THE α-METHOD FOR SOLVING AND REFINING SYMMETRICAL AND HOMOGENEOUS INEQUALITIES

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Abstract. The new α-method for solving and refining symmetrical and homogenous algebraic inequalities in three variables was proposed in [1]. In this work we present more details about this method, and apply it to solve and generalise a wide range of inequalities published in Crux.

1. Introduction

Recently, M. Drăgan has proposed a new method for solving and refining symmetrical and homogeneous algebraic inequalities in three variables [1]. This method links the symmetric sums of three variables in a way which enables fine comparisons between expressions which involve them.

In the first part of this paper we present the "α-method", which we apply in the second part to solve some inequalities published in this journal.

Theorem 1.1. *[The α-method] Let x, y, z be positive numbers. We denote:*

$$s = x + y + z, \quad p = xyz, \quad \alpha = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9,$$

$$t_{1,2} = \frac{\alpha^2 + 18\alpha - 27 \pm \sqrt{(\alpha - 1)(\alpha - 9)^3}}{8}$$

$$t_3 = 4\alpha - 9, \quad t_4 = \frac{\alpha^3}{4\alpha - 9}, \quad t = \frac{(x + y + z)^3}{xyz} \geq 27.$$

The following inequalities hold:

$$t_3 \leq t_1 \leq t \leq t_2 \leq t_4. \tag{1}$$

Proof. We first prove that $t_3 \leq t \leq t_4$. By Schur's inequality we have

$$s^3 + 9p \geq 4s \sum xy \quad \text{or} \quad t = \frac{s^3}{p} \geq 4\alpha - 9 \quad \text{or} \quad t \geq t_3.$$

Also from Schur we get $\left(\sum \frac{1}{x}\right)^3 + \frac{9}{xyz} \geq 4 \sum \frac{1}{xy} \sum \frac{1}{x}$, or $\frac{\alpha^3}{4\alpha - 9} = t_4 \geq t$.

We now prove that $t_1 \leq t \leq t_2$.

Indeed, one can check that x, y, z are the positive roots of the equation $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0$, where $\sigma_1, \sigma_2, \sigma_3$ are the symmetric sums

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + yz + zx, \quad \sigma_3 = xyz.$$

Then, it is known that [4]:

$$18\sigma_1\sigma_2\sigma_3 - 4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 - 4\sigma_2^3 - 27\sigma_3^2 \geq 0.$$

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If we replace $\sigma_1 = s$, $\sigma_2 = \dfrac{p\alpha}{s}$, $\sigma_3 = p$, this becomes

$$18\alpha - 4t + \alpha^2 - \frac{4\alpha^3}{t} - 27 \geq 0,$$

which is equivalent to $t \in [t_1, t_2]$.

□

Using the previous notations, the symmetric and homogeneous inequalities in three variables of the form $F(a, b, c) \geq 0$, can be rewritten as $f(t) = g(t, \alpha) \geq 0$, $t \in [t_1, t_2]$ or $t \in [t_3, t_4]$ with $\alpha \geq 9$ fixed.

The monotonicity of the function f (defined in various problems) on the intervals $[t_1, t_2]$ or $[t_3, t_4]$ will allow us to prove many symmetric inequalities. In particular, we recover some results published in Crux Mathematicorum.

2. Applications

Application 2.1. Let a, b, c be positive numbers and $\alpha = (\sum a)\left(\sum \frac{1}{a}\right)$. Prove that

$$\frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2} \leq \frac{4\alpha^3}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8.$$

Solution. Writing the left-hand side in the α notation we obtain

$$\frac{4\sum a^2}{\sum ab} + \frac{2\sum ab}{\sum a^2} = \frac{4(s^3 - 2p\alpha)}{p\alpha} + \frac{2p\alpha}{s^3 - 2p\alpha} = \frac{4t}{\alpha} - 8 + \frac{2\alpha}{t - 2\alpha}.$$

Let us consider the function $f : [t_3, t_4] \to \mathbb{R}$ defined as

$$f(t) = \frac{4t}{\alpha} - 8 + \frac{2\alpha}{t - 2\alpha}. \tag{2}$$

Clearly for $t \geq t_3$, we have $f'(t) = \dfrac{4}{\alpha} - \dfrac{2\alpha}{(t-2\alpha)^2} \geq \dfrac{4}{\alpha} - \dfrac{2\alpha}{(2\alpha-9)^2}$.

But $\dfrac{4}{\alpha} - \dfrac{2\alpha}{(2\alpha-9)^2} \geq 0$ if $\alpha \geq \dfrac{36+9\sqrt{2}}{7} \simeq 6,36$, so true as $\alpha \geq 9$.

