The best constant in some geometric inequalities

Mihály Bencze and Marius Drăgan

Abstract

The purpose of this article is to find a way to determine the best constant for some geometric inequalities.

1 Introduction

First recall the fundamental triangle theorem of Blundon (see [**1**]):

Theorem 1. *For any triangle ABC, the inequality $s_1 \leq s_{\leq} s_2$ holds, where s_1, s_2 represent the semiperimeter of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ which have the same circumradius R and irradius r as the triangle ABC, where*

$$s_1^2 = 2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}$$
$$s_2^2 = 2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3}.$$

Also recall the Blundon theorem for acute triangles (see [**4**]):

Theorem 2. *For any acute triangle holds*

$s_1 \leq s \leq s_2$ *if* $2 \leq \dfrac{R}{r} \leq \sqrt{2}+1$ *and* $s_3 \leq s \leq s_2$ *if* $\dfrac{R}{r} \geq \sqrt{2}+1$, *where s_1, s_2 represent the semiperimeter of isosceles triangles from Theorem 1 and s_3 the semiperimeter of a right angle triangle, where* $s_3 = 2R + r$.

We will search the best constant k for the inequality of type

$$E(a,b,c,k) \geq \quad \text{or} \quad \leq 0 \tag{1}$$

where E are symmetric and homogeneous in a,b,c.
We reduce the inequality (1) at inequality of form

$$F(R,r,s) \geq \quad \text{or} \quad \leq k \tag{2}$$

We consider R,r in (2) constant and s variable.
If we denote $x = \dfrac{R}{r}$ inequality (2) is equivalent with

$$F(x,1,s) \geq \quad \text{or} \quad \leq k \tag{3}$$

defined for $x \geq 2$.
Let $f : [s_1, s_2] \to \mathbb{R},\ f(s) = F(x,1,s),\ x \geq 2$.

For inequality $F(x,1,s) \geq k$ or $f(s) \geq k$.

If f is increasing we obtain $f(s) \geq f(s_1) \geq k$, so $k \leq f(s_1) = h(x)$, $\forall\, x \geq 2$. So the best constant is $k_1 = \inf\limits_{x \geq 2} h(x)$.

If f is decreasing then $f(s) \geq f(s_2) \geq k$, hence $k \leq f(s_2) = g(x)$, $\forall\, x \geq 2$, and the best constant is $k_2 = \inf\limits_{x \geq 2} g(x)$.

For inequality $F(x,1,s) \leq k$ or $f(s) \leq k$, if f is increasing we have $f(s) \leq f(s_2) \leq k$ or $k \geq f(s_2) = u(x)$, $\forall\, x \geq 2$.
So the best constant is $k_3 = \sup\limits_{x \geq 2} u(x)$.

If f is decreasing we have $f(s) \leq f(s_1) \leq k$ or $k \geq f(s_1) = v(x)$, $\forall\, x \geq 2$, so the best constant is $k_4 = \sup\limits_{x \geq 2} v(x)$. In the same way we proceed in the case of acute triangle using Theorem 2.

In the following we present some applications.

I. Find the best constant α such that the inequality

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} \leq \alpha + \frac{3-4\alpha}{2}\,\frac{r}{R},$$

is true in every ABC triangle.

The following identities are known:

$$\sum \frac{a}{b+c} = \frac{2s^2 - 2r^2 - 2Rr}{s^2 + r^2 + 2Rr} \quad \text{and} \quad \sum \frac{bc}{(a+b)(a+c)} = \frac{s^2 - 2Rr + r^2}{s^2 + 2Rr + r^2}$$

So

$$\sum \left(\frac{a}{b+c}\right)^2 = \left(\sum \frac{a}{b+c}\right)^2 - 2\sum \frac{bc}{(a+b)(a+c)}$$
$$= \frac{2[s^4 - (6r^2 + 4Rr)s^2 + 6R^2r^2 + 4Rr^3 + r^4]}{(s^2 + r^2 + 2Rr)^2}.$$

