SOME RELATIONS BETWEEN THE TANGENT LENGTHS OF A BICENTRIC QUADRILATERAL

MARIUS DRĂGAN AND MIHÁLY BENCZE

ABSTRACT. The purpose of this article is to give some new identities and inequalities involving the sides a, b, c, d, the inradius r and circumradius R, the semiperimeter s and the tangent lengths t_1, t_2, t_3, t_4 of a bicentric quadrilateral.

In the following we will call the distances from the vertices to the points of the tangency at the sides with $\mathcal{C}(I, r)$ the tangent lengths.

Let M, N, P, Q the points where AB, BC, CD, DA cut the $\mathcal{C}(I, r)$. We denote the tangent lengths with $AM = t_1$, $BN = t_2$, $CP = t_3$, $DQ = t_4$.

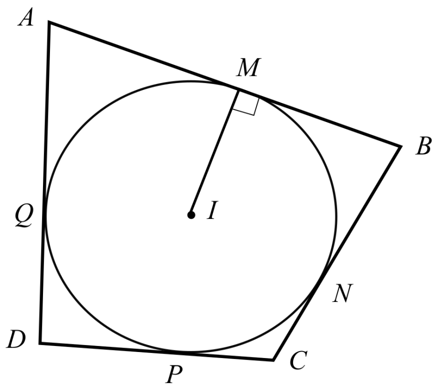

Figure 1

Lemma 1. *In every bicentric quadrilateral the following equalities are true:*

i) $t_1 + t_2 + t_3 + t_4 = s$;

ii) $t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2r\left(\sqrt{4R^2 + r^2} - r\right)$;

iii) $\sum\limits_{1\le i<j\le 4} t_it_j = 2r\sqrt{4R^2 + r^2}$;

iv) $\sum\limits_{1\le i<j<k\le 4} t_it_jt_k = r^2 s$;

v) $t_1t_2t_3t_4 = r^4$.

Proof. We have

i) $t_1 + t_2 = a$, $t_2 + t_3 = b$, $t_3 + t_4 = c$, $t_4 + t_1 = d$.

Suming up these equalities we obtain the statement.

ii) Using $\tan \dfrac{A}{2} = \dfrac{MI}{AM} = \sqrt{\dfrac{ad}{bc}}$ we obtain $\dfrac{r}{t_1} = \sqrt{\dfrac{ad}{bc}}$ or

$$t_1 = r\sqrt{\frac{bc}{ad}}\,,\ t_2 = r\sqrt{\frac{cd}{ba}}\,,\ t_3 = r\sqrt{\frac{ad}{bc}}\,,\ t_4 = r\sqrt{\frac{ab}{cd}}\,.$$

2010 *Mathematics Subject Classification.* ????
Key words and phrases. ???

So

$$t_1t_2+t_2t_3+t_3t_4+t_4t_1=r^2\left(\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}\right)$$
$$=r^2\left(\frac{(a+c)^2}{ac}-2+\frac{(b+d)^2}{bd}-2\right)=r^2\left(s^2\frac{ac+bd}{abcd}-4\right)$$
$$=r^2\left(\frac{ac+bd}{r^2}-4\right)=r^2\left(\frac{ac+bd-4r^2}{r^2}\right)$$
$$=ac+bd-4r^2=2r\sqrt{4R^2+r^2}-2r^2$$

iii) We have $t_1t_3=t_2t_4=r^2$. So from *ii*) we have

$$\sum_{1\le i<j\le 4}t_it_j=2r\sqrt{4R^2+r^2}-2r^2+2r^2=2r\sqrt{4R^2+r^2}$$

iv)
$$\sum_{1\le i<j\le 4}t_it_jt_k=r^2\left(\sqrt{\frac{ab}{cd}}+\sqrt{\frac{cb}{da}}+\sqrt{\frac{dc}{ab}}+\sqrt{\frac{ad}{bc}}\right)$$
$$=r^3\left(\frac{ab+dc}{\sqrt{abcd}}+\frac{bc+ad}{\sqrt{abcd}}\right)=r^3\frac{ab+dc+bc+ad}{rs}=r^2s$$

v) $t_1t_3=t_2t_4=r^2$ from which $t_1t_2t_3t_4=r^4$. □

Theorem 1. *Let $ABCD$ a bicentric quadrilateral. Then t_1,t_2,t_3,t_4 are the roots of a four degree equation with the coefficient depending only on a,b,c,d.*

Proof. t_1,t_2,t_3,t_4 are the roots of the equation $t^4-\sigma_1t^3+\sigma_2t^2-\sigma_3t+\sigma_4=0$, where according to the lemma

$$\sigma_1=s,\ \sigma_2=2r\sqrt{4R^2+r^2}.$$

But $ac+bd=2r\sqrt{4R^2+r^2}+2r^2$ or

$$\sigma_2=ac+bd-2r^2-ac+bd-\frac{2abcd}{s^2}$$
$$\sigma_3=r^2s=\frac{abcd}{s}$$
$$\sigma_4=4r^4=\frac{4a^2b^2c^2d^2}{s^4}.$$

So $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ depend only on a,b,c,d. We obtain the equation

$$t^4-\frac{\sum a}{2}t^3+\left(ac+bd-\frac{8abcd}{\left(\sum a\right)^2}\right)t^2-\frac{2abcd}{\sum a}t+\frac{16a^2b^2c^2d^2}{\left(\sum a\right)^4}=0$$

with the roots t_1,t_2,t_3,t_4. □

Theorem 2. *Let $ABCD$ a bicentric quadrilateral. Then t_1,t_2,t_3,t_4 are the roots of a four degree equation with the coefficient depending only on R,r and s.*