Since $f'(t) \geq 0$, $\forall\, t \in [t_3, t_4]$, function f is increasing on $[t_3, t_4]$. Hence

$$f(t) \leq f(t_4) = \frac{4\alpha^2}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8.$$

Remark 2.1. Since

$$\frac{4\alpha^2}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8 - \alpha + 3 = \frac{-(\alpha-9)(11\alpha^2-66\alpha+108)}{(4\alpha-9)(\alpha^2-8\alpha+18)} \leq 0,\ \forall\, \alpha \geq 9,$$

we obtain

$$\frac{4\alpha^2}{4\alpha-9} + \frac{8\alpha-18}{\alpha^2-8\alpha+18} - 8 \leq \alpha - 3 = \sum \frac{b+c}{a}.$$

Application 2.1 represents a refinement of Problem 4497 (Crux Math. dec. 2019), author Hoong Le Nhat Tung, with the statement:

Consider a, b, c positive numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

Application 2.2. Let a, b, c be positive real numbers and define the expression $\alpha = (\sum a)\left(\sum \frac{1}{a}\right)$. Prove that

$$\sum \frac{1}{a^2} - 27\left(\sum \frac{ab}{c}\right)^{-2} \geq \frac{1}{3}\sum\left(\frac{1}{a}-\frac{1}{b}\right)^2 + \frac{4(a+b+c)(\alpha-9)(4\alpha-9)}{3abc(2\alpha-9)^2}.$$

Solution. Clearly, we have the identities

$$\sum \frac{1}{a^2} = \left(\sum \frac{1}{a}\right)^2 - 2\sum \frac{1}{bc} = \frac{\alpha^2}{s^2} - \frac{2s}{p}$$

$$\left(\frac{\sum ab}{c}\right)^{-2} = \left(\frac{\prod a}{\sum a^2b^2}\right)^{-2} = \frac{s^4}{(p\alpha^2 - 2s^3)^2}$$

$$\sum\left(\frac{1}{a}-\frac{1}{b}\right)^2 = \frac{2\alpha^2}{s^2} - \frac{6s}{p}.$$

In this notation, for $t \in [t_3, t_4]$ we obtain

$$\begin{aligned}
&\sum \frac{1}{a^2} - 27\left(\sum \frac{ab}{c}\right)^{-2} - \frac{1}{3}\sum\left(\frac{1}{a}-\frac{1}{b}\right)^2 \\
&= \frac{1}{3}\frac{\alpha^2}{s^2} - \frac{27s^4}{(p\alpha^2-2s^3)^2} \\
&= \frac{s}{p}\left(\frac{\alpha^2}{3t} - \frac{27t}{(\alpha^2-2t)^2}\right) = \frac{s}{p}f(t),
\end{aligned}$$

where the function $f : [t_3, t_4] \to \mathbb{R}$ is defined by

$$f(t) = \frac{\alpha^2}{3t} - \frac{27t}{(\alpha^2-2t)^2}. \tag{3}$$

If follows that $f'(t) = \frac{-\alpha^2}{3t^2} - \frac{27(\alpha^2+2t)}{(\alpha^2-2t)^3}$, $t \in [t_3, t_4]$.

Since $t \leq t_4 = \frac{\alpha^3}{4\alpha-9} \leq \frac{\alpha^2}{2}$, $\forall\, \alpha \geq 9$, we have $\alpha^2 - 2t > 0$.
Since $f'(t) < 0$, $\forall\, t \in [t_3, t_4]$, function f is decreasing on $[t_3, t_4]$ or

$$f(t) \geq f(t_4) = \frac{4(\alpha-9)(4\alpha-9)(a+b+c)}{3(2\alpha-9)^2abc}, \ \forall\, \alpha \geq 9,$$

which proves the inequality.

Remark 2.2. Application 2.2 is a refinement of Problem 2930, 3/2023 from Crux Mathematicorum, authored by José Díaz Barrero:

Suppose that a, b, c are positive real numbers. Prove that

$$\begin{aligned}
&\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} - 27\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^2 \geq \\
&\geq \frac{1}{3}\left[\left(\frac{1}{a}-\frac{1}{b}\right)^2 + \left(\frac{1}{b}-\frac{1}{c}\right)^2 + \left(\frac{1}{c}-\frac{1}{a}\right)^2\right]
\end{aligned}$$

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Application 2.3. Let a, b and c be positive real numbers such that $\sum a^2 = 1$ and consider $\alpha = (\sum a)\left(\sum \frac{1}{a}\right)$. Prove that

$$\sum a^2 \sum \frac{1}{a^2} \geq 3 + \frac{2\sum a^3}{abc} + \frac{(\alpha-9)\left(\alpha-5-\sqrt{(\alpha-1)(\alpha-9)}\right)}{4}.$$

