinh z \cosh z, \quad u + w = 2\sqrt{p} \cosh z.$$

Let  $g(x) = (m + x)x^d = mx^d + x^\kappa$ . Combining (3) with the two power-difference estimates, and using  $d - 1 = -1/r$ , gives

$$\begin{aligned} \frac{g(u) - g(w)}{u - w} &\leq p^{-1/(2r)} \left( md + \frac{\kappa}{2} (u + w) \right) \\ &= p^{-1/(2r)} \left( r + \frac{\kappa}{2m} (1 - p) \right). \end{aligned}$$

Because  $m \geq 1$ ,  $\kappa < 2$ , and  $r \geq 2$ , we have  $\kappa/(2m) \leq r$ . Also,  $p(2 - p) \leq 1$ . It follows that

$$r + \frac{\kappa}{2m} (1 - p) \leq r(2 - p) \leq \frac{r}{p}.$$

Since  $0 < p < 1$  and  $1 + 1/(2r) < 2$ , we conclude that

$$\frac{g(u) - g(w)}{u - w} \leq r p^{-1-1/(2r)} < \frac{r}{p^2}. \quad (6)$$

On the other hand, the reciprocal terms have the exact difference

$$\begin{aligned} \frac{m + w}{w^2} - \frac{m + u}{u^2} &= (u - w) \frac{m(u + w) + uw}{u^2 w^2} \\ &= (u - w) \frac{2m + 1}{p^2}. \end{aligned}$$

Therefore, using  $\kappa r = 2m + 1$  and (6),

$$\begin{aligned} J(w) - J(u) &= (u - w) \frac{2m + 1}{p^2} - \kappa(g(u) - g(w)) \\ &> 0. \end{aligned}$$

Finally,  $J(v) \geq J(w) > J(u)$ . Together with the case  $u = 1$ , this proves the lemma and its equality statement.  $\square$

## 2.2 The endpoint inequality and its sharpness

**Theorem 2.2.** *The largest exponent in Problem 5167 is  $k = 3/2$ . At this exponent, equality holds if and only if  $a = b = c = 1$ .*

*Proof.* By symmetry, choose the labels so that  $a \geq 1$  and  $b \geq 1$ . The constraint determines

$$c = \frac{3 - ab}{a + b} > 0.$$

Thus  $ab < 3$ , and  $c \leq 1$  because  $ab \geq 1$  and  $a + b \geq 2$ . Set

$$f(x) = \frac{1}{x} - x^{3/2}, \quad F(a, b) = f(a) + f(b) + f\left(\frac{3 - ab}{a + b}\right).$$

We will show that  $F(a, b) \geq 0$  on  $\{(a, b) : a \geq 1, b \geq 1, ab < 3\}$ .

Use Lemma 2.1 with  $m = 1$ , so that

$$h(x) = x^{-2} + \frac{3}{2}\sqrt{x}, \quad J(x) = (1 + x)h(x).$$

Since  $f'(x) = -h(x)$  and  $\partial c / \partial a = -(b + c)/(a + b)$ , we have

$$\frac{\partial F}{\partial a} = -h(a) + h(c)\frac{b + c}{a + b}.$$

The relations  $b \geq 1$  and  $a \geq c$  imply

$$\frac{b + c}{a + b} - \frac{1 + c}{1 + a} = \frac{(b - 1)(a - c)}{(a + b)(1 + a)} \geq 0.$$

Moreover,

$$ac + a + c \leq ac + b(a + c) = 3,$$

so the lemma applies with  $u = a$ ,  $v = c$ . Because  $h(c) > 0$ , it follows that

$$\frac{\partial F}{\partial a} \geq -h(a) + h(c)\frac{1 + c}{1 + a} = \frac{J(c) - J(a)}{1 + a} \geq 0. \quad (7)$$

Interchanging  $a$  and  $b$  proves  $\partial F / \partial b \geq 0$  as well.

Reducing  $a$  to 1 with  $b$  fixed preserves  $ab < 3$ . Reducing  $b$  to 1 afterwards also remains in the domain. Integration of the nonnegative partial derivatives along these two segments therefore gives

$$F(a, b) \geq F(1, b) \geq F(1, 1) = 0.$$

Hence

$$\boxed{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^{3/2} + b^{3/2} + c^{3/2}.} \quad (8)$$

If either  $a > 1$  or  $b > 1$ , the corresponding coordinate segment has positive length, and the lemma makes its derivative strictly positive where that coordinate exceeds 1. Equality is therefore possible only at  $a = b = 1$ , which forces  $c = 1$ . The converse is immediate.

To prove sharpness, fix  $k > 3/2$  and take

$$a = b = 1 + \varepsilon, \quad c = \frac{3 - (1 + \varepsilon)^2}{2(1 + \varepsilon)}, \quad 0 < \varepsilon < \sqrt{3} - 1.$$

These triples satisfy  $ab + bc + ca = 3$ , with  $a, b > 1$  and  $0 < c < 1$ . For the fixed exponent  $k$ , Taylor expansion at  $\varepsilon = 0$  gives

$$\begin{aligned} c &= 1 - 2\varepsilon + \frac{3}{2}\varepsilon^2 + O(\varepsilon^3), \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= 3 + \frac{9}{2}\varepsilon^2 + O(\varepsilon^3), \\ a^k + b^k + c^k &= 3 + \left(3k^2 - \frac{3}{2}k\right)\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

Consequently,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - (a^k + b^k + c^k) = -\frac{3}{2}(2k - 3)(k + 1)\varepsilon^2 + O(\varepsilon^3).$$

The coefficient of  $\varepsilon^2$  is negative when  $k > 3/2$ . Dividing by  $\varepsilon^2$  and taking the limit shows that every such  $k$  fails for all sufficiently small positive  $\varepsilon$ . Together with (8), this proves the claimed largest exponent.  $\square$

### 3 The $n$ -variable generalization

The natural extension of the original problem is the following: for a positive integer  $n$ , determine all  $k > 0$  for which

$$\sum_{i=1}^n \frac{1}{x_i} \geq \sum_{i=1}^n x_i^k \tag{9}$$

holds for every positive  $n$ -tuple satisfying

$$\sum_{1 \leq i < j \leq n} x_i x_j = \binom{n}{2}, \quad \text{at most one } x_i \text{ is less than 1.} \tag{10}$$

The normalization is chosen so that the all-one tuple satisfies the constraint in every dimension.

**Theorem 3.1.** *For each integer  $n \geq 2$ , inequality (9) holds for every tuple satisfying (10) if and only if*

$$0 < k \leq K_n, \quad K_n = \frac{2n - 3}{n - 1} = 2 - \frac{1}{n - 1}.$$

*For  $n \geq 3$ , equality for any such exponent occurs exactly when  $x_1 = \dots = x_n = 1$ . For  $n = 2$  and  $0 < k < 1$ , equality occurs exactly when  $x_1 = x_2 = 1$ ; for  $n = 2$  and  $k = 1$ , every pair with  $x_1 x_2 = 1$  gives equality. No positive exponent is admissible when  $n = 1$ .*

**Proof. The small dimensions.** If  $n = 1$ , the pairwise constraint is  $0 = 0$ , and the condition on variables below 1 is automatic. Taking  $x_1 = 2$  disproves  $1/x_1 \geq x_1^k$  for every  $k > 0$ . If  $n = 2$ , the constraint is  $x_1 x_2 = 1$  and

$$\frac{1}{x_1} + \frac{1}{x_2} = x_1 + x_2.$$

Thus the endpoint  $K_2 = 1$  works, with equality for every pair satisfying the constraint. The argument below will also show that it is optimal and that all smaller positive exponents work.

**A necessary upper bound for  $n \geq 2$ .** Fix  $n \geq 2$  and a positive exponent  $k$ . For sufficiently small  $\varepsilon > 0$ , set

$$x_1 = \cdots = x_{n-1} = t = 1 + \varepsilon, \quad x_n = c(\varepsilon) = \frac{n - (n-2)t^2}{2t}.$$

Then

$$\begin{aligned} \sum_{i < j} x_i x_j &= \binom{n-1}{2} t^2 + (n-1)tc(\varepsilon) \\ &= \frac{(n-1)(n-2)}{2} t^2 + \frac{n-1}{2} (n - (n-2)t^2) \\ &= \binom{n}{2}. \end{aligned}$$

For  $n \geq 3$ , choosing  $0 < \varepsilon < \sqrt{n/(n-2)} - 1$  guarantees  $c(\varepsilon) > 0$ . For  $n = 2$ ,  $c(\varepsilon) = 1/t > 0$ . In both cases,

$$1 - c(\varepsilon) = \frac{(t-1)((n-2)t+n)}{2t} > 0,$$

so all these tuples satisfy (10).

Write

$$c(\varepsilon) = \frac{n}{2(1+\varepsilon)} - \frac{n-2}{2}(1+\varepsilon).$$

Direct differentiation gives

$$c(0) = 1, \quad c'(0) = -(n-1), \quad c''(0) = n.$$

Define

$$\Delta(\varepsilon) = \frac{n-1}{1+\varepsilon} + \frac{1}{c(\varepsilon)} - (n-1)(1+\varepsilon)^k - c(\varepsilon)^k.$$

We have  $\Delta(0) = \Delta'(0) = 0$ . The second derivatives of the reciprocal sum and the power sum at zero are, respectively,

$$\begin{aligned} 2(n-1) + 2(n-1)^2 - n &= n(2n-3), \\ k(k-1)((n-1) + (n-1)^2) + kn &= nk((n-1)k - (n-2)). \end{aligned}$$

Subtracting and factoring yields

$$\Delta''(0) = -n(k+1)((n-1)k - (2n-3)). \quad (11)$$

Since  $\Delta$  is smooth near zero,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\Delta(\varepsilon)}{\varepsilon^2} = \frac{1}{2} \Delta''(0).$$

For  $k > K_n = (2n-3)/(n-1)$ , this limit is negative. Consequently,  $\Delta(\varepsilon) < 0$  for all sufficiently small positive  $\varepsilon$ . Every exponent greater than  $K_n$  therefore fails on an admissible tuple.