Proof. From lemma we have

$$t^4-st^3+2r\sqrt{4R^2+r^2t^2}-r^2st+r^4=0$$

the equation having the roots t_1,t_2,t_3,t_4. □

Corollary 1. *Let $ABCD$ a bicentric quadrilateral. Then t_1, t_3 are the roots of the equation $st^2 - x_1 t + sr^2 = 0$ and t_2, t_4 are the roots of the equation $st^2 - x_2 t + sr^2 = 0$.*

Proof. We have $\dfrac{t_1^2}{r^2} = \dfrac{bc}{ad}$ or $\dfrac{t_1^2 + r^2}{r^2} = \dfrac{x_1}{ad}$. Also $\dfrac{t_1^2 + r^2}{t_1^2} = \dfrac{x_1}{bc}$.
Multiplying these two equalities, we obtain

$$\left(\frac{t_1^2 + r^2}{rt_1}\right)^2 = \left(\frac{x_1}{sr}\right)^2 \quad \text{or} \quad (t_1^2 + r^2)s = x_1 t, \quad \text{or}$$

$$\begin{aligned} st_1^2 - x_1 t_1 + r^2 s &= 0 \\ st_2^2 - x_2 t_1 + r^2 s &= 0 \\ st_3^2 - x_1 t_1 + r^2 s &= 0 \\ st_4^2 - x_2 t_1 + r^2 s &= 0, \end{aligned}$$

from which it follows the statement. □

Corollary 2. *In every bicentric quadrilateral the following equality is true:*

$$\begin{aligned} & t^4 - st^3 + 2r\sqrt{4R^2 + r^2}\, t^2 - r^2 st + r^4 \\ & = \left(t^2 - \frac{x_1}{s}\, t + r^2\right)\left(t^2 - \frac{x_2}{s}\, t + r^2\right). \end{aligned}$$

Proof. We have

$$\begin{aligned} & \left(t^2 - \frac{x_1}{s}\, t + r^2\right)\left(t^2 - \frac{x_2}{s}\, t + r^2\right) \\ & = t^4 - \left(\frac{x_1 + x_2}{s}\right) t^3 + \left(2r^2 + \frac{x_1 x_2}{s^2}\right) t^2 - \frac{r^2}{s}(x_1 + x_2) t + r^4 \\ & = t^4 - st^3 + \left(2r^2 + \frac{16R^2 r^2}{2r\left(\sqrt{4R^2 + r^2} + r\right)}\right) t^2 - r^2 st + r^4 \\ & = t^4 - st^3 + 2r\sqrt{4R^2 + r^2}\, t^2 - r^2 st + r^4. \end{aligned}$$

In the following we will write t_1, t_2, t_3, t_4 according only to x_1, x_2, r. □

Corollary 3. *In every bicentric quadrilateral we have*

$$\{t_1, t_3\} = \frac{x_1 \pm \sqrt{x_1^2 - 4r^2 s^2}}{2s} \quad \textit{and} \quad \left\{\frac{x_2 \pm \sqrt{x_2^2 - 4r^2 s^2}}{2s}\right\}$$

since $s = \sqrt{x_1 + x_2}$.

Proof. It follows from Corollary 1. □

Remark 1. Since $x_1 = ab + cd$, $x_2 = ac + bd$, it follows that t_1, t_2, t_3, t_4 may be written according only to a, b, c, d.

Corollary 4. *In every bicentric quadrilateral we have*

$$\begin{aligned} & (a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 \\ & = (t_1 - t_3)^4 (t_2 - t_4)^4 \left[(t_1 - t_3)^2 - (t_2 - t_4)^2\right]. \end{aligned}$$

Proof. Since $a = t_1 + t_2$, $b = t_2 + t_3$, $c = t_3 + t_4$, $d = t_4 + t_1$, then

$$(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$$
$$= (t_1-t_3)^2(t_1+t_2-t_3-t_4)(t_2-t_4)^2(t_2-t_4)^2(t_2+t_3-t_1-t_4)^2(t_3-t_1)^2$$
$$= (t_1-t_3)^4(t_2-t_4)^4\left[(t_1-t_3)^2-(t_2-t_4)^2\right]^2 .$$

□

In the following we will write t_1, t_2, t_3, t_4 using only a, b, c, d.

Corollary 5. *In every bicentric quadrilateral the following equality is true:*

$$t_1 = \frac{bc}{s}\,,\ t_2 = \frac{cd}{s}\,,\ t_3 = \frac{da}{s}\,,\ t_4 = \frac{ab}{s}\,.$$

Proof. We have

$$t_1 = r\sqrt{\frac{bc}{ad}} = \frac{F}{s}\sqrt{\frac{bc}{ad}} = \frac{\sqrt{abcd}}{s}\sqrt{\frac{bc}{ad}} = \frac{bc}{s}\,.$$

□

Corollary 6. *In every bicentric quadrilateral the following equality is true:*

$$(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$$
$$= 16s^2r^4\left[\left(\sqrt{4R^2+r^2}\right)^2 - s^2\right]\left[s^2 - 8r\left(\sqrt{4R^2+r^2}-r\right)\right].$$

Proof. We compute

$$s^8(t_1-t_3)^4(t_2-t_3)^4 = (ad-bc)^4(ab-dc)^4$$
$$= \left[(ad+bc)^2 - 4abcd\right]^2\left[(ab+dc)^2-4abcd\right]^2$$
$$= \left[(x_1^2-4F)^2(x_2^2-4F)^2\right]^2 = \left[(x_1x_2)^2 - 4F^2(x_1^2+x_2^2)+16F^4\right]^2$$
$$= \left[4r^2\left(\sqrt{4R^2+r^2}-r\right)^2s^4 - 4s^6r^2 + 8s^2r^2\cdot 2r\left(\sqrt{4R^2+r^2}-r\right)s^2 + 16s^4r^4\right]^2$$
$$= 4^2s^8r^4\left[\left(\sqrt{4R^2+r^2}-r\right)^2 - s^2 + 4r\left(\sqrt{4R^2+r^2}-r\right)+4r^2\right]$$
$$= 4^2r^4s^8\left[\left(\sqrt{4R^2+r^2}+r\right)^2 - s^2\right]^2 .$$

