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A generalization of Minkowski's Inequality

Introduction

The purpose of this paper is to prove the inequality :

$$\left(\sum_{i=1}^{n} (a_i + b_i)^q\right)^{\frac{1}{p}} \leq 2^{\frac{q}{p}-1}\left[\left(\sum_{i=1}^{n} a_i^q\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{p}}\right]$$

where a_i and b_i are positive real numbers, and p, q are real numbers such that

$$p \geq q > 1.$$

Equality obtains if and only if

$$a_i = b_i.$$

The motivation for the paper is found in [1].

The proof will be a straightforward application of the standard method of finding the maximum value of a function with n variables. To this end, define

$$f(x_1, x_2, ..., x_n) = \left(\sum_{i=1}^{n} (x_i + b_i)^q\right)^{\frac{1}{p}} - 2^{\frac{q}{p}-1}\left[\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{p}}\right],$$

where we consider the b_i constant.

Setting the n first partial derivatives to zero, we find the critical point:

$$f_{x_j} = \frac{1}{p}\left(\sum_{i=1}^{n} (x_i + b_i)^q\right)^{\frac{1}{p}-1} q(x_j + b_j)^{q-1} - 2^{\frac{q}{p}-1}\frac{1}{p}\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{p}-1} q\, x_j^{q-1}$$

Setting all these to zero, and dividing the resulting equations by each other, we obtain

$$\frac{x_j + b_j}{x_k + b_k} = \frac{x_j}{x_k}.$$

Rearranging these, we obtain

$$\frac{x_j}{b_j} = \frac{x_k}{b_k}$$

for all choices of k and j. Let us denote the common ratio of all these fractions by λ. We aim to show $\lambda = 1$. To achieve this, we substitute

$$x_i = \lambda\ b_i$$

in the equation for the first partial derivative, and set it to zero. Carrying out the operations and simplifying, we obtain

$$(1+\lambda)^{\frac{q}{p}-1} = 2^{\frac{q}{p}-1}$$

or $\lambda = 1$, which implies that

$$x_i = \ b_i$$

for all indices i, so the critical point

$$(\ b_1,\ b_2, ...,\ b_n)$$

is unique. We readily verify that

$$f(\ b_1,\ b_2, ...,\ b_n) = 0.$$

We will now show that this is a maximum point by showing that all eigenvalues of the hessian at the critical point are strictly negative. We have to compute the second partial derivatives, first the mixed partials:

$$f_{x_k x_j} = \frac{q^2}{p}\left(\frac{1}{p}-1\right)\left[\left(\sum_{i=1}^{n}(\ x_i+\ b_i)^q\right)^{\frac{1}{p}-2}(\ x_j+\ b_j)^{q-1}(\ x_k+\ b_k)^{q-1}-\right.$$

$$\left.-2^{\frac{q}{p}-1}\left(\sum_{i=1}^{n}\ x_i^q\right)^{\frac{1}{p}-2}\ x_k^{q-1}\ x_j^{q-1}\right]$$

The non-mixed partials:

$$f_{x_j x_j} =$$

$$= \frac{q}{p}\left[\left(\frac{1}{p}-1\right)\left(\sum_{i=1}^{n}(\ x_i+\ b_i)^q\right)^{\frac{1}{p}-2} q(\ x_j+\ b_j)^{2(q-1)}+\right.$$

$$+\left(\sum_{i=1}^{n}(\ x_i+\ b_i)^q\right)^{\frac{1}{p}-1}(q-1)(\ x_j+\ b_j)^{q-2}-$$

$$-2^{\frac{q}{p}-1}\left(\frac{1}{p}-1\right)\left(\sum_{i=1}^{n}\ x_i^q\right)^{\frac{1}{p}-2}q\ x_j^{2(q-1)}-$$

$$-2^{\frac{q}{p}-1}(q-1)\left(\sum_{i=1}^{n}\ x_i^q\right)^{\frac{1}{p}-1}x_j^{q-2}\Bigg]$$

We will evaluate these partials at the critical point:

$$f_{x_k x_j}(\ b_1,\ b_2,...,\ b_n)=$$

$$=-\frac{q^2(1-p)}{p^2}\Bigg[2^{\frac{q}{p}-2q}\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-2}2^{q-1}\ b_j^{q-1}2^{q-1}\ b_k^{q-1}-$$

$$-2^{\frac{q}{p}-1}\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-2}b_k^{q-1}\ b_j^{q-1}\Bigg]=$$

$$=\frac{q^2(p-1)}{p^2}2^{\frac{q}{p}-2}\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-2}(\ b_j\ b_k)^{q-1}$$

Now the values of the non mixed partials at the critical point:

$$f_{x_j x_j}(\ b_1,\ b_2,...,\ b_n)=$$

$$=\frac{q}{p}\Bigg[\left(\frac{1}{p}-1\right)2^{\frac{q}{p}-2q}q\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-2}2^{2(q-1)}\ b_j^{2(q-1)}+$$

$$+2^{\frac{q}{p}-q}\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-1}(q-1)2^{q-2}\ b_j^{q-2}-$$

$$-2^{\frac{q}{p}-1}\left(\frac{1}{p}-1\right)\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-2}q\ b_j^{2(q-1)}-$$

$$-2^{\frac{q}{p}-1}(q-1)\left(\sum_{i=1}^{n}\ b_i^q\right)^{\frac{1}{p}-1}b_j^{q-2}\Bigg]=$$

$$= \frac{q}{p} 2^{\frac{q}{p}-2} \left(\sum_{i=1}^{n} \; b_i^q \right)^{\frac{1}{p}-2} \; b_j^{q-2} \left[\frac{q(p-1)}{p} \; b_j^q + (1-q) \left(\sum_{i=1}^{n} \; b_i^q \right) \right]$$

We are now ready to compute the eigenvalues of the hessian. To do this, we refer to Faddeev-Sominsky, problem 227 of page 40 of [2]. The entries of that matrix on the main diagonal are called $x_1, \; x_2, ..., \; x_n$, and the ij-th entry is denoted by $A_j \; B_i$. The determinant of this matrix is

$$\prod_{i=1}^{n} (\; x_i - \; A_i \; B_i) \left(1 + \sum_{i=1}^{n} \frac{A_i \; B_i}{x_i - \; A_i \; B_i} \right)$$

Let us denote

$$C = \frac{q^2(p-1)}{p^2} 2^{\frac{q}{p}-2} \left(\sum_{i=1}^{n} \; b_i^q \right)^{\frac{1}{p}-2} \; b_j^{q-2}$$

In the determinant formula above, we will substitute

$$A_i = \; b_i^{q-1}$$

and

$$B_i = C \; b_i^{q-1}$$

This substitution matches the entries of our hessian. From here we easily obtain the eigenvalues as

$$-\frac{q}{p} 2^{\frac{q}{p}-2} \left(\sum_{i=1}^{n} \; b_i^q \right)^{\frac{1}{p}-1} \; (q-1) \; b_i^{q-2}.$$

Under our conditions, these are clearly all strictly negative, concluding our proof. If p and q are equal, Minkowski's inequality obtains.

References :

[1] Karen Adams: A Minkowski-like inequality over R^n, Journal of Mathematical Sciences and Mathematics Education Vol. 12 No. 2

[2] D. Faddeev, I.Sominsky: Problems in Higher Algerba, MIR Publishers,

Moscow, 1968

[3] D.S. Mitrinovic: Analytic Inequalities, Grundlehren der mathematischen Wissenschaften, 1970