**The endpoint for  $n \geq 3$ .** Fix  $n \geq 3$ , and put

$$N = \binom{n}{2}, \quad r = n - 1, \quad m = n - 2, \quad \kappa = K_n.$$

Relabel the variables as  $a_1, \dots, a_r, c$ , with all  $a_i \geq 1$ . This is possible because at most one of the original variables is below 1. Define

$$S = \sum_{i=1}^r a_i, \quad Q = \sum_{1 \leq i < j \leq r} a_i a_j.$$

The constraint  $Q + cS = N$  gives

$$c = \frac{N - Q}{S} > 0.$$

The independent coordinates lie in

$$\mathcal{D} = \{(a_1, \dots, a_r) : a_i \geq 1 \text{ for every } i, Q < N\}.$$

Since  $Q + S \geq \binom{r}{2} + r = N$ , we also have  $c \leq 1$  throughout this domain.

Set

$$f(x) = \frac{1}{x} - x^\kappa, \quad F(a_1, \dots, a_r) = \sum_{i=1}^r f(a_i) + f(c).$$

Fix one coordinate  $a = a_i$ . Let  $B$  be the sum of the other  $m$  independent coordinates, and let  $R$  be the sum of their pairwise products. Then

$$B \geq m, \quad R \geq \binom{m}{2}, \quad N = ac + B(a + c) + R.$$

It follows that

$$ac + m(a + c) \leq N - \binom{m}{2} = 2m + 1. \tag{12}$$

Thus Lemma 2.1 applies with  $u = a$ ,  $v = c$ .

Because  $S = a + B$  and  $Q = aB + R$ , differentiation yields

$$\frac{\partial c}{\partial a} = -\frac{B + c}{B + a}, \quad \frac{\partial F}{\partial a} = -h(a) + h(c) \frac{B + c}{B + a},$$

where  $h(x) = x^{-2} + \kappa x^{\kappa-1}$  is precisely the function in Lemma 2.1 for  $m = n - 2$ . The relations  $a \geq 1 \geq c$  and  $B \geq m$  give

$$\frac{B + c}{B + a} - \frac{m + c}{m + a} = \frac{(B - m)(a - c)}{(B + a)(m + a)} \geq 0.$$

Since  $h(c) > 0$ , the lemma implies

$$\frac{\partial F}{\partial a} \geq -h(a) + h(c) \frac{m + c}{m + a} = \frac{J(c) - J(a)}{m + a} \geq 0. \tag{13}$$

These derivative formulas are smooth wherever  $c > 0$ , and hence remain valid at coordinates equal to 1 by their smooth extension. The calculation applies to each independent coordinate.

Now decrease the independent coordinates one at a time to 1. Every coordinate segment stays in  $\mathcal{D}$ : decreasing a coordinate does not increase  $Q$ , and all independent coordinates remain at least 1. Integration of (13) gives

$$F(a_1, \dots, a_r) \geq F(1, \dots, 1) = 0,$$

because at the final point

$$c = \frac{N - \binom{r}{2}}{r} = 1.$$

This proves the endpoint inequality

$$\sum_{i=1}^n \frac{1}{x_i} \geq \sum_{i=1}^n x_i^{K_n} \quad (n \geq 3). \quad (14)$$

If some independent coordinate exceeds 1, the lemma makes the corresponding derivative strictly positive on the interior of the segment reducing that coordinate. Thus equality in (14) occurs exactly at the all-one tuple.

**Smaller positive exponents and equality.** For every  $n \geq 2$ , put  $P = \prod_{i=1}^n x_i$ . Applying the arithmetic–geometric mean inequality to the  $N = \binom{n}{2}$  pairwise products gives

$$1 = \frac{1}{N} \sum_{i < j} x_i x_j \geq \left( \prod_{i < j} x_i x_j \right)^{1/N} = P^{2/n},$$

since each variable occurs in  $n - 1$  pairwise products. Therefore  $P \leq 1$ . Applying the same inequality to the reciprocals then yields

$$L := \sum_{i=1}^n \frac{1}{x_i} \geq n P^{-1/n} \geq n.$$

For  $0 < k < K_n$ , let  $\theta = k/K_n \in (0, 1)$ . Weighted arithmetic–geometric mean gives, for each  $i$ ,

$$x_i^k = 1^{1-\theta} (x_i^{K_n})^\theta \leq (1 - \theta) + \theta x_i^{K_n}.$$

The endpoint is already established for every  $n \geq 2$ . Hence

$$\begin{aligned} \sum_{i=1}^n x_i^k &\leq n(1 - \theta) + \theta \sum_{i=1}^n x_i^{K_n} \\ &\leq n(1 - \theta) + \theta L \\ &\leq L. \end{aligned}$$

Because  $0 < \theta < 1$ , equality in the first inequality requires  $x_i^{K_n} = 1$  for each  $i$ , or equivalently  $x_i = 1$  for every  $i$ . Conversely, the all-one tuple gives equality for every admissible exponent. Together with the previously established endpoint equality cases and the obstruction (11), this proves all assertions of the theorem.  $\square$

## 4 A further sharp family with reciprocal powers

Throughout this section,  $n \geq 3$  and the positive variables satisfy (10). Write

$$d = d_n := \frac{n-2}{n-1}, \quad L_p := \sum_{i=1}^n x_i^{-p} \quad (p \geq 0), \quad S_k := \sum_{i=1}^n x_i^k.$$

In particular,  $L_0 = n$ . The reciprocal exponent need not equal 1; in fact, it can be any nonnegative real number.

**Lemma 4.1** (Pair domination). *Relabel the variables as  $a_1, \dots, a_{n-1}, c$ , with every  $a_i \geq 1$ . Then  $0 < c \leq 1$  and  $a_i c \leq 1$  for each  $i$ . If the tuple is not the all-one tuple, then  $c < 1$  and  $a_i c < 1$  for every  $i$ .*

*Proof.* The fact that  $c \leq 1$  was established in the proof of Theorem 3.1. Put  $m = n - 2$ . For each  $a = a_i$ , equation (12) gives

$$ac + m(a + c) \leq 2m + 1.$$

Set  $\rho = \sqrt{ac}$ . Since  $a + c \geq 2\rho$ ,

$$\rho^2 + 2m\rho \leq 2m + 1.$$

The left side is strictly increasing for  $\rho > 0$ , and equals  $2m + 1$  at  $\rho = 1$ . Hence  $ac \leq 1$ . If  $ac = 1$ , equality in the preceding estimates forces  $a + c = 2$ , so  $a = c = 1$ . All the remaining variables are at least 1, and the pairwise normalization then forces them all to equal 1. Similarly,  $c = 1$  would make every variable at least 1, and hence would force the all-one tuple. This proves the strict assertions.  $\square$

**Lemma 4.2** (A concave power-sum bound). *Under the same hypotheses,*

$$\sum_{i=1}^n x_i^d \leq n, \quad d = \frac{n-2}{n-1}, \tag{15}$$

*with equality if and only if every variable equals 1.*

*Proof.* We first establish the comparison needed for the coordinate argument. Let  $m \geq 1$ ,  $r = m + 1$ ,  $d = m/r$ , and

$$\Phi(x) = (m + x)x^{d-1}.$$

Differentiation gives

$$\Phi'(x) = dx^{d-2}(x - 1),$$

so  $\Phi$  is strictly decreasing on  $(0, 1]$ . Suppose  $u \geq 1$ ,  $v > 0$ , and  $uv + m(u + v) \leq 2m + 1$ . Then

$$0 < v \leq w(u) := \frac{2m + 1 - mu}{m + u} \leq 1.$$

For  $u = 1$ , the desired comparison  $\Phi(v) \geq \Phi(u)$  follows at once. For  $u > 1$ , the definition of  $w$  gives

$$(m + u)(m + w) = r^2, \quad 1 - uw = \frac{m(u - 1)^2}{m + u} > 0.$$

On the interval where  $w(u) > 0$ , direct differentiation yields

$$\begin{aligned} \frac{d}{du} \log \frac{\Phi(w(u))}{\Phi(u)} &= \frac{d(u + w - 2uw)}{uw(m + u)} \\ &= \frac{(2m + 1)(1 - uw)}{r uw(m + u)} > 0 \quad (u > 1). \end{aligned}$$

The ratio equals 1 at  $u = 1$ . Thus  $\Phi(w(u)) > \Phi(u)$  when  $u > 1$ , and  $v \leq w$  gives  $\Phi(v) \geq \Phi(w) > \Phi(u)$ . Consequently,

$$\Phi(v) \geq \Phi(u), \quad (16)$$

with equality only at  $u = v = 1$ .