Also

$$s^4\left[(t_1-t_3)^2-(t_2-t_4)^2\right]^2 = \left[(ad-bc)^2-(ab-cd)^2\right]^2$$
$$= \left[(ad+bc)^2 - 4abcd - (ab+cd)^2 + 4abcd\right]^2 = (x_1^2-x_2^2)^2$$
$$= (x_1^2+x_2^2)^2 - 4x_1^2x_2^2 = \left[(x_1+x_2)^2 - 2x_1x_2\right]^2 - 4x_1^2x_2^2$$
$$= (s^4-2x_1x_2)^2 - 4x_1^2x_2^2 = s^8 - 4s^4x_1x_2$$
$$= s^4\left[s^4 - 8r\left(\sqrt{4R^2+r^2}-r\right)s^2\right]$$

(1) $$= s^6\left[s^2 - 8r\left(\sqrt{4R^2+r^2}-r\right)\right].$$

We obtain

$$(t_1-t_3)^4(t_2-t_4)^4 = 16r^4\left[\left(\sqrt{4R^2+r^2}+r\right)^2 - s^2\right]^2$$

and

$$\left[(t_1-t_3)^2-(t_2-t_4)^2\right]=s^2\left[s^2-8r\left(\sqrt{4R^2+r^2}-r\right)\right]$$

From (1) and Corollary 4 we obtain the statement. □

Let $M=AB\cap\mathcal{C}(I,r)$. We have:

Lemma 2. *Let $ABCD$ a bicentric quadrilateral. Then* $MO=\sqrt{R^2-\dfrac{bc^2d}{s^2}}$.

Proof. We have $R^2-MO^2=t_1t_2$ or $MO=\sqrt{R^2-\dfrac{bc^2d}{s^2}}$. □

Corollary 7. *In every bicentric quadrilateral the following equality is true:*

$$MO^2+NO^2+PO^2+QO^2=\left(\sqrt{4R^2+r^2}-r\right)^2.$$

Proof. From Corollary 6 we have

$$MO^2+NO^2+PO^2+QO^2=4R^2-\left(\frac{a^2bd}{s^2}+\frac{ab^2c}{s^2}+\frac{bc^2d}{s^2}+\frac{ad^2c}{s^2}\right)$$

$$=4R^2-\frac{ac(b^2+d^2)+bd(a^2+c^2)}{s^2}=4R^2-\frac{ac(s^2-2bd)+bd(s^2-2ac)}{s^2}$$

$$=4R^2-\frac{s^2(ac+bd)-4abcd}{s^2}=4R^2-ac-bd+4r^2$$

$$=4R^2-2r\sqrt{4R^2+r^2}-2r^2+4r^2=\left(\sqrt{4R^2+r^2}-r\right)^2.$$

□

Theorem 3 (Fuss)**.** *In every bicentric quadrilateral the following equality is true:* $\overline{d}^2=R^2+r^2-r\sqrt{4R^2+r^2}$.

Proof. From sine theorem in triangle MOA, we have

$$\frac{R}{\sin\alpha}=\frac{a}{\sin(\pi-2\alpha)}\quad\text{or}\quad\cos\alpha=\frac{a}{2R}.$$

From sine theorem in triangle MOB we have $\dfrac{BO}{\sin\beta}=\dfrac{MO}{\sin\alpha}$ or

$$\sin\beta=\frac{BO}{MO}\sin\alpha=\frac{R}{MO}\sqrt{1-\frac{a^2}{4R^2}}=\frac{1}{2MO}\sqrt{4R^2-a^2}.$$

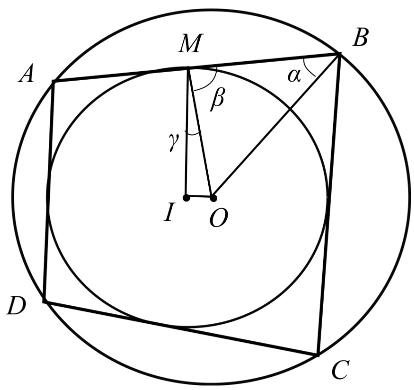

Figure 2

We have

$$\cos\gamma = \cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta = \frac{1}{2MO}\sqrt{4R^2 - a^2}.$$

From cosine theorem in MIO triangle we have

$$\overline{d}^2 = MI^2 + MO^2 - 2MI\,MO\cos\gamma = r^2 + R^2 - \frac{bc^2d}{s^2} - r\sqrt{4R^2 - a^2}$$

In the same way

$$\overline{d} = R^2 + r^2 - \frac{ad^2c}{s^2} - r\sqrt{4R^2 - b^2}$$
$$\overline{d} = R^2 + r^2 - \frac{ba^2d}{s^2} - r\sqrt{4R^2 - c^2} \quad \text{and}$$
$$\overline{d}^2 = R^2 + r^2 - \frac{cd^2a}{s^2} - r\sqrt{4R^2 - d^2}$$

By summing up we get

$$4\overline{d}^2 = 4R^2 + 4r^2 - \frac{1}{s^2}\left(a^2bd + bc^2d + ad^2c + ab^2c\right)$$
$$(2) \qquad -r\left(\sqrt{4R^2 - a^2} + \sqrt{4R^2 - b^2} + \sqrt{4R^2 - c^2} + \sqrt{4R^2 - d^2}\right)$$

We have

$$\frac{1}{s^2}\left(a^2bd + bc^2d + ad^2c + ab^2d\right) = \frac{1}{s^2}\left[(a^2 + c^2)bd + (b^2 + d^2)ac\right]$$
$$(3) \qquad = \frac{1}{s^2}\left(s^2(ac + bd) - 4s^2r^2\right) = ac + bd - 4r^2$$

□

To prove (2) we will prove the following lemma:

Lemma 3. *In every bicentric quadrilateral the following equality is true:*

$$\sqrt{4R^2 - a^2} + \sqrt{4R^2 - b^2} + \sqrt{4R^2 - c^2} + \sqrt{4R^2 - d^2} = 2\sqrt{4R^2 + r^2} + 2r$$

Proof.