We consider the function $f : (0, +\infty) \to \mathbb{R}$

$$f(t) = \frac{2[t^2 - (6r^2 + 4Rr)t + 6R^2r^2 + 4Rr^3 + r^4]}{(t + r^2 + 2Rr)^2} \quad \text{with}$$
$$f'(t) = \frac{4R[(2R + 2r)t - 5R^2r - 6Rr^2 - 2r^3]}{(t + 2Rr + r^2)^3}.$$

From Gerretsen inequality we have $t = s^2 \geq 16Rr - 5r^2$. We will prove $16Rr - 5r^2 \geq \dfrac{5R^2r + 6Rr^2 + 2r^2}{2R + 2r}$ which is equivalent with

$$(2x + 2)(16x - 5) \geq 5x^2 + 6x + 2, \ \forall\, x \geq 2.$$

This inequality is true if $x > \dfrac{2\sqrt{97} - 8}{27} \simeq 0,43$, which is true since $x \geq 2$. We deduce that $f'(t) > 0$ or f is increasing. Then $f(t) \leq f(s_2^2)$, therefore one has

$$\sum \frac{a^2}{(b+c)^2} \leq \frac{29R^3 - 5R^2r - 8Rr^2 - 4r^3 + (7R + 2r)\sqrt{R(R - 2r)^3}}{2R(3R + 2r)^2}.$$

The inequality from the statement is equivalent to

$$\frac{29x^3 - 5x^2 - 8x - 4 + (7x + 2)\sqrt{x(x - 2)^3}}{2x(3x + 2)^2} \leq \alpha + \left(\frac{3}{2} - 2\alpha\right)\frac{1}{x}, \forall\, x \geq 2$$

or

$$\alpha \geq \frac{x}{x-2}\left[\frac{29x^3-5x^2-8x-4+(7x+2)\sqrt{x(x-2)^3}}{2x(3x+2)^2}-\frac{3}{2x}\right]$$
$$=\frac{29x^2+26x+8+(7x+2)\sqrt{x(x-2)}}{2(3x+2)^2}, \quad \forall\, x \geq 2.$$

So the best constant is

$$\alpha_0=\sup_{x\geq 2}\frac{29x^2+26x+8+(7x+2)\sqrt{x(x-2)}}{2(3x+2)^2} \overset{WA}{=} 2.$$

Remark 1. In every ABC triangle is true

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}+\frac{5r}{2R}\leq 2.$$

II. Find the best constant α such that the inequality

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}\geq \alpha+\frac{3-4\alpha}{2}\frac{r}{R}$$

is true in every ABC triangle.
We saw in **I** that f is increasing. So

$$f(t)\geq f(s_1^2)=\frac{29R^3-5R^2r-8Rr^2-4r^3-(7R+2r)\sqrt{R(R-2r)^3}}{2R(3R+2r)^2}$$

therefore

$$\frac{29x^3-5x^2-8x-4-(7x+2)\sqrt{x(x-2)^3}}{2x(3x+2)^2}\geq \alpha+\left(\frac{3}{2}-2\alpha\right)\frac{1}{x}, \forall\, x\geq 2$$

or

$$\alpha\leq \frac{29x^2+26x+8-(7x+2)\sqrt{x(x-2)}}{2(3x+2)^2}, \forall\, x\geq 2$$

So the best constant is

$$\alpha_1=\min_{x\geq 2}\frac{29x^2+26x+8-(7x+2)\sqrt{x(x-2)}}{2(3x+2)^2} \overset{WA}{=} \frac{11}{9}.$$

Remark 2. In every ABC triangle is true

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}+\frac{17r}{18R}\geq\frac{11}{9}.$$

III. Find the best positive constant α such that the inequality

$$\alpha\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)+8(1-\alpha)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq 1,$$

is true in every ABC triangle.