Solution. One can write

$$\sum a^2 \sum \frac{1}{a^2} - 3 - \frac{2\sum a^3}{abc} =$$
$$= \frac{s^3-2\alpha p}{s} \cdot \frac{\alpha^2-2s^3}{ps^2} - 3 - \frac{2(s^3-3\alpha p+3p)}{p}.$$

Consider the function $f : [t_1, t_2] \to \mathbb{R}$ defined as

$$f(t) = \frac{(t-2\alpha)(\alpha^2-2t)}{t} - 3 - 2(t-3\alpha+3),$$

for which $f'(t) = \dfrac{2(\alpha^3-2t^2)}{t}$ and $t_0 = \sqrt{\dfrac{\alpha^3}{2}}$ is a critical point.
We have $t_2 \leq t_0$. We distinguish two cases, according to Wolphram Alpha.

Case 1. $9 \leq \alpha \leq \alpha_0 = 3\left(3+\sqrt{3}+\sqrt{9+6\sqrt{3}}\right) \simeq 27.40$.
In this case $t_0 \leq t_1 \leq t_2$, hence f is decreasing, so

$$f(t) \geq f(t_2) = \frac{(\alpha-9)(\alpha-5-\sqrt{(\alpha-1)(\alpha-9)}}{4}.$$

Case 2. $\alpha > \alpha_0$.
In this case $t_1 \leq t_0 \leq t_2$, hence $f(t) \geq \min\{f(t_1), f(t_2)\} = f(t_2)$ since

$$f(t_1) = \frac{(\alpha-9)(\alpha-5+\sqrt{(\alpha-1)(\alpha-5)}}{4}.$$

So in both cases $f(t) \geq \dfrac{(\alpha-9)(\alpha-5-\sqrt{(\alpha-1)(\alpha-5)}}{4} \geq 0$.

Remark 2.3. Application 2.3 is a refinement of Problem 2532 in Crux Mathematicorum by Ho-joo Lee with the statement:

Suppose that a, b, c are positive real numbers satisfying $a^2+b^2+c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3+b^3+c^3)}{abc}.$$

Application 2.4. Let a, b, c be the side lengths of a triangle with inradius r and circumradius R, semiperimeter s and $\alpha = s\sum \frac{1}{s-a}$. Prove that

$$\frac{12\alpha^3+28\alpha^2-75\alpha+27}{18\alpha^3+12\alpha^2-31\alpha+9} \leq \sum_{cyclic} \frac{a}{2a+b+c} \leq \frac{55\alpha-111}{75\alpha-163}.$$

Solution. Consider the substitution $a = y+z$, $b = z+x$, $c = x+y$.
We have $\dfrac{R}{r} = \dfrac{\prod(y+z)}{4xyz}$, $\alpha = s\sum \dfrac{1}{s-a} = \sum x \cdot \sum \dfrac{1}{x}$.

$$\sum_{cyclic} \frac{a}{2a+b+c} = \sum \frac{y+z}{2x+3y+3z}$$
$$= \frac{\sum\limits_{cyclic} (y+z)(2y+3z+3x)(2z+3x+3y)}{\prod(2x+3y+3z)}$$
$$= \frac{12\sigma_1^3 + 7\sigma_2\sigma_1 - 3\sigma_3}{18\sigma_1^3 + 3\sigma_1\sigma_2 - \sigma_3} = \frac{12s^3 + 7p\alpha - 3p}{18s^3 + 3p\alpha - p} = \frac{12t + 7\alpha - 3}{18t + 3\alpha - 1}.$$

Considering the function $f : [t_3, t_4] \to \mathbb{R}$ defined as
$$f(t) = \frac{12t + 7\alpha - 3}{18t + 3\alpha - 1},$$
we get $f'(t) = \dfrac{-6(15\alpha - 7)}{(3\alpha + 18t - 1)^2} < 0$, so f is decreasing for $t \in [t_3, t_4]$ or $f(t_4) \le f(t) \le f(t_3)$, which can be written as
$$\frac{12\alpha^3 + 28\alpha^2 - 75\alpha + 27}{18\alpha^3 + 12\alpha^2 - 31\alpha + 9} \le f(t) \le \frac{55\alpha - 111}{75\alpha - 163}.$$
The desired inequality is obtained.