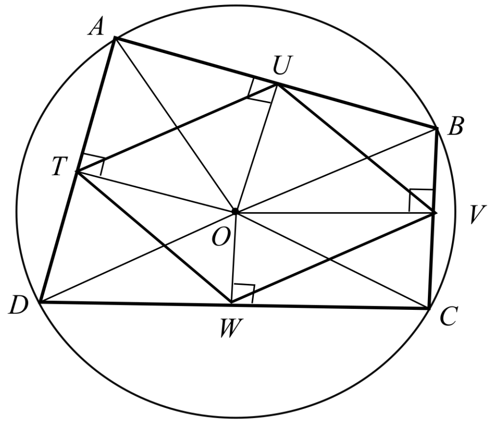

Figure 3

Let $OU \perp AB$, $OV \perp BC$, $OW \perp DC$, $OT \perp AD$.
We denote $OU = x$, $OV = y$, $OW = z$, $OT = t$, $BD = d_1$, $AC = d_2$, $UT = VW = \dfrac{d_1}{2}$, $UV = TW = \dfrac{d_2}{2}$, $OA = OB = OC = OD = R$.
From Ptolemy theorem in cyclic quadrilateral $AUOT$, $BUOV$, $CVOW$, $OTDW$, we have $\dfrac{xd}{2} + \dfrac{at}{2} = \dfrac{d_1 R}{2}$, $\dfrac{xb}{2} + \dfrac{ya}{2} = \dfrac{d_2 R}{2}$, $\dfrac{zb}{2} + \dfrac{yc}{2} = \dfrac{d_1 R}{2}$, $\dfrac{zd}{2} + \dfrac{ct}{2} = \dfrac{d_2 R}{2}$. Adding these equalities we obtain

$$x(b+d) + y(a+c) + z(b+d) + t(a+c) = 2R(d_1 + d_2)$$

or $x + y + z + t = \dfrac{2R(d_1+d_2)}{s}$.

We denote $\alpha = d_1 + d_2$. But from Ptolemy theorem we have $\dfrac{d_1}{d_2} = \dfrac{x_2}{x_1}$ and $d_1 d_2 = x_3 = 2r\left(\sqrt{4R^2+r^2} + r\right)$. So $\dfrac{d_1}{\alpha} = \dfrac{x_2}{s^2}$ and $\dfrac{d_2}{\alpha} = \dfrac{x_1}{s^2}$.
By multiplying we obtain $\dfrac{d_1 d_2}{\alpha^2} = \dfrac{x_1 x_2}{s^4}$. But $x_1 x_2 = \dfrac{16R^2 r^2 s^2}{x_3}$.
We obtain $\dfrac{x_3}{\alpha^2} = \dfrac{16R^2 r^2}{x_3 s^2}$ or $\alpha = \dfrac{x_3 s}{4Rr}$.
So we obtain $x + y + z + t = \sqrt{4R^2 + r^2} + r$.
But $x = \dfrac{1}{2}\sqrt{4R^2 - a^2}$. So $\dfrac{1}{2}\sum \sqrt{4R^2 - a^2} = \sqrt{4R^2 + r} + r$ or

$$\sum \sqrt{4R^2 - a^2} = 2\sqrt{4R^2 + r^2} + 2r \tag{4}$$

If we replace (3) and (4) in (2) we obtain

$$4\overline{d}^2 = 4R^2 + 4r^2 - 2r\sqrt{4R^2+r^2} - 2r^2 + 4r^2 - 2r\sqrt{4R^2+r^2} - 2r^2 \quad \text{or}$$

$$4\overline{d}^2 = 4R^2 + 4r^2 - 4r\sqrt{4R^2+r^2} \qquad \text{or} \qquad \overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2+r^2}\,.$$

□

Corollary 8. *Let M, N, P, Q the points where AB, BC, CD, DA cut $\mathcal{C}(I, r)$. Then*

$$MN = 2r\sqrt{\frac{t_2}{t_2+t_4}}\,, \quad NP = 2r\sqrt{\frac{t_3}{t_1+t_3}}\,,$$
$$PQ = 2r\sqrt{\frac{t_4}{t_2+t_4}}\,, \quad MQ = 2r\sqrt{\frac{t_1}{t_1+t_3}}\,.$$

Proof. We have $AI = \dfrac{r}{\sin\frac{A}{2}}$. Since $AMIQ$ is cyclic, from Ptolemy theorem we have $AM \cdot QI + MI \cdot AQ = MQ \cdot AI$ or $MQ \cdot AI = 2t_1 r$, or

$$MQ = \frac{2t_1 r}{r/\sin\frac{A}{2}} = 2t_1 \sin\frac{A}{2} = 2t_1\sqrt{\frac{ad}{ad+bc}}$$
$$= 2t_1\sqrt{\frac{t_3}{t_1+t_3}} = 2\sqrt{t_1 t_3}\sqrt{\frac{t_1}{t_1+t_3}} = 2r\sqrt{\frac{t_1}{t_1+t_3}}$$