Return to the normalized  $n$ -tuple and use the independent coordinates  $a_1, \dots, a_{n-1}$  from the proof of Theorem 3.1. Let

$$F_0 = n - \sum_{i=1}^{n-1} a_i^d - c^d.$$

For one coordinate  $a = a_i$ , let  $B$  be the sum of the other  $m = n - 2$  independent coordinates. The same derivative and ratio comparisons as before give

$$\begin{aligned} \frac{\partial F_0}{\partial a} &= -da^{d-1} + dc^{d-1} \frac{B + c}{B + a} \\ &\geq -da^{d-1} + dc^{d-1} \frac{m + c}{m + a} \\ &= \frac{d(\Phi(c) - \Phi(a))}{m + a} \geq 0. \end{aligned}$$

Here  $ac + m(a + c) \leq 2m + 1$ , so (16) applies. Decreasing the independent coordinates successively to 1 stays in the domain and gives  $F_0 \geq F_0(1, \dots, 1) = 0$ . If any independent coordinate exceeds 1, the corresponding derivative is strictly positive on the interior of its segment. Hence equality occurs only at the all-one tuple.  $\square$

**Theorem 4.3** (The full reciprocal-power family). *Fix  $n \geq 3$  and  $p \geq 0$ . The inequality*

$$\sum_{i=1}^n x_i^{-p} \geq \sum_{i=1}^n x_i^k \quad (17)$$

*holds for every tuple satisfying (10) if and only if*

$$\boxed{0 < k \leq p + d_n, \quad d_n = \frac{n-2}{n-1}.}$$

*Equality in (17) for any admissible exponent holds exactly at the all-one tuple. Moreover, the endpoint has the remainder estimate*

$$\sum_{i=1}^n x_i^{-p} - \sum_{i=1}^n x_i^{p+d} \geq \frac{2p+d}{d} \left( n - \sum_{i=1}^n x_i^d \right) \geq 0. \quad (18)$$

*Proof. The endpoint and its remainder.* The all-one tuple is immediate, so assume the tuple is nontrivial. Use the labels  $a_1, \dots, a_{n-1}, c$  of Lemma 4.1, and set

$$L = -\log c > 0, \quad t_i = \log a_i \geq 0.$$

Pair domination implies  $t_i < L$  for every  $i$ . Put  $s_0 = d/2$  and  $s = p + d/2$ , and define

$$R(t) = \frac{\sinh(st)}{\sinh(s_0 t)} \quad (t > 0), \quad R(0) = \frac{s}{s_0}.$$

The value at zero is the continuous limiting value. The function  $R$  is nondecreasing on  $[0, \infty)$ . Indeed, for  $t > 0$ ,

$$\frac{d}{dt} \log R(t) = s \coth(st) - s_0 \coth(s_0 t) \geq 0,$$

because  $s \geq s_0 > 0$  and  $q \coth q$  is increasing for  $q > 0$ :

$$\frac{d}{dq} (q \coth q) = \frac{\sinh q \cosh q - q}{\sinh^2 q} \geq 0.$$

The numerator is nonnegative since  $\sinh(2q) \geq 2q$ . In particular,

$$R(L) \geq R(0) = \frac{2p+d}{d}, \quad R(L) \geq R(t_i).$$

For any  $x > 0$ ,

$$x^{-p} - x^{p+d} = -2x^{d/2} \sinh((p+d/2) \log x).$$

Define

$$H = n - \sum_{i=1}^n x_i^d, \quad G_p = \sum_{i=1}^n x_i^{-p} - \sum_{i=1}^n x_i^{p+d}.$$

By Lemma 4.2,  $H > 0$  for this nontrivial tuple. The preceding identity gives the exact decomposition

$$\begin{aligned} G_p &= R(L)H \\ &\quad + 2 \sum_{i=1}^{n-1} a_i^{d/2} (R(L) - R(t_i)) \sinh(s_0 t_i). \end{aligned} \tag{19}$$

Every term in the sum is nonnegative. Thus  $G_p \geq R(L)H \geq (2p+d)H/d$ , proving (18) and strictness away from the all-one tuple.

**Smaller positive exponents.** The pairwise arithmetic–geometric mean estimate gives  $P := \prod_i x_i \leq 1$ . Hence, for every  $p \geq 0$ ,

$$L_p \geq nP^{-p/n} \geq n.$$

For  $0 < k < p+d$ , put  $\theta = k/(p+d) \in (0, 1)$ . Weighted arithmetic–geometric mean gives

$$x_i^k \leq 1 - \theta + \theta x_i^{p+d}.$$

Summing and using the endpoint inequality yields

$$S_k \leq n(1 - \theta) + \theta S_{p+d} \leq n(1 - \theta) + \theta L_p \leq L_p.$$

Equality in the first estimate requires  $x_i^{p+d} = 1$  for each  $i$ , so all variables must equal 1.

**Sharpness.** Take the same admissible perturbation

$$x_1 = \cdots = x_{n-1} = 1 + \varepsilon, \quad x_n = c(\varepsilon) = \frac{n - (n-2)(1 + \varepsilon)^2}{2(1 + \varepsilon)}.$$

Let  $\Delta_{p,k}(\varepsilon) = L_p - S_k$  on this family. Then  $\Delta_{p,k}(0) = \Delta'_{p,k}(0) = 0$ , while

$$\begin{aligned} \Delta''_{p,k}(0) &= n [p((n-1)p + n - 2) - k((n-1)k - (n-2))] \\ &= -n(p+k)((n-1)(k-p) - (n-2)). \end{aligned} \tag{20}$$

For  $k > p + d$ , the last expression is negative. Taylor's formula then makes  $\Delta_{p,k}(\varepsilon) < 0$  for all sufficiently small positive  $\varepsilon$ . This excludes every larger exponent.  $\square$

For completeness, if  $n = 2$  and  $p > 0$ , the complete range is  $0 < k \leq p$ . At  $k = p$ , every pair with  $x_1 x_2 = 1$  gives equality; for  $0 < k < p$ , equality occurs only at  $(1, 1)$ . Indeed, the endpoint is the identity  $x_1^{-p} + x_2^{-p} = x_1^p + x_2^p$ , and interpolation gives the smaller exponents. Formula (20) excludes  $k > p$ . For  $n = 2$ ,  $p = 0$ , no positive exponent is admissible, by the same formula or strict arithmetic-geometric mean applied to  $x_1^k + x_2^k$ . For  $n = 1$ , taking  $x_1 = 2$  excludes every  $k > 0$  for every  $p \geq 0$ .

## 5 Quantitative improvements and sharp extremal bounds

The nonnegative deficit in (15) already gives a remainder at the endpoint through (18). Below the endpoint, a second remainder measures logarithmic deviation from the equality tuple.

**Proposition 5.1** (A quantitative interpolation inequality). *For  $n \geq 3$ ,  $p \geq 0$ , set  $d = d_n$ ,  $K = p + d$ , and  $\theta = k/K$ , where  $0 < k \leq K$ . Under (10),*

$$\begin{aligned} L_p - S_k &\geq \theta \frac{2p + d}{d} (n - S_d) + (1 - \theta)(L_p - n) \\ &\quad + \frac{\theta(1 - \theta)K^2}{2} \sum_{i=1}^n \min\{1, x_i^K\} (\log x_i)^2. \end{aligned} \tag{21}$$

*All three terms on the right are nonnegative.*

*Proof.* For a real number  $z$  and  $0 \leq \theta \leq 1$ , we claim

$$1 - \theta + \theta e^z - e^{\theta z} \geq \frac{\theta(1 - \theta)}{2} \min\{1, e^z\} z^2. \tag{22}$$

Let  $b_z = \min\{1, e^z\}$ . On the interval joining 0 and  $z$ , the function  $e^t - b_z t^2/2$  is convex, because its second derivative is  $e^t - b_z \geq 0$ . Applying convexity at the point  $\theta z$  proves (22). If  $0 < \theta < 1$  and  $z \neq 0$ , the estimate is strict, because the adjusted exponential is strictly convex on the interior of that nondegenerate interval.

Use this estimate with  $z = K \log x_i$  and sum over  $i$ . The exact identity

$$L_p - S_k = \theta(L_p - S_K) + (1 - \theta)(L_p - n) + \sum_{i=1}^n (1 - \theta + \theta x_i^K - x_i^k)$$

then gives the logarithmic remainder. Finally, apply (18) to  $L_p - S_K$ . The other deficits are nonnegative by Lemma 4.2 and  $L_p \geq n$ .  $\square$

In particular, if  $0 < k < K$  and all variables satisfy  $x_i \geq \eta$  for a fixed  $0 < \eta \leq 1$ , then

$$L_p - S_k \geq \frac{k(K-k)}{2} \eta^K \sum_{i=1}^n (\log x_i)^2. \quad (23)$$

Indeed,  $\min\{1, x_i^K\} \geq \eta^K$ . Thus, below the endpoint, a small deficit quantitatively forces every variable to be close to 1.

Such a positive quadratic logarithmic remainder cannot hold at the endpoint, even near the equality tuple. On the perturbation used in (20), with  $k = K$ , the difference satisfies  $\Delta_{p,K}(\varepsilon) = O(\varepsilon^3)$ , because its value and first two derivatives vanish at zero. On the other hand,

$$\sum_{i=1}^n (\log x_i)^2 = n(n-1)\varepsilon^2 + O(\varepsilon^3).$$

Their ratio tends to zero. Consequently, no fixed  $C > 0$  can make  $L_p - S_K \geq C \sum_i (\log x_i)^2$  valid on every admissible tuple in a neighborhood of the all-one tuple. This explains why the endpoint and subcritical remainder estimates have different forms.