We know identities: $\sum\tan\frac{A}{2}=\frac{4R+r}{s}$, $\prod\sin\frac{A}{2}=\frac{r}{4R}$.

The inequality from statement may be written as

$$\alpha\left[\frac{(4R+r)^2}{s^2}-2\right]+(8-8\alpha)\frac{r}{4R}\geq 1. \tag{4}$$

We consider the function $f:[s_1,s_2]\to\mathbb{R}$

$$f(s)=\alpha\left[\frac{(4R+r)^2}{s^2}-2\right]+(8-8\alpha)\frac{r}{4R},$$

which is decreasing in s with R,r fixed.

So from Blundon's inequality we obtain

$$f(s)\geq f(s_2) \tag{5}$$

From (4) and (5) it follows that $f(s_2)\geq 1$ or

$$\alpha\left[\frac{(4x+1)^2}{s_2^2(x)}-2\right]+\frac{2}{x}-\frac{2\alpha}{x}\geq 1,\ \forall\, x\geq 2,$$

where $s_2(x)=2x^2+10x-1-2\sqrt{x(x-2)^3}$.

We obtain after perform some calculation

$$\alpha\geq u(x)=\frac{2x^2+10x-1+2\sqrt{x(x-2)^3}}{12x^2+8x-1-(4x+4)\sqrt{x(x-2)}},\ \forall\, x\geq 2$$

So $\alpha\geq\sup_{x\geq 2}u(x)\overset{WA}{=}\frac{1}{2}$.

Therefore the best constant is $\alpha=\frac{1}{2}$.

Remark 3. If we take $\alpha = \frac{1}{2}$, we obtain $\sum \tan \frac{A}{2} + 8 \prod \sin \frac{A}{2} \geq 2$, which represent a problem of Leon Banoff from Crux Mathematicorum no 5 (1984).

IV. Find the best constants $\alpha, \beta, \gamma \geq -2$ such that the inequality

$$\left(\frac{r}{r_a}\right)^3 + \left(\frac{r}{r_b}\right)^3 + \left(\frac{r}{r_c}\right)^3 \leq \frac{\alpha R + \beta r}{R + \gamma r},$$

is true in every ABC triangle.
First we will prove that

$$\left(\frac{r}{r_a}\right)^3 + \left(\frac{r}{r_b}\right)^3 + \left(\frac{r}{r_c}\right)^3 \leq \frac{8R - 13r}{8R + 11r} \tag{6}$$

We have

$$\left(\frac{r}{r_a}\right)^3 + \left(\frac{r}{r_b}\right)^3 + \left(\frac{r}{r_c}\right)^3 = \frac{(s-a)^3 + (s-b)^3 + (s-c)^3}{s^3} \tag{7}$$

$$= \frac{9s^3 - 3s^2 \sum a + 3s \sum a^2 - \sum a^3}{s^3}$$

$$= \frac{3s^3 + 6s(s^2 - r^2 - 4Rr) - 2s(s^2 - 6Rr - 3r)}{s^3} = 1 - \frac{12Rr}{s^2}$$

From (6) and (7) it follows that we need to prove that

$$1 - \frac{12Rr}{s^2} \leq \frac{8R - 13r}{8R + 11r} \quad \text{or} \quad s^2 \leq \frac{8R^2 + 11Rr}{2}. \tag{8}$$

By Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r$. So to prove (8), it remains to show that

$$8R^2 + 8Rr + 6r \leq 8R^2 + 11Rr \quad \text{or} \quad 3r(R - 2r) \geq 0,$$

which clearly holds.

In the following we will prove that the best inequality of the type in the statement is the inequality (6).