□

Corollary 9. *In every bicentric quadrilateral, $MP \perp QN$ with the notation from above.*

Proof. From Corollary 8 we have $MQ^2 + NP^2 = MN^2 + QP^2$. So $MNPQ$ is orthodiagonal. □

Corollary 10. *In every bicentric quadrilateral we have*

$$MN \cdot NP \cdot PQ \cdot QM = \frac{2r^5\left(\sqrt{4R^2+r^2}+r\right)}{R^2}.$$

Proof. From Corollary 8 we have

$$MN \cdot NP \cdot PQ \cdot QN = \frac{16\sqrt{t_1t_2t_3t_4}\,r^4}{(t_1+t_3)(t_2+t_4)} = \frac{16r^5}{2r\left(\sqrt{4R^2+r^2}-r\right)} = \frac{2r^5}{R^2}\left(\sqrt{4R^2+r^2}+r\right).$$

□

Corollary 11. *In every bicentric quadrilateral the following inequality is true:*

$$4r\sqrt{1+\sqrt{\frac{2r}{\sqrt{4R^2+r^2}-r}}} \le MN + NP + PQ + QM \le$$
$$\le \frac{r\left(\sqrt{4R^2+r^2}+2\sqrt{2}R+r\right)}{2} \le 4\sqrt{2}\,r.$$

Proof. According to Corollary 8 we have $f : [s_1, s_2] \to \mathbb{R}$,

$$\begin{aligned}
f(s) &= MN + NP + PQ + QM \\
&= 2r\left(\sqrt{\frac{t_2}{t_2+t_4}} + \sqrt{\frac{t_4}{t_2+t_4}} + \sqrt{\frac{t_1}{t_1+t_3}} + \sqrt{\frac{t_3}{t_1+t_3}}\right) \\
&= 2r\left(\frac{\sqrt{ab}+\sqrt{cd}}{\sqrt{ab+cd}} + \frac{\sqrt{ad}+\sqrt{bc}}{\sqrt{ad+bc}}\right) = 2r\left(\sqrt{\frac{x_1+2sr}{x_1}} + \sqrt{\frac{x_2+2sr}{x_2}}\right) \\
&= 2r\sqrt{2 + \frac{2sr(x_1+x_2)}{x_1x_2} + 2\sqrt{\frac{x_1x_2 + 2sr(x_1+x_2) + 4s^2r^2}{x_1x_2}}} \\
&= 2r\sqrt{2 + \frac{2sr\,s^2\,x_3}{16R^2r^2s^2} + 2\sqrt{1 + \frac{2sr\,s^2\,x_3}{16R^2r^2s^2} + \frac{4s^2r^2x_3}{16R^2r^2s^2}}} \\
&= 2r\sqrt{2 + \frac{sx_3}{8R^2r} + 2\sqrt{1 + \frac{sx_3}{8R^2r} + \frac{x_3}{4R^2}}},
\end{aligned}$$

which is increasing in s. So

$$f(s_1) \le f(s) \le f(s_2)$$

or

$$M_1N_1 + N_1P_1 + P_1Q_1 + Q_1M_1 \le MN + NP + PQ + QN \le$$
$$\le M_2N_2 + N_2P_2 + P_2Q_2 + Q_2M_2,$$

where M_1, N_1, P_1, Q_1 represent the intersection of the incircle with the sides of $A_1B_1C_1D_1$ from Blundon theorem and M_2, N_2, P_2, Q_2 the intersection of the sides $A_2B_2C_2D_2$ with incircle $\mathcal{C}(I,r)$.
We have

$$M_1N_1 + N_1P_1 + P_1Q_1 + Q_1M_1 = 2r\left(\frac{\sqrt{a_1b_1}+\sqrt{c_1d_1}}{\sqrt{a_1b_1+c_1d_1}} + \frac{\sqrt{a_1d_1}+\sqrt{b_1c_1}}{\sqrt{a_1d_1+b_1c_1}}\right)$$

From Blundon-Eddy theorem we have

$$\begin{aligned} a_1 = c_1 &= \sqrt{R^2-(r-d)^2} + \sqrt{R^2-(r+d)^2} \\ b_1 &= 2\sqrt{R^2-(r-d)^2}, \quad d_1 = 2\sqrt{R^2-(r+d)^2} \end{aligned}$$

So

$$M_1N_1 + N_1P_1 + P_1Q_1 + Q_1M_1 = 2\frac{\sqrt{b_1}+\sqrt{d_1}}{\sqrt{b_1+d_1}} = 2\sqrt{\frac{b_1+d_1+2\sqrt{b_1d_1}}{b_1+d_1}}$$

$$= 2\sqrt{1+\frac{2\sqrt{4r^2}}{s_1}} = 2\sqrt{1+\frac{4r}{\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}}} = 2\sqrt{1+\sqrt{\frac{2r}{\sqrt{4R^2+r^2}}}}$$

Also

$$M_2N_2 + N_2P_2 + P_2Q_2 + Q_2M_2 = 2r\left(\frac{\sqrt{a_2b_2}+\sqrt{c_2d_2}}{\sqrt{a_2b_2+c_2d_2}} + \frac{\sqrt{a_2d_2}+\sqrt{b_2c_2}}{\sqrt{a_2d_2+b_2c_2}}\right).$$

From Blundon theorem we have

$$a_2 = b_2 = \frac{2R}{R+d}\sqrt{(R+d)^2-r^2}, \quad c_2 = d_2 = \frac{2R}{R-d}\sqrt{(R-d)^2-r^2}$$

So

$$\begin{aligned} &M_2N_2 + N_2P_2 + P_2Q_2 + Q_2M_2 \\ &= \frac{a_2+c_2}{\sqrt{a_2^2+c_2^2}} + \sqrt{2} = \frac{a_2+c_2}{\sqrt{(a_2+c_2)^2-2a_2c_2}} = \frac{s_2}{\sqrt{s_2^2-\frac{16R^2r^2}{R^2-d^2}}} \end{aligned}$$