**Proposition 5.2** (Sharp bounds for coordinates and arithmetic sums). *For  $n \geq 3$ , under (10),*

$$\max_i x_i < 2 + \frac{1}{n-2}, \quad n \leq \sum_{i=1}^n x_i < n + \frac{1}{n-2}. \quad (24)$$

*The lower sum bound is attained exactly at the all-one tuple. Both upper constants are sharp suprema, but neither is attained.*

*Proof.* Put  $m = n-2$ ,  $r = n-1$ , and relabel the variables as  $a_1, \dots, a_r, c$ , with  $a_i \geq 1$  and  $0 < c \leq 1$ . The inequality  $a_i c + m(a_i + c) \leq 2m + 1$  gives  $ma_i < 2m + 1$ , since  $c > 0$ . Thus every  $a_i < 2 + 1/m$ ; the same bound is automatic for  $c \leq 1$ .

Let  $T = \sum_{i=1}^r (a_i - 1)$  and  $E = \sum_{i < j} (a_i - 1)(a_j - 1) \geq 0$ . Expanding the normalization gives

$$mT + E + c(r + T) = r.$$

Hence

$$T \leq \frac{r(1-c)}{m+c}, \quad \sum_{i=1}^n x_i = r + c + T \leq g(c) := r + c + \frac{r(1-c)}{m+c}.$$

Since

$$g'(c) = 1 - \frac{r^2}{(m+c)^2} \leq 0 \quad (0 \leq c \leq 1),$$

with strict inequality for  $c < 1$ , positivity of  $c$  gives

$$\sum_{i=1}^n x_i < g(0) = n + \frac{1}{m}.$$

For the lower bound, Cauchy's inequality gives

$$n(n-1) = \left( \sum_i x_i \right)^2 - \sum_i x_i^2 \leq \frac{n-1}{n} \left( \sum_i x_i \right)^2.$$

Thus  $\sum_i x_i \geq n$ , with equality only when all variables are equal; the normalization then makes their common value 1.

To establish sharpness of both upper constants, take  $m$  variables equal to 1, one variable equal to  $c \in (0, 1)$ , and the last equal to

$$u(c) = \frac{2m+1-mc}{m+c}.$$

They satisfy the normalization, and  $u(c) > 1$ . As  $c \rightarrow 0^+$ , the maximum tends to  $2 + 1/m$  and the sum tends to  $n + 1/m$ . Neither limit tuple is positive in every coordinate, which is consistent with the strict upper bounds.  $\square$

## 6 Homogeneous and multiplicity-weighted extensions

### 6.1 Removing the unit normalization

For positive  $y_1, \dots, y_n$  with  $n \geq 3$ , define their pairwise scale

$$Q = \left( \frac{1}{\binom{n}{2}} \sum_{i < j} y_i y_j \right)^{1/2} > 0.$$

Assume at most one  $y_i$  is below  $Q$ . Applying Theorem 4.3 to  $x_i = y_i/Q$  gives

$$\boxed{\sum_{i=1}^n y_i^k \leq Q^{p+k} \sum_{i=1}^n y_i^{-p}, \quad p \geq 0, \quad 0 < k \leq p + d_n.} \quad (25)$$

Indeed,  $L_p(x) = Q^p \sum_i y_i^{-p}$  and  $S_k(x) = Q^{-k} \sum_i y_i^k$ . The exponent range is sharp, because the normalized examples have  $Q = 1$ . The multiplicative constant 1 is also sharp: equality holds when every  $y_i = Q$ .

Equivalently, if  $\sum_{i < j} y_i y_j = \binom{n}{2} s^2$  and at most one  $y_i < s$ , one can replace  $Q$  by  $s$  in (25). This formulation allows the theorem to be used with any positive normalization scale.

### 6.2 Integer multiplicities

A weighted formulation follows by grouping equal variables. The weights here are positive integers, not arbitrary real weights, and the constraint must include products within each group.

**Theorem 6.1** (Grouped-variable extension). *Let  $w_1, \dots, w_s$  be positive integers. Set*

$$N = \sum_{j=1}^s w_j \geq 3,$$

*and suppose at least one weight equals 1. Let  $y_1, \dots, y_s > 0$  satisfy*

$$\sum_{j=1}^s \binom{w_j}{2} y_j^2 + \sum_{i < j} w_i w_j y_i y_j = \binom{N}{2}. \quad (26)$$

*Require  $y_j \geq 1$  whenever  $w_j \geq 2$ , and allow at most one of the variables with  $w_j = 1$  to be below 1. For each  $p \geq 0$ , the inequality*

$$\sum_{j=1}^s w_j y_j^{-p} \geq \sum_{j=1}^s w_j y_j^k$$

*holds for all such data exactly when*

$$0 < k \leq p + \frac{N-2}{N-1}.$$

*Equality occurs exactly when all  $y_j = 1$ .*

*Proof.* Replace each  $y_j$  by  $w_j$  identical copies. This produces an  $N$ -tuple whose pairwise sum is the left side of (26). A group with multiplicity at least two cannot be below 1, and among the singleton groups only one may be below 1. Thus the expanded tuple satisfies the hypotheses of Theorem 4.3. Its power sums are exactly the weighted sums in the statement.

For sharpness, choose a singleton group, give it the value

$$c(\varepsilon) = \frac{N - (N-2)(1+\varepsilon)^2}{2(1+\varepsilon)},$$

and give every other group the value  $1+\varepsilon$ . The expanded tuple has  $N-1$  entries equal to  $1+\varepsilon$  and one entry equal to  $c(\varepsilon)$ . It is the sharpness family from (20), with dimension  $N$ . Equality follows from the all-one equality case of the same theorem.  $\square$

If there is no singleton group, the imposed restrictions make every  $y_j \geq 1$ . The normalization then forces every  $y_j = 1$ . In this degenerate case every positive exponent works, and there is no finite largest exponent. The singleton assumption in Theorem 6.1 is therefore essential for its sharpness claim.

## 7 Applications

The following applications illustrate different uses of the preceding results. Unless a different scale is specified, the variables satisfy (10).

## 7.1 Maximizing concave power sums

The case  $p = 0$  gives  $S_d \leq n$ , and in fact  $S_k \leq n$  for every  $0 < k \leq d$ . The maximum is attained only at the all-one tuple. The endpoint  $d$  is optimal. For example,

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3 \quad (n = 3), \quad \sum_{i=1}^4 x_i^{2/3} \leq 4 \quad (n = 4).$$

These are conditional power-sum bounds: the restriction on numbers below 1 is part of their hypotheses.

## 7.2 Reciprocal roots dominating the arithmetic sum

Choose  $p = 1/(n-1)$ , so  $p + d = 1$ . Then

$$\sum_{i=1}^n x_i^{-1/(n-1)} \geq \sum_{i=1}^n x_i. \quad (27)$$

The reciprocal exponent  $1/(n-1)$  is the smallest positive exponent that works for this comparison: any smaller  $p$  would have  $1 > p + d$ , and hence would fail by (20). For instance, in dimensions 3, 4, and 5, the left sides involve reciprocal square roots, cube roots, and fourth roots, respectively. The two-variable version is the identity for pairs with product 1.

## 7.3 Higher inverse powers and radical inequalities

For  $n = 3$ ,  $d = 1/2$ . Therefore, for every  $p \geq 0$ ,

$$a^{-p} + b^{-p} + c^{-p} \geq a^{p+1/2} + b^{p+1/2} + c^{p+1/2}.$$

Examples include

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} \geq a + b + c, \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^{5/2} + b^{5/2} + c^{5/2}.$$

For  $n = 5$  and  $p = 2$ , the largest exponent on the right is  $11/4$ . Every displayed endpoint is sharp and has equality only at the all-one tuple.

## 7.4 Selecting the dimension for prescribed exponents

Fix  $p \geq 0$  and  $k > 0$ . The complete dimension criterion is:

- If  $0 < k \leq p$ , the inequality holds in every dimension  $n \geq 2$ .
- If  $p < k < p + 1$ , it holds exactly when

$$n \geq 1 + \left\lceil \frac{1}{p+1-k} \right\rceil.$$

- If  $k \geq p + 1$ , it holds in no finite dimension.

Dimension 1 is always excluded. The middle criterion is just the rearrangement of  $k \leq p + 1 - 1/(n-1)$ . For example, when  $p = 1$ , the exponent  $7/4$  requires  $n \geq 5$ , while  $19/10$  requires  $n \geq 11$ .

## 7.5 Conditional inequalities between means

For positive data  $y_1, \dots, y_n$ , define

$$M_t = \left( \frac{1}{n} \sum_{i=1}^n y_i^t \right)^{1/t} \quad (t \neq 0), \quad G = \left( \prod_{i=1}^n y_i \right)^{1/n}.$$

Let  $A = M_1$  and  $H = M_{-1}$ , and let  $Q$  be the pairwise scale defined in Section 6. For  $n \geq 3$  and at most one  $y_i < Q$ , the concave endpoint gives  $M_d \leq Q$ . Arithmetic-geometric mean applied to  $y_i^d$  and to  $y_i^{-1}$  also gives  $G \leq M_d$  and  $H \leq G$ . Finally,

$$A^2 - Q^2 = \frac{1}{n(n-1)} \sum_{i=1}^n (y_i - A)^2 \geq 0.$$

Consequently,

$$H \leq G \leq M_d \leq Q \leq A. \quad (28)$$

For  $p > 0$ , the homogeneous endpoint additionally yields

$$M_{-p}^p M_{p+d}^{p+d} \leq Q^{2p+d}. \quad (29)$$

In particular,  $H M_{K_n}^{K_n} \leq Q^{K_n+1}$ . Equality in these conditional endpoint comparisons occurs when all data are equal.