Remark 2.4. Since $\forall\, \alpha \ge 9$ and $\dfrac{R}{r} = \dfrac{\prod(y+z)}{4xyz} = \dfrac{p\alpha - p}{4p} = \dfrac{\alpha - 1}{4}$, one has the inequalities
$$\frac{6}{\alpha - 1} \le \frac{12\alpha^3 + 28\alpha^2 - 75\alpha + 27}{18\alpha^3 + 12\alpha^2 - 31\alpha + 9}, \quad \frac{55\alpha - 111}{75\alpha - 163} \le \frac{3\alpha - 3}{32},$$
hence we obtain
$$\frac{3}{2}\,\frac{r}{R} \le \frac{12\alpha^3 + 28\alpha^2 - 75\alpha + 27}{18\alpha^3 + 12\alpha^2 - 31\alpha + 9} \le \sum_{cyclic} \frac{a}{2a+b+c} \le \frac{55\alpha - 111}{75\alpha - 163} \le \frac{3}{8}\,\frac{R}{r}.$$
Hence, Application 2.4 is a refinement of Problem 4508 from Crux Mathematicorum, nov. 2020, by George Apostolopoulos, with the statement:
Let a, b, c be the side lengths of a triangle with inradius r and circumradius R. Prove that
$$\frac{3}{2}\,\frac{r}{R} \le \sum_{cyclic} \frac{a}{2a+b+c} \le \frac{3}{8}\,\frac{R}{r}.$$

3. Problems

The following problems can be solved by the proposed method.

Problem 3.1. Let a, b, c be positive real numbers and $\alpha = \left(\sum a\right)\left(\sum \dfrac{1}{a}\right)$.
Prove that
$$\frac{\left(\sum a\right)^2}{\sum a^2} + \frac{1}{2}\left(\frac{\sum a^3}{abc} - \frac{\sum a^2}{\sum ab}\right) \ge 4 + \frac{(\alpha - 9)(2\alpha^2 - 15\alpha + 9)}{2\alpha(2\alpha - 9)} \ge 4$$

(Refinement of Klamkin's inequality, Problem 03(2005) Crux Mathematicorum, author Pham Von Thuan).

Problem 3.2. Let x, y, z be positive real numbers satisfiyng $\sum x^2 = 1$ and $\alpha = (\sum x)\left(\sum \frac{1}{x}\right)$. Prove that

$$a) \quad \sum \frac{1}{x} - \sum x \geq 2\sqrt{3} +$$

$$+\frac{xyz}{(x+y+z)^2}\left(2\alpha^2 - 13\alpha + 9 - 2\sqrt{3}\sqrt{8\alpha^2 - 54\alpha + 81}\right) \geq 2\sqrt{3};$$

$$b) \quad \sum \frac{1}{x} + \sum x \geq$$

$$4\sqrt{3} + \frac{1}{xyz(x+y+z)^2}\left(2\alpha^2 - 5\alpha - 9 - 4\sqrt{24\alpha^2 - 162\alpha + 243}\right) \geq 4\sqrt{3}.$$

(Refinement of Problem 2946, Crux Mathematicorum no 4(2004), author Panos E. Tsaoussoghou).

Problem 3.3. Let x, y and z be positive numbers and $\alpha = (\sum x)\left(\sum \frac{1}{x}\right)$. Prove that

$$\sqrt[3]{\frac{x^3+y^3+z^3}{xyz}} + \sqrt{\frac{xy+yz+zx}{x^2+y^2+z^2}} \geq \sqrt[3]{\alpha - 6} + \sqrt{\frac{4\alpha - 9}{\alpha^2 - 8\alpha + 18}} \geq \sqrt[3]{3} + 1$$

(Refinement of problem 3467, Crux Mathematicorum oct. 2009, author Tuon Le).

Problem 3.4. Let h_a, h_b, h_c the altitudes, r the inradius of triangle and s the semiperimeter of a triangle, and define $\alpha = s\sum \frac{1}{s-a}$. Prove that

$$\frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} \geq \frac{3}{5} + \frac{6(\alpha - 9)}{90\alpha - 185} \geq \frac{3}{5}$$

(Refinement of Problem 3993, Crux Mathematicorum dec.(2014), author Dragoljub Miloševic).

Problem 3.5. Suppose x, y and z are real numbers and $\alpha = (\sum x)\left(\sum \frac{1}{x}\right)$. Prove that

$$\begin{aligned}(x^3+y^3+z^3)^2 + 3(xyz)^2 &\geq 4(y^3z^3 + z^3x^3 + x^3z^3) + x^2y^2z^2 f(\alpha) \geq \\ &\geq 4(x^3y^3 + y^3z^3 + z^3x^3),\end{aligned}$$

where

$$f(\alpha) = \frac{1}{32}(\alpha - 9)\left(\alpha^3 - 11\alpha^2 + 35\alpha - 57 - (\alpha^2 - 6\alpha + 13)\sqrt{(\alpha - 1)(\alpha - 9)}\right).$$

(Refinement of Problem 2839, Crux Mathematicorum sept.(2003), author Murray Klamkin).

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