We have

$$\begin{aligned} a_2c_2 &= \frac{4R^2}{R^2-d^2}\sqrt{[(R+d)^2-r^2][(R-d)^2-r^2]} \\ &= \frac{4R^2}{R^2-d^2}\sqrt{[(R-r)^2-d^2][(R+r)^2-d^2]} \\ &= \frac{4R^2}{R^2-d^2}\sqrt{\left(r\sqrt{4R^2+r^2}-2Rr\right)\left(r\sqrt{4R^2+r^2}+2Rr\right)} = \frac{4R^2r^2}{R^2-d^2} \end{aligned}$$

So

$$M_2N_2+N_2P_2+P_2Q_2+Q_2M_2 = \frac{\sqrt{4R^2+r^2}+r}{\sqrt{\left(\sqrt{4R^2+r^2}+r\right)^2-\frac{8R^2r^2}{R^2-d^2}}} + \sqrt{2}$$

We have

$$\left(\sqrt{4R^2+r^2}+r\right)^2-\frac{8^2r^2}{R^2-d^2}$$
$$=4R^2+2r^2+2r\sqrt{4R^2+r^2}-\frac{8R^2r^2}{r\left(\sqrt{4R^2+r^2}-r\right)}$$
$$=4R^2+2r^2+2r\sqrt{4R^2+r^2}-2r\left(\sqrt{4R^2+r^2}+r\right)=4R^2$$

We obtain

$$M_2N_2+N_2P_2+P_2Q_2+Q_2M_2=\frac{\left(\sqrt{4R^2+r^2}+r+2\sqrt{2}R\right)r}{R}.$$

□

Corollary 12. *In every bicentric quadrilateral the following inequality is true:*

$$MN+NP+PQ+QM\geq\frac{8r^2}{R}$$

Proof. From Corollary 11, if we denote $\frac{R}{r}=x\geq\sqrt{x}$, it will be sufficient to prove that

$$\sqrt{1+\sqrt{\frac{2}{\sqrt{4x^2+1}-1}}}\geq\frac{2}{x}\quad\text{or}\quad x^2+x^2\sqrt{\frac{2}{\sqrt{4x^2+1}-1}}\geq 4,$$

or

$$x^2\sqrt{\frac{2}{\sqrt{4x^2+1}-1}}\geq 4-x^2. \tag{5}$$

If $x\geq\sqrt{2}$ the inequality is true.
We will prove that inequality (5) is true for each $\sqrt{2}\leq x\leq 2$.

If we denote $x^2=y$, we will prove that $y\sqrt{\frac{2}{\sqrt{4y+1}-1}}\geq 4-y$, $\forall\, y\in[2,4]$
or $2y^2\geq(4-y)^2\sqrt{4y+1}-(4-y)^2$, $\forall\, y\in[2,4]$,
or $3y^2-8y+16)^2\geq(4-y)^4(4y+1)$, $\forall\, y\in[2,4]$,
or $(y-2)(y^3-16y^2+72y-128)\leq 0$, $\forall\, y\in[2,4]$,
or $y^3-16y^2+72y-128\leq 0$, $\forall\, y\in[2,4]$.
Since $y^3-16y^2+72y-128=0$ has only real roots $y_0\simeq 10,148$ and $f:\mathbb{R}\to\mathbb{R}$ is an increasing function, it follows that $f(y)\leq 0$, $\forall\, y\in[2,4]$. □

Corollary 13. *In every bicentric quadrilateral the following inequality is true:*

$$\frac{32r^4\left(\sqrt{4R^2+r^2}-2r\right)}{\sqrt{4R^2+r^2}-r}\geq MN^4+NP^4+PQ^4+QM^4\geq$$
$$\geq\frac{r^4\left(24R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}\right)}{R^4}$$

Proof. From Corollary 8 we have $f : [s_1, s_2] \to \mathbb{R}$

$$f(s) = MN^4 + NP^4 + PQ^4 + QM^4 = 16r^4\left[\frac{t_1^2+t_3^2}{(t_1+t_3)^2} + \frac{t_2^2+t_4^2}{(t_2+t_4)^2}\right]$$
$$= 16r^4\left[\frac{\frac{ad}{bc}+\frac{bc}{ad}}{\frac{ad}{bc}+\frac{bc}{ad}+2} + \frac{\frac{ab}{cd}+\frac{cd}{ab}}{\frac{ab}{cd}+\frac{cd}{ab}+2}\right]$$
$$= 16r^4\left[\frac{(ad+bc)^2-2abcd}{(ad+bc)^2} + \frac{(ab+cd)^2-2abcd}{(ab+dc)^2}\right]$$
$$= 16r^4\left(\frac{x_1^2-2r^2s^2}{x_1^2} + \frac{x_2^2-2r^2s^2}{x_2^2}\right) = 16r^4\left(2 - \frac{2r^2s^2(x_1^2+x_2^2)}{x_1^2x_2^2}\right)$$
$$= 16r^4\left[2 - \frac{2r^2s^2(s^4-2x_1x_2)}{x_1^2x_2^2}\right] = 16r^4\left[2 - \frac{2r^2s^2\left(s^4 - \frac{32R^2r^2s^2}{x_3}\right)}{\frac{256R^2r^2s^2}{x_3^2}}\right]$$
$$= 16r^4\left[2 - \frac{x_3(x_3s^4 - 32R^2r^2s^2)}{128R^2}\right].$$