## 7.6 A reciprocal dual formulation

Put  $y_i = 1/x_i$ . The hypotheses become

$$\sum_{i < j} \frac{1}{y_i y_j} = \binom{n}{2}, \quad \text{at most one } y_i > 1.$$

The endpoint family becomes

$$\sum_{i=1}^n y_i^p \geq \sum_{i=1}^n y_i^{-(p+d_n)} \quad (n \geq 3, p \geq 0).$$

This is useful when the natural data have at most one entry above 1 rather than at most one below 1. It is exactly the original family under inversion, so the exponent range and equality cases are inherited without additional assumptions.

## 7.7 Extremal sums and coordinate bounds

Proposition 5.2 gives explicit feasible ranges. For three variables,

$$\max\{a, b, c\} < 3, \quad 3 \leq a + b + c < 4.$$

For four variables,

$$\max_i x_i < \frac{5}{2}, \quad 4 \leq \sum_{i=1}^4 x_i < \frac{9}{2}.$$

The upper numbers are suprema, not maxima, since the limiting configuration has one zero coordinate. These distinctions are important in optimization problems over strictly positive variables.

## 7.8 Uniform stability away from zero

Take  $n = 4$ ,  $p = 1$ ,  $k = 1$ , so  $K = 5/3$ . If every variable is at least  $1/2$ , equation (23) gives

$$\sum_{i=1}^4 \frac{1}{x_i} - \sum_{i=1}^4 x_i \geq \frac{2^{-5/3}}{3} \sum_{i=1}^4 (\log x_i)^2.$$

Thus a deficit of at most  $\varepsilon$  implies

$$\sum_{i=1}^4 (\log x_i)^2 \leq 3 \cdot 2^{5/3} \varepsilon.$$

This is a quantitative form of the equality statement, not merely a qualitative assertion that equality requires all variables to be 1.

## 7.9 Grouped data with a repeated coordinate

Use multiplicities  $(2, 1, 1)$ , so the expanded dimension is  $N = 4$ . For  $a \geq 1$ ,  $b, c > 0$ , at most one of  $b, c$  below 1, and

$$a^2 + 2ab + 2ac + bc = 6,$$

Theorem 6.1 gives

$$\frac{2}{a} + \frac{1}{b} + \frac{1}{c} \geq 2a^{5/3} + b^{5/3} + c^{5/3}.$$

The sharp endpoint is determined by the expanded dimension 4, not by the number 3 of distinct coordinate symbols. Equality holds only when  $a = b = c = 1$ .

## 7.10 A nonunit normalization

Let  $y_1, \dots, y_4 > 0$  satisfy  $\sum_{i < j} y_i y_j = 24$ , with at most one  $y_i < 2$ . Then  $Q = 2$ , and the homogeneous endpoint with  $p = 1$  is

$$\sum_{i=1}^4 y_i^{5/3} \leq 2^{8/3} \sum_{i=1}^4 \frac{1}{y_i}.$$

The factor  $2^{8/3}$  is best possible, since equality holds at  $(2, 2, 2, 2)$ . This example shows why the normalization scale must appear with the exponent  $p + k$ .

# 8 Related problems with solutions

The following problems are variations developed from the preceding results. Their solutions also illustrate how to recognize the relevant normalization, reciprocal exponent, and effective dimension.

**Problem 8.1** (An optimal concave power). Let  $a, b, c > 0$ ,  $ab + bc + ca = 3$ , with at most one of  $a, b, c$  below 1. Find the largest  $\alpha > 0$  for which

$$a^\alpha + b^\alpha + c^\alpha \leq 3$$

holds for every such triple.

*Solution.* The answer is  $\alpha = 1/2$ . This is the  $n = 3, p = 0$  case of Theorem 4.3; equality holds only at  $(1, 1, 1)$ . For a direct sharpness check, on the admissible family with the first two entries  $1 + \varepsilon$ , the second derivative at zero of  $3 - (a^\alpha + b^\alpha + c^\alpha)$  equals  $-3\alpha(2\alpha - 1)$ . It is negative for every  $\alpha > 1/2$ , so each larger exponent fails for sufficiently small positive  $\varepsilon$ .  $\square$

**Problem 8.2** (The least reciprocal exponent). Let  $x_1, \dots, x_4 > 0$  satisfy  $\sum_{i < j} x_i x_j = 6$ , with at most one variable below 1. Find the smallest  $p > 0$  such that

$$\sum_{i=1}^4 x_i^{-p} \geq \sum_{i=1}^4 x_i$$

holds for every admissible tuple.

*Solution.* Here  $d_4 = 2/3$  and the right-hand exponent is  $k = 1$ . The necessary and sufficient condition is  $1 \leq p + 2/3$ , so the smallest value is  $p = 1/3$ . At this value equality holds only at  $(1, 1, 1, 1)$ . If  $p < 1/3$ , formula (20) gives the second derivative  $-4(p + 1)(1 - 3p) < 0$  on the admissible perturbation, which proves failure rather than merely showing that the theorem cannot be applied.  $\square$

**Problem 8.3** (Inverse squares in five variables). Let  $x_1, \dots, x_5 > 0$  satisfy  $\sum_{i < j} x_i x_j = 10$ , with at most one variable below 1. Determine the largest  $k > 0$  for which

$$\sum_{i=1}^5 \frac{1}{x_i^2} \geq \sum_{i=1}^5 x_i^k$$

holds universally.

*Solution.* Take  $n = 5, p = 2$ , and  $d_5 = 3/4$ . The answer is  $k = 2 + 3/4 = 11/4$ , with equality only at the all-one tuple. Every larger  $k$  fails because the second derivative in (20) is  $-5(2 + k)(4k - 11) < 0$ . Thus both validity and optimality follow from the proved family.  $\square$

**Problem 8.4** (The best factor at a different scale). Let  $y_1, \dots, y_4 > 0$  satisfy  $\sum_{i < j} y_i y_j = 24$ , with at most one variable below 2. Find the smallest constant  $C$  for which

$$\sum_{i=1}^4 y_i^{5/3} \leq C \sum_{i=1}^4 \frac{1}{y_i}$$

holds for every such tuple.

*Solution.* Put  $x_i = y_i/2$ . Then  $\sum_{i < j} x_i x_j = 6$ , with at most one  $x_i < 1$ . The endpoint  $K_4 = 5/3$  gives

$$2^{-5/3} \sum_i y_i^{5/3} \leq 2 \sum_i \frac{1}{y_i},$$

so  $C = 2^{8/3}$  works. At  $(y_1, \dots, y_4) = (2, 2, 2, 2)$  the ratio of the power sum to the reciprocal sum is  $2^{8/3}$ , so no smaller constant works. Equality occurs only at this tuple.  $\square$

**Problem 8.5** (A sharp arithmetic range). Under the hypotheses of the original three-variable problem, determine the infimum and supremum of  $a + b + c$ , and decide whether they are attained.

*Solution.* Proposition 5.2 gives  $3 \leq a + b + c < 4$ . The infimum 3 is attained only at  $(1, 1, 1)$ . To approach the supremum, take  $b = 1$ ,  $0 < c < 1$ , and

$$a = \frac{3 - c}{1 + c}.$$

Then  $a > 1$  and  $ab + bc + ca = 3$ . As  $c \rightarrow 0^+$ ,  $a \rightarrow 3$  and  $a + b + c \rightarrow 4$ . Thus the supremum is 4, but it is not attained for positive triples.  $\square$

**Problem 8.6** (A repeated-variable version). Let  $a \geq 1$  and  $b > 0$  satisfy  $a^2 + 2ab = 3$ . Find the largest  $k > 0$  such that

$$\frac{2}{a} + \frac{1}{b} \geq 2a^k + b^k$$

holds for every admissible pair.

*Solution.* The expanded triple  $(a, a, b)$  satisfies the original normalization and has at most one entry below 1. Therefore  $k = 3/2$  works, with equality only at  $a = b = 1$ . For sharpness, set  $a = 1 + \varepsilon$  and  $b = (3 - a^2)/(2a)$ . This is precisely the sharpness family for Problem 5167, so every  $k > 3/2$  fails for sufficiently small positive  $\varepsilon$ .  $\square$

**Problem 8.7** (A dimension threshold). For which positive integers  $n$  does

$$\sum_{i=1}^n \frac{1}{x_i} \geq \sum_{i=1}^n x_i^{7/4}$$

hold for every tuple satisfying (10)?

*Solution.* Dimension 1 is excluded by taking  $x_1 = 2$ . For  $n \geq 2$ , Theorem 3.1 gives the exact condition

$$\frac{7}{4} \leq 2 - \frac{1}{n-1} \iff n \geq 5.$$

Thus the answer is exactly the integers  $n \geq 5$ , with equality only at the all-one tuple. For  $n = 2, 3, 4$ , the perturbed admissible tuple used in (11) supplies counterexamples, since  $7/4 > K_n$ .  $\square$

**Problem 8.8** (A product of means). Let  $y_1, \dots, y_4 > 0$  and define

$$Q = \left( \frac{1}{6} \sum_{i < j} y_i y_j \right)^{1/2}, \quad H = \frac{4}{\sum_i y_i^{-1}}, \quad M_{5/3} = \left( \frac{1}{4} \sum_i y_i^{5/3} \right)^{3/5}.$$

Assume at most one  $y_i < Q$ . Prove

$$H M_{5/3}^{5/3} \leq Q^{8/3},$$

and determine the equality case.

*Solution.* Normalize by  $x_i = y_i/Q$  and apply the four-variable endpoint with  $p = 1$ . The homogeneous inequality is

$$\sum_i y_i^{5/3} \leq Q^{8/3} \sum_i y_i^{-1}.$$

Divide by 4 and multiply by  $H = 4/\sum_i y_i^{-1}$  to obtain the claimed product estimate. Equality holds exactly when all  $y_i = Q$ , or equivalently when all four numbers are equal. This also shows that the multiplicative constant 1 is optimal.  $\square$

**Problem 8.9** (A logarithmic remainder). Let  $a, b, c > 0$ ,  $ab + bc + ca = 3$ , with at most one variable below 1. Put  $L = 1/a + 1/b + 1/c$  and define

$$W(x) = \min\{1, x^{3/2}\}(\log x)^2 \quad (x > 0).$$

Prove

$$\begin{aligned} L - (a + b + c) &\geq \frac{1}{3}(L - 3) + \frac{10}{3}(3 - \sqrt{a} - \sqrt{b} - \sqrt{c}) \\ &\quad + \frac{1}{4}(W(a) + W(b) + W(c)). \end{aligned}$$

*Solution.* In Proposition 5.1, choose  $n = 3$ ,  $p = 1$ ,  $d = 1/2$ ,  $K = 3/2$ , and  $k = 1$ . Then  $\theta = 2/3$ ,

$$\theta \frac{2p + d}{d} = \frac{10}{3}, \quad 1 - \theta = \frac{1}{3}, \quad \frac{\theta(1 - \theta)K^2}{2} = \frac{1}{4}.$$

Substitution gives exactly the proposed estimate. All terms on the right are nonnegative. Equality holds only at  $(1, 1, 1)$ : since  $0 < \theta < 1$ , the exponential remainder estimate (22) is strict whenever a variable differs from 1.  $\square$

**Problem 8.10** (Why the restriction is necessary). Show that the condition on the number of variables below 1 cannot be omitted from the original sharp inequality with exponent  $3/2$ .

*Solution.* Take

$$a = b = \frac{1}{2}, \quad c = \frac{11}{4}.$$

Then  $ab + bc + ca = 1/4 + 11/8 + 11/8 = 3$ , but two variables are below 1. The reciprocal sum equals  $48/11$ , while

$$a^{3/2} + b^{3/2} + c^{3/2} = \frac{\sqrt{2}}{2} + \frac{11\sqrt{11}}{8} > \frac{37}{8} > \frac{48}{11}.$$

Here  $\sqrt{2} > 1$  and  $\sqrt{11} > 3$  give the first strict inequality, and  $37 \cdot 11 > 48 \cdot 8$  gives the second. Thus the desired inequality is reversed on this positive triple.  $\square$

## 9 Normalization by elementary symmetric sums

The pairwise normalization can be replaced by any elementary symmetric sum. For  $0 \leq j \leq n$ , write

$$e_j(x_1, \dots, x_n) = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=j}} \prod_{i \in I} x_i, \quad e_0 = 1.$$

An elementary symmetric sum of degree greater than the number of its arguments is understood to be zero. The following theorem includes arithmetic, pairwise, and product normalizations in one statement.

**Theorem 9.1** (All symmetric orders). Let  $n \geq 2$ ,  $1 \leq j \leq n$ , and  $p \geq 0$ . Suppose that

$$e_j(x_1, \dots, x_n) = \binom{n}{j}, \quad x_i > 0, \quad \text{at most one } x_i < 1. \quad (30)$$

Then the inequality

$$\sum_{i=1}^n x_i^{-p} \geq \sum_{i=1}^n x_i^k$$

holds for every such tuple if and only if

$$\boxed{0 < k \leq K_{n,j}(p) := p + \frac{n-j}{n-1}}. \quad (31)$$

If  $p = 0$  and  $j = n$ , this range is empty. For  $0 < k < K_{n,j}(p)$ , equality holds only at the all-one tuple. At the endpoint, the equality cases are as follows:

- If  $2 \leq j \leq n-1$ , equality holds only at the all-one tuple.
- If  $j = 1$  and  $p = 0$ , every tuple satisfying the hypotheses gives equality; if  $j = 1$  and  $p > 0$ , only the all-one tuple does.
- If  $j = n$  and  $p > 0$ , equality holds exactly at permutations of  $(t, t^{-1}, 1, \dots, 1)$ , where  $t > 0$ .

### 9.1 A comparison valid for every positive parameter

The argument for the concave power bound extends to positive real parameters, not just the integer  $n-2$  used earlier.

**Lemma 9.2.** Let  $\mu > 0$ ,  $d = \mu/(\mu+1)$ , and  $\Phi(x) = (\mu+x)x^{d-1}$  for  $x > 0$ . If  $u \geq 1$ ,  $v > 0$ , and  $uv + \mu(u+v) \leq 1 + 2\mu$ , then  $\Phi(v) \geq \Phi(u)$ , with equality only at  $u = v = 1$ .

*Proof.* Differentiation gives  $\Phi'(x) = dx^{d-2}(x-1)$ , so  $\Phi$  is strictly decreasing on  $(0, 1]$ . The hypotheses imply

$$0 < v \leq w(u) := \frac{1 + 2\mu - \mu u}{\mu + u} \leq 1.$$

For  $u = 1$ , strict decrease proves the assertion. For  $u > 1$ , we have  $0 < w < 1$  and

$$uw + \mu(u+w) = 1 + 2\mu, \quad 1 - uw = \frac{\mu(u-1)^2}{\mu+u} > 0.$$

Direct differentiation, followed by the first identity, yields

$$\begin{aligned} \frac{d}{du} \log \frac{\Phi(w(u))}{\Phi(u)} &= \frac{d(u+w-2uw)}{uw(\mu+u)} \\ &= \frac{(1+2\mu)(1-uw)}{(\mu+1)uw(\mu+u)} > 0. \end{aligned}$$

The interval from 1 to the given  $u$  lies in the domain  $w > 0$ . Since the ratio equals 1 at  $u = 1$ , we obtain  $\Phi(v) \geq \Phi(w(u)) > \Phi(u)$  for  $u > 1$ . This also proves the stated equality case.  $\square$

## 9.2 Proof for the intermediate symmetric orders

*The intermediate-order endpoint.* Relabel the variables as  $a_1, \dots, a_{n-1}, c$ , with every  $a_i \geq 1$ . Writing  $a = (a_1, \dots, a_{n-1})$ , the constraint gives

$$c = \frac{\binom{n}{j} - e_j(a)}{e_{j-1}(a)} > 0.$$

The denominator is positive. Moreover,  $e_j(a) + e_{j-1}(a) = e_j(a_1, \dots, a_{n-1}, 1) \geq \binom{n}{j}$ , so  $c \leq 1$ . The independent coordinate domain is given by  $a_i \geq 1$  and  $e_j(a) < \binom{n}{j}$ . Decreasing any coordinate to 1 remains in this domain.

Fix a coordinate  $a_i$ , abbreviate it as  $u$ , and let  $z$  denote the other  $n-2$  independent coordinates. Set

$$A = e_{j-2}(z), \quad B = e_{j-1}(z), \quad C = e_j(z), \quad \lambda = \frac{B}{A}, \quad \mu = \frac{n-j}{j-1}.$$

Here  $A, B, \mu$  are positive. Counting each monomial of degree  $j-1$  exactly  $j-1$  times gives

$$(j-1)B = \sum_{\substack{I \subseteq \{1, \dots, n-2\} \\ |I|=j-2}} \left( \prod_{i \in I} z_i \right) \sum_{\ell \notin I} z_\ell \geq (n-j)A.$$

Thus  $\lambda \geq \mu$ . Let

$$A_0 = \binom{n-2}{j-2}, \quad B_0 = \binom{n-2}{j-1}, \quad C_0 = \binom{n-2}{j}.$$

Since the coordinates in  $z$  are at least 1,  $A \geq A_0$  and  $C \geq C_0$ . Also  $B_0/A_0 = \mu$  and  $\binom{n}{j} = A_0 + 2B_0 + C_0$ . The normalization takes the form  $Auc + B(u+c) + C = \binom{n}{j}$ . Consequently,

$$uc + \mu(u+c) \leq uc + \lambda(u+c) = \frac{\binom{n}{j} - C}{A} \leq 1 + 2\mu. \quad (32)$$

The last numerator is positive, so the stated lower bounds on  $A, C$  justify the final estimate.