From Blundon-Eddy inequality we have

$$x_3s^2 \geq x_3 \cdot 8r\left(\sqrt{4R^2+r^2} - r\right)$$
$$= 16r^2\left(\sqrt{4R^2+r^2}+r\right)\left(\sqrt{4R^2+r^2}-r\right) = 64R^2r^2 \geq 32R^2r^2,$$

or $x_3s^2 > 32R^2r^2$.
So, if we consider the function $g : [s_1, s_2] \to \mathbb{R}$,

$$g(s) = x_3s^4 - 32R^2r^2s^2 = s^2(x_3s^2 - 32R^2r^2)$$

g is an increasing function as a product of two positive increasing functions. It follows that $f : [s_1, s_2] \to \mathbb{R}$, $f(s) = 16r^4\left(2 - \dfrac{x_3g(s)}{128R^2}\right)$ is a decreasing function on $[s_1, s_2]$, or $f(s_2) \leq f(s) \leq f(s_1)$, $\forall\, s \in [s_1, s_2]$, or

$$M_1N_1^4 + N_1P_1^4 + P_1Q_1^4 + Q_1M_1^4 \geq MN^4 + NP^4 + PQ^4 + QM^4 \geq$$
$$\geq M_2N_2^4 + N_2P_2^4 + P_2Q_2^4 + Q_2M_2^4.$$

We have

$$M_1N_1^4 + N_1P_1^4 + P_1Q_1^4 + Q_1M_1^4$$
$$= 16r^4\left(\frac{a_1^2d_1^2+b_1^2c_1^2}{(a_1d_1+b_1c_1)^2} + \frac{a_1^2b_1^2+c_1^2d_1^2}{(a_1b_1+c_1d_1)^2}\right)$$

$$32r^4\frac{b_1^2+d_1^2}{(b_1+d_1)^2} \pm 32r^4\frac{(b_1+d_1)^2-2b_1d_1}{(b_1+d_1)^2} = 32r^4\,\frac{s_1^2-8r^2}{s_1^2}$$
$$= \frac{32r^4\left(8r\left(\sqrt{4R^2+r^2}-r\right)-8r^2\right)}{8r\left(\sqrt{4R^2+r^2}-r\right)} = \frac{32r^4\left(\sqrt{4R^2+r^2}-2r\right)}{\sqrt{4R^2+r^2}-r}.$$

Also

$$\begin{aligned}&M_1N_1^4+N_1P_1^4+P_1Q_1^4+Q_1M_1^4\\&=16r^4\left(\frac{a_2^2d_2^2+b_2^2c_2^2}{(a_2d_2+b_2c_2)^2}+\frac{a_2^2b_2^2+c_2^2d_2^2}{(a_2b_2+c_2d_2)^2}\right)\\&=16r^4\left(\frac{2a_2^2c_2^2}{4a_2^2c_2^2}+\frac{a_2^4+c_2^4}{(a_2^2+c_2^2)^2}\right)=16r^4\left(\frac{1}{2}+\frac{a_2^4+c_2^4}{(a_2^2+c_2^2)^2}\right).\end{aligned}$$

We have $a_2+c_2=s_2=\sqrt{4R^2+r^2}+r$. Also

$$a_2c_2=\frac{4R^2r^2}{R^2-d^2}=r\left(\sqrt{4R^2+r^2}+r\right).$$

We have

$$\begin{aligned}a_2^2+c_2^2&=(a_2+c_2)^2-2a_2c_2\\&=\left(\sqrt{4R^2+r^2}+r\right)^2-2r\left(\sqrt{4R^2+r^2}+r\right)\\&=\left(\sqrt{4R^2+r^2}+r\right)\left(\sqrt{4R^2+r^2}-r\right)=4R^2\end{aligned}$$

Also

$$\begin{aligned}a_2^4+c_2^4&=(a_2^2+c_2^2)^2-2a_2^2c_2^2=16R^4-2r^2\left(\sqrt{4R^2+r^2}+r\right)^2\\&=16R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}\end{aligned}$$

So

$$\begin{aligned}&M_2N_2^4+N_2P_2^4+P_2Q_2^4+Q_2M_2^4\\&=16r^4\left(\frac{1}{2}+\frac{16R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}}{16R^4}\right)\\&=r^4\frac{24R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}}{R^4},\end{aligned}$$

from which we obtain the statement. □

Corollary 14. *In every bicentric quadrilateral the following inequality is true:*

$$16r^4\leq MN^4+NP^4+PQ^4+QM^4\leq 8R^2r^2$$

Proof. From Corollary 13 it results that we have to prove

$$MN^4+NP^4+PQ^4+QM^4\geq 16r^4.$$

It will be sufficient to prove that, if we denote $x=\frac{R}{r}\geq\sqrt{2}$,

$$\frac{24x^4-8x^2-4\sqrt{4x^2+1}}{x^4}\geq 16,\ \forall\, x\geq\sqrt{2}$$

or $(2x^4-2x^2-1)^2\geq 4x^2+1, \forall\, x\geq\sqrt{2}$, or $4x^6(x^2-2)\geq 0$, which is true. The right side of the inequality from the statement is equivalent using Corollary 13 with $32\left(\sqrt{4x^2+1}-2\right)\leq 8x^2\left(\sqrt{4x^2+1}-1\right)$ or

$$(x^2-4)\sqrt{4x^2+1}\geq x^2-8,\ \forall\, x\geq\sqrt{2}.$$

If $4\leq x^2\leq 8$, the inequality is true.

If $2 \le x^2 \le 4$, the inequality is equivalent (if we denote $x^2 = y$) to $(8-y)^2 \ge (4-y)^2(4y+1), \forall\, y \in [2,4]$, or $(y-2)(y^2-6y+6) \le 0, \forall\, y \in [2,4]$, or $y^2 - 6y + 6 \le 0$, or $y \in \left[3-\sqrt{3}, 3+\sqrt{3}\right]$.