Putting  $\rho = \sqrt{uc}$  and using  $u+c \geq 2\rho$ , we get  $\rho^2 + 2\mu\rho \leq 1 + 2\mu$ , hence  $uc \leq 1$ . If  $uc = 1$ , equality forces  $u+c = 2$ , thus  $u = c = 1$ . Then every original variable is at least 1, and the normalization forces all of them to be 1. Therefore, away from that tuple,

$$0 < c < 1, \quad a_i c < 1 \quad \text{for every } i. \quad (33)$$

Put

$$d = \frac{\mu}{\mu+1} = \frac{n-j}{n-1}, \quad F_0 = n - \sum_{i=1}^{n-1} a_i^d - c^d.$$

Holding  $z$  fixed and differentiating the normalization gives

$$\frac{\partial c}{\partial u} = -\frac{\lambda + c}{\lambda + u}.$$

Since  $u \geq 1 \geq c$  and  $\lambda \geq \mu$ ,

$$\frac{\lambda + c}{\lambda + u} - \frac{\mu + c}{\mu + u} = \frac{(\lambda - \mu)(u - c)}{(\lambda + u)(\mu + u)} \geq 0.$$

Lemma 9.2 now implies

$$\begin{aligned}\frac{\partial F_0}{\partial u} &= -du^{d-1} + dc^{d-1} \frac{\lambda + c}{\lambda + u} \\ &\geq \frac{d(\Phi(c) - \Phi(u))}{\mu + u} \geq 0.\end{aligned}$$

Reducing the independent coordinates successively to 1 proves

$$H := n - \sum_{i=1}^n x_i^d \geq 0, \quad (34)$$

with equality only at the all-one tuple. Indeed, any coordinate initially greater than 1 produces a segment with strictly positive derivative. At the final point  $c = 1$  by Pascal's identity.

For completeness, the transfer to inverse powers can be written explicitly. Let

$$\ell = -\log c, \quad t_i = \log a_i, \quad R(t) = \frac{\sinh((p + d/2)t)}{\sinh((d/2)t)}, \quad R(0) = \frac{2p + d}{d}.$$

The function  $R$  is nondecreasing on  $[0, \infty)$ , and strictly increasing on  $(0, \infty)$  if  $p > 0$ . This follows by differentiating its logarithm and using the increasing function  $q \coth q$ , as in the proof of Theorem 4.3. By (33),  $\ell \geq t_i$ . The same algebraic decomposition as (19) gives

$$\begin{aligned}\sum_i x_i^{-p} - \sum_i x_i^{p+d} &= R(\ell)H + 2 \sum_{i=1}^{n-1} a_i^{d/2} (R(\ell) - R(t_i)) \sinh(dt_i/2) \\ &\geq \frac{2p + d}{d} \left( n - \sum_i x_i^d \right) \geq 0.\end{aligned} \quad (35)$$

All its terms have the required signs. The strictness in (34) proves the endpoint equality case.  $\square$

### 9.3 Arithmetic and product normalizations

*The endpoint cases  $j = 1$  and  $j = n$ .* If  $j = 1$ , then  $\sum_i x_i = n$ , and  $d = 1$ . After the same relabelling, each  $a_i + c \leq 2$ , since the other  $n - 2$  variables are at least 1. Thus  $a_i c \leq 1$ , with equality forcing the all-one tuple. Also  $H = n - \sum_i x_i = 0$ . The decomposition preceding (35) is an algebraic identity valid for every  $d > 0$ , so it applies with  $d = 1$  and proves the endpoint  $k = p + 1$ . For  $p = 0$  this is simply the identity  $n = \sum_i x_i$ . For  $p > 0$  and a nontrivial tuple, at least one  $t_i > 0$  and  $\ell > t_i$ , so strict increase of  $R$  makes its corresponding summand positive. Equality then occurs only at the all-one tuple.

If  $j = n$ , the constraint is  $\prod_i x_i = 1$ . Write  $a_i = e^{t_i}$ , with  $t_i \geq 0$ , and  $c = e^{-\ell}$ , where  $\ell = \sum_i t_i$ . For  $p > 0$ ,

$$\sum_i x_i^{-p} - \sum_i x_i^p = 2 \left( \sinh(p\ell) - \sum_{i=1}^{n-1} \sinh(pt_i) \right) \geq 0. \quad (36)$$

Here  $\sinh$  is superadditive on nonnegative arguments, because

$$\sinh(u + v) - \sinh u - \sinh v = \sinh u (\cosh v - 1) + \sinh v (\cosh u - 1) \geq 0.$$

The last inequality is strict if both  $u, v > 0$ . It follows that equality in (36) holds exactly when at most one  $t_i$  is positive, giving the stated reciprocal-pair pattern. For  $p = 0$ , the sharpness argument below excludes every positive exponent.  $\square$

## 9.4 Interpolation and sharpness in every order

*Completion of the proof of Theorem 9.1.* Let  $P = \prod_i x_i$ . Each variable occurs in  $\binom{n-1}{j-1}$  of the monomials defining  $e_j$ . Their arithmetic–geometric mean inequality gives

$$1 = \frac{e_j(x)}{\binom{n}{j}} \geq P^{j/n}.$$

Thus  $P \leq 1$  and  $L_p = \sum_i x_i^{-p} \geq nP^{-p/n} \geq n$ . If  $0 < k < K = K_{n,j}(p)$ , set  $\theta = k/K \in (0, 1)$ . The endpoint inequality and weighted arithmetic–geometric mean yield

$$\sum_i x_i^k \leq n(1 - \theta) + \theta \sum_i x_i^K \leq n(1 - \theta) + \theta L_p \leq L_p.$$

Equality in the first inequality requires every  $x_i^K = 1$ , so all variables must equal 1.

To prove sharpness for all  $j$ , put  $t = 1 + \varepsilon$  and choose

$$x_1 = \cdots = x_{n-1} = t, \quad c = x_n = \frac{n}{jt^{j-1}} - \frac{n-j}{j}t. \quad (37)$$

Indeed,

$$e_j(x) = \binom{n-1}{j} t^j + \binom{n-1}{j-1} t^{j-1} c = \binom{n}{j}.$$

The function  $c(t)$  is strictly decreasing for  $t > 0$  and  $c(1) = 1$ . For  $j < n$ , take  $1 < t < (n/(n-j))^{1/j}$  to ensure  $c > 0$ ; for  $j = n$ ,  $c = t^{-(n-1)} > 0$  automatically. Thus, for sufficiently small positive  $\varepsilon$ , exactly one variable is below 1. Differentiation with respect to  $\varepsilon$  at zero gives

$$c(0) = 1, \quad c'(0) = -(n-1), \quad c''(0) = n(j-1).$$

For  $\Delta_{p,k}(\varepsilon) = \sum_i x_i^{-p} - \sum_i x_i^k$ , we have  $\Delta_{p,k}(0) = \Delta'_{p,k}(0) = 0$  and

$$\begin{aligned} \left. \frac{d^2}{d\varepsilon^2} \sum_i x_i^{-p} \right|_0 &= np((n-1)(p+1) - (j-1)), \\ \left. \frac{d^2}{d\varepsilon^2} \sum_i x_i^k \right|_0 &= nk((n-1)(k-1) + (j-1)). \end{aligned}$$

Subtracting and factoring gives

$$\Delta''_{p,k}(0) = -n(p+k)((n-1)(k-p) - (n-j)). \quad (38)$$

This is strictly negative whenever  $k > p + (n-j)/(n-1)$ . Taylor's formula then makes  $\Delta_{p,k}(\varepsilon) < 0$  for all sufficiently small positive  $\varepsilon$ . This excludes every larger exponent and completes the proof, including the empty range when  $p = 0$ ,  $j = n$ .  $\square$

For  $n = 1$ , the arithmetic normalization  $e_1(x_1) = 1$  forces  $x_1 = 1$ , so every positive exponent works. This differs from the empty pairwise normalization  $e_2(x_1) = 0$ , which imposed no restriction in the original extension.

## 10 Further consequences and solved variations

### 10.1 A hierarchy of sharp endpoints

For fixed  $n$  and  $p$ , the sharp endpoints decrease by  $1/(n-1)$  as the order  $j$  increases. For example, in four variables they are

$$\frac{j}{K_{4,j}(p)} \left| \begin{array}{cccc} 1 & 2 & 3 & 4 \\ p+1 & p+\frac{2}{3} & p+\frac{1}{3} & p \end{array} \right.$$

The entries refer to different normalization surfaces, not to a chain of inequalities for one fixed tuple. For  $p = 0$  and  $j < n$ , the theorem gives the sharp concave bound

$$\sum_i x_i^{(n-j)/(n-1)} \leq n.$$

For instance,  $e_3(x_1, x_2, x_3, x_4) = 4$  implies  $\sum_i x_i^{1/3} \leq 4$  under the one-small-variable restriction. In five variables,  $e_3 = 10$  gives sharp reciprocal exponent  $3/2$ , whereas  $e_4 = 5$  gives sharp reciprocal exponent  $5/4$ .

### 10.2 Homogeneous means and a product estimate

For positive  $y_1, \dots, y_n$ , define

$$E_j = \left( \frac{e_j(y)}{\binom{n}{j}} \right)^{1/j}.$$

If at most one  $y_i < E_j$ , normalization by  $x_i = y_i/E_j$  gives

$$\sum_i y_i^k \leq E_j^{p+k} \sum_i y_i^{-p} \quad \text{for } 0 < k \leq p + \frac{n-j}{n-1}. \quad (39)$$

For every admissible exponent, the multiplicative constant 1 in this form is best possible, as shown by any all-equal tuple. If  $j < n$  and  $d = (n-j)/(n-1)$ , this also gives  $M_d \leq E_j$ , where  $M_d = (n^{-1} \sum_i y_i^d)^{1/d}$ . The equality cases follow by scaling those in Theorem 9.1.

A particularly different endpoint arises at the product order. Let  $A = n^{-1} \sum_i y_i$ ,  $H = n/(\sum_i y_i^{-1})$ , and  $G = (\prod_i y_i)^{1/n}$ . If at most one  $y_i < G$ , taking  $j = n$  and  $p = k = 1$  gives the sharp estimate

$$AH \leq G^2. \quad (40)$$

Equality is not restricted to all-equal tuples: it holds at permutations of  $(Gt, G/t, G, \dots, G)$ ,  $t > 0$ . Thus this product inequality complements, rather than replaces, the usual separate comparisons involving arithmetic and harmonic means.