If $x^2 \ge 8$ the inequality is equivalent to $(y-4)^2(4x^2+1) \ge (y-8)^2$, $\forall\, y \ge 8$, or $(y-2)(y^2-6y+6) \ge 0$, which is true since $y \ge 8$. □

In the following we refine the right side of the inequality from Corollary 15.

Corollary 15. *In every bicentric quadrilateral the following inequality is true:*

$$MN^4 + NP^4 + PQ^4 + QM^4 \le \sqrt[4]{2^{13}}\sqrt{R^3}\sqrt{r}.$$

Proof. In the same way as in Corollary 14, to prove the inequality from statement, it will be sufficient to prove that

$$32\left(\sqrt{4x^2+1}-2\right) \le \sqrt[4]{2^{13}}\sqrt{x^3}\left(\sqrt{4x^2+1}-1\right),\ \forall\, x \ge \sqrt{2},$$

which is verified using Wolphram Alpha. □

Theorem 4 (Blundon-Eddy)**.** *In every bicentric quadrilateral the following inequality is true:*

$$\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)} \le s \le \sqrt{4R^2+r^2}+r.$$

Proof. The left side of the inequality from the statement, using Lemma 1 from equalities $s = t_1+t_2+t_3+t_4$ and $(t_1+t_3)(t_2+t_4) = 2r\left(\sqrt{4R^2+r^2}-r\right)$, is equivalent to $4(t_1+t_3)(t_2+t_4) \le (t_1+t_3+t_2+t_4)$ which is true according to AM-GM inequality.

Also

$$\sqrt{4R^2+r^2}-r = \frac{(t_1+t_3)(t_2+t_4)}{2r}$$

or

$$\sqrt{4R^2+r^2}+r = \frac{(t_1+t_3)(t_2+t_4)}{2r}+2r$$

or, since $r = \sqrt[4]{t_1t_2t_3t_4}$, we have

$$\sqrt{4R^2+r^2}+r = \frac{(t_1+t_3)(t_2+t_4)}{2\sqrt[4]{t_1t_2t_3t_4}}+2\sqrt[4]{t_1t_2t_3t_4}.$$

So the right side of the inequality from the statement is equivalent to

$$2\sqrt[4]{t_1t_2t_3t_4}(t_1+t_2+t_3+t_4) \le (t_1+t_3)(t_2+t_4)+4\sqrt{t_1t_2t_3t_4}$$

or, since $t_1t_3 = t_2t_4$,

$$2\sqrt{t_1t_3}\left(t_1+t_2+t_3+\frac{t_1t_3}{t_2}\right) \le (t_1+t_3)\left(t_3+\frac{t_1t_3}{t_2}\right)+4t_1t_3$$

or

$$2\sqrt{t_1t_3}\left[t_1t_3+t_2^2+t_2(t_1+t_3)\right] \le (t_1+t_3)(t_2^2+t_1t_3)+4t_1t_3t_2.$$

If we denote $x = t_2$, $y = t_1+t_3$, $z = \sqrt{t_1t_3}$, we obtain

$$2z(x^2+xy+z^2) \le y(x^2+z^2)+4z^2x$$

or $\quad (y-2z)(x^2+z^2-xz) \ge 0$,

which is true since, from AM-GM, $t_1+t_3 \ge 2\sqrt{t_1t_3}$ or $y \ge 2z$. □

References

[1] Drăghicescu, I.C., *Poligoane bicentrice*, Ed. Univ. Bucureşti, 2009
[2] Mihalca, D., Chitescu, I. şi Chiriţă M., *Geometria patrulaterului*, Ed. Teora, 1998
[3] Drăgan, M. şi Maftei I.V., *Inegalităţi matematice structurate pe tipuri de inegalităţi clasice*, Ed. Didactică şi Pedagogică, 2015
[4] Drăgan, M., Maftei, I.V. şi Rădulescu, S., *Inegalităţi matematice* (*Extinderi şi generalizări*), Ed. Didactică şi Pedagogică, Ed. Didactică şi Pedagogică, 2012
[5] Vasić, P.M. and Peĉarić, J.E., *On Jensen Inequalities*, Univ. Beograd Publ. Electrotehn. Fak., Ser. Mat. Fiz., no. 50–54, 1979
[6] Bencze, M. and Drăgan, M., *600 Selected Problems for Mathematical Contests*, Studis Publ. House, Iaşi, Romania, 2017
[7] Mitrinoviĉ, D.S., Peĉarić, J.E. and Volonec, V., *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ., Boston, 1989
[8] Mitrinoviĉ, D.S., *Analytic Inequalities*, Springer Verlag, Berlin, 1970
[9] Jensen, J.L., *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math., 1906
[10] Nicolescu, L. şi Boskoff, V., *Probleme practice de geometrie*, Ed. Tehnică, Bucureşti, 1990
[11] Bottema, O., *Geometric Inequalities*, Groningen, 1969
[12] Bottema, O., *Inequalities for R, r and s*, Univ. Beograd Publ., Electrotehn. Fak., Ser. Mat. Fiz., no. 338–352, 1971, 27–36
[13] Blundon, W.J., *Inequalities with the Triangle*, Canadian Math. Bulletin, no. 8, 1965, 615–626
[14] *Revista de Matematică din Timişoara – R.M.T.*, 2000–2018
[15] *Recreaţii Matematice*, 2000–2018
[16] *Gazeta Matematică*, 1990–2018
[17] *Octogon Mathematical Magazine*, 2000–2018

Liceul "Mircea cel Bătrân",
Bucureşti, Romania
Email address: marius.dragan2005@yahoo.com

505600 Hărmanului 6,
Săcele-Négyfalu, Romania
Email address: benczemihaly@gmail.com