### 10.3 Integer multiplicities at every symmetric order

Let  $w_1, \dots, w_s$  be positive integers with  $N = \sum_\ell w_\ell \geq 2$ . The elementary symmetric sums of the expanded tuple containing  $w_\ell$  copies of  $y_\ell$  are the coefficients of

$$\prod_{\ell=1}^s (1 + zy_\ell)^{w_\ell}.$$

Fix  $1 \leq j \leq N$ , and require its coefficient of  $z^j$  to equal  $\binom{N}{j}$ . Assume every group of multiplicity at least 2 has  $y_\ell \geq 1$ , and at most one singleton group is below 1. If there is at least one singleton group, the sharp range is

$$\sum_{\ell=1}^s w_\ell y_\ell^{-p} \geq \sum_{\ell=1}^s w_\ell y_\ell^k, \quad 0 < k \leq p + \frac{N-j}{N-1}. \quad (41)$$

Validity follows by expanding the tuple. For sharpness, give a chosen singleton the value  $c$  from (37), with  $n = N$ , and give every other group the value  $t$ . The expanded tuple is precisely that counterexample family. Equality cases are obtained from the expanded tuple in Theorem 9.1; in particular, a nontrivial product-endpoint equality needs two singleton groups. If there are no singleton groups, all entries are at least 1 and the normalization forces all of them to be 1. In that restricted case, every positive exponent works and no finite largest exponent exists.

## 10.4 Remainders for intermediate symmetric orders

For  $2 \leq j \leq n-1$ , the proof gives more than the endpoint inequality: (35) controls the deficit by the concave power-sum deficit. In addition, let  $K = p + (n-j)/(n-1)$  and  $0 < k < K$ , and set  $\theta = k/K$ . The same pointwise exponential remainder estimate used in Proposition 5.1 gives

$$\begin{aligned} \sum_i x_i^{-p} - \sum_i x_i^k &\geq \theta \frac{2p+d}{d} \left( n - \sum_i x_i^d \right) + (1-\theta) \left( \sum_i x_i^{-p} - n \right) \\ &\quad + \frac{\theta(1-\theta)K^2}{2} \sum_i \min\{1, x_i^K\} (\log x_i)^2, \quad d = \frac{n-j}{n-1}. \end{aligned}$$

Every term on the right is nonnegative. If  $x_i \geq \eta$  for every  $i$ , where  $0 < \eta \leq 1$ , this yields

$$\sum_i x_i^{-p} - \sum_i x_i^k \geq \frac{k(K-k)}{2} \eta^K \sum_i (\log x_i)^2.$$

At the endpoint, however, no positive coefficient multiplying  $\sum_i (\log x_i)^2$  can hold uniformly near the all-one tuple: (38) vanishes when  $k = K$ , so the endpoint deficit along (37) is  $O(\varepsilon^3)$ , whereas the logarithmic square sum is  $n(n-1)\varepsilon^2 + O(\varepsilon^3)$ .

## 10.5 Five further problems with complete solutions

**Problem 10.1** (A triple-product normalization in four variables). Let  $x_1, x_2, x_3, x_4 > 0$  satisfy

$$x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = 4,$$

with at most one variable below 1. Determine the largest  $k > 0$  for which  $\sum_i x_i^{-1} \geq \sum_i x_i^k$  always holds.

*Solution.* Here  $n = 4$ ,  $j = 3$ , and  $p = 1$ , so the answer is  $k = 4/3$  by Theorem 9.1. Equality holds only at  $(1, 1, 1, 1)$ . To exhibit the obstruction explicitly, put the first three variables equal to  $t = 1 + \varepsilon$  and the last equal to  $c = 4/(3t^2) - t/3$ . Then  $t^3 + 3t^2c = 4$  and  $0 < c < 1$  for small positive  $\varepsilon$ . The second derivative of the difference at zero is  $-4(1+k)(3(k-1)-1)$ , which is negative for  $k > 4/3$ . Every larger exponent therefore fails on this admissible family.  $\square$

**Problem 10.2** (An arithmetic normalization). Let  $a, b, c > 0$ ,  $a + b + c = 3$ , with at most one variable below 1. Find the largest  $k > 0$  such that  $1/a + 1/b + 1/c \geq a^k + b^k + c^k$  always holds.

*Solution.* The arithmetic-order case  $n = 3$ ,  $j = 1$ ,  $p = 1$  gives  $k = 2$ , with equality only at  $(1, 1, 1)$ . For sharpness, use  $a = b = 1 + \varepsilon$  and  $c = 1 - 2\varepsilon$ , where  $0 < \varepsilon < 1/2$ . The difference has second derivative  $-6(1+k)(k-2)$  at zero. For every  $k > 2$  it is negative, so sufficiently small positive  $\varepsilon$  reverses the inequality. The constant 2 is thus optimal.  $\square$

**Problem 10.3** (An exceptional product equality). Let  $a, b, c > 0$ ,  $abc = 1$ , with at most one variable below 1. Prove  $1/a + 1/b + 1/c \geq a + b + c$ , and find every equality case.

*Solution.* Relabel so that  $a, b \geq 1$ . Then  $c = 1/(ab) \leq 1$ , and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - (a + b + c) = ab + bc + ca - a - b - c = -(a-1)(b-1)(c-1) \geq 0.$$

The factorization uses  $abc = 1$ . Equality requires  $a = 1$ ,  $b = 1$ , or  $c = 1$ ; the last alternative forces  $a = b = 1$ . Thus all equality triples are permutations of  $(t, t^{-1}, 1)$  with  $t > 0$ . The exponent 1 is also largest for the reciprocal inequality under this normalization: (38) with  $n = j = 3$ ,  $p = 1$  is  $-6(1+k)(k-1)$ , negative for  $k > 1$ .  $\square$

**Problem 10.4** (Fourth-order normalization and inverse squares). Let  $x_1, \dots, x_5 > 0$  satisfy  $e_4(x_1, \dots, x_5) = 5$ , with at most one variable below 1. Determine the largest  $k > 0$  for which  $\sum_i x_i^{-2} \geq \sum_i x_i^k$  holds universally.

*Solution.* Taking  $n = 5$ ,  $j = 4$ , and  $p = 2$  gives  $k = 9/4$ . Equality holds exactly at the all-one tuple. The explicit sharpness family has four entries  $t = 1 + \varepsilon$  and last entry  $c = 5/(4t^3) - t/4$ , since  $t^4 + 4t^3c = 5$ . Its difference has second derivative  $-5(2+k)(4(k-2)-1)$  at zero, which is negative whenever  $k > 9/4$ . This proves both validity and optimality.  $\square$

**Problem 10.5** (A weighted cubic constraint). Suppose  $a \geq 1$ ,  $b, c > 0$ , and at most one of  $b, c$  is below 1. Under the constraint

$$a^2b + a^2c + 2abc = 4,$$

find the largest  $k > 0$  for which

$$\frac{2}{a} + \frac{1}{b} + \frac{1}{c} \geq 2a^k + b^k + c^k$$

always holds.

*Solution.* Expand the pair of copies of  $a$ : the tuple  $(a, a, b, c)$  has  $e_3 = 4$  and at most one entry below 1. Thus  $k = 4/3$  works by Theorem 9.1, with equality only when  $a = b = c = 1$ . For sharpness set  $a = b = t = 1 + \varepsilon$  and  $c = 4/(3t^2) - t/3$ . Then the expanded tuple has three entries  $t$  and one entry  $c$ , exactly the sharpness family of the first problem in this subsection. Consequently, every  $k > 4/3$  fails for sufficiently small positive  $\varepsilon$ .  $\square$

## 11 Concluding remarks

Problem 5167 has the sharp exponent  $3/2$ . Its normalized  $n$ -variable reciprocal extension has endpoint  $K_n = 2 - 1/(n - 1)$ . The broader reciprocal-power family has endpoint  $p + d_n = p + (n - 2)/(n - 1)$  for every  $p \geq 0$  and  $n \geq 3$ ; the two small dimensions are handled separately. The case  $p = 0$  gives the optimal concave power-sum bound, and the case  $p = 1$  recovers the reciprocal inequality. For fixed  $p$ , the endpoints approach  $p + 1$  as the dimension increases, but that limiting exponent is not admissible in any finite dimension.

The restriction that at most one variable is below the normalization scale is central: it gives both the coordinate comparisons and the logarithmic domination used to transfer the concave bound to all inverse powers. Scaling and integer multiplicities yield further formulations without losing this restriction.

The exact decomposition (19) and the quantitative interpolation estimate (21) explain how deficits control departure from equality. The endpoint degeneracy also explains why a positive quadratic logarithmic remainder is available below, but not at, the endpoint. The sharp arithmetic and coordinate bounds distinguish maxima from unattained suprema. The applications and fifteen solved variations illustrate the exponent ranges, equality cases, and the necessity of the stated hypotheses.

The elementary-symmetric normalization  $e_j = \binom{n}{j}$  has the sharp endpoint  $p + (n - j)/(n - 1)$  for  $n \geq 2$ . This places arithmetic, pairwise, and product constraints within a single framework while retaining their different equality patterns. The homogeneous and integer-multiplicity forms extend to every symmetric order, and the intermediate orders also admit quantitative remainder bounds.

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