We suppose that it exists other constants $\alpha, \beta, \gamma \in \mathbb{R}$ and $\gamma \geq -2$ such that

$$\left(\frac{r}{r_a}\right)^3 + \left(\frac{r}{r_b}\right)^3 + \left(\frac{r}{r_c}\right)^3 \leq \frac{\alpha R + \beta r}{R + \gamma r} \leq \frac{8R - 13r}{8R + 11r} \tag{9}$$

is true in every ABC triangle or

$$1 - \frac{12Rr}{s^2} \leq \frac{\alpha R + \beta r}{R + \gamma r} \leq \frac{8R - 13r}{8R + 11r}. \tag{10}$$

According to Blundon's inequality and (10), we obtain

$$1 - \frac{12Rr}{2R^2 + 10Rr - r^2 + 2\sqrt{R(R - 2r)^3}} \leq \frac{\alpha R + \beta r}{R + \gamma r} \leq \frac{8R - 13r}{8R + 11r}. \tag{11}$$

In the case of isosceles triangle with sides $b = c = 1$, $a = 0$, from (11) since $R = \frac{1}{2}$, $r = 0$ it follows that $1 \leq \alpha \leq 1$ or $\alpha = 1$.
In the case of equilateral triangle $\frac{R}{r} = 2$, from (11)

$$\frac{1}{9} \leq \frac{2 + \beta}{2 + \gamma} \leq \frac{1}{9} \quad \text{or} \quad \gamma = 9\beta + 16 \geq -2 \quad \text{or} \quad \beta \geq -2$$

So inequality (11) may be written as

$$1 - \frac{12x}{2x^2 + 10x - 1 + 2\sqrt{x(x - 2)^3}} \leq \frac{x + \beta}{x + 9\beta + 16} \leq \frac{8x - 13}{8x + 11}. \tag{12}$$

The second part of inequality (12) $(x \geq 2 \geq -9\beta - 16)$, may be written as:

$8x^2 + 8\beta x + 11x + 11\beta \leq 8x^2 + (72\beta + 128)x - 13x - 13(9\beta + 16)$
or

$(64\beta + 104)x - 13(9\beta + 16) - 11\beta \geq 0$, $\forall\, x \geq 2$. So

$$64\beta \geq -104 \quad \text{or} \quad \beta \geq -\frac{13}{8}. \tag{13}$$

The first side of (12) may be written as

$$\frac{2\beta+4}{x+9\beta+16} \leq \frac{3x}{2x^2+10x-1+2\sqrt{x(x-2)^3}}\,,\ \forall\, x \geq 2$$

or

$$(x-2)[(4\beta+5)x+\beta+2]+(4\beta+8)\sqrt{x(x-2)^3} \leq 0,\ \forall\, x \geq 2$$

or

$$(4\beta+5)x+\beta+2+(4\beta+8)\sqrt{x(x-2)} \leq 0,\ \forall\, x \geq 2$$

or

$$4\beta+5+\frac{\beta+2}{x}+(4\beta+8)\sqrt{1-\frac{2}{x}} \leq 0,\ \forall\, x \geq 2$$

Taking $x \to \infty$ we obtain

$$\beta \leq -\frac{13}{8}. \tag{14}$$

From (13 and (14) we get that (6) is the best inequality of the type in the "statement".

V. Find the best constant α such that the inequality

$$\frac{bc}{(s-a)^2}+\frac{ca}{(s-b)^2}+\frac{ab}{(s-c)^2} \leq \alpha\frac{R}{r}+12-2\alpha,$$

is true in every acute triangle.

We have

$$\sum\frac{bc}{(s-a)^2}=\frac{r-8R}{r}+\frac{(4R+r)^2}{rs^2}.$$

So the inequality from statement is equivalent to

$$F(s,R,r)=\left[\frac{(4R+r)^3}{rs^2}-11-\frac{8R}{r}\right]\frac{r}{R-2r} \leq \alpha.$$

Since F is decreasing we obtain

$$F(s,R,r) \leq F(s_1,R,r) \leq \alpha_1 \quad \text{if} \quad 2 \leq \frac{R}{r} \leq \sqrt{2}+1$$

$$F(s,R,r) \leq F(s_3,R,r) \leq \alpha_2 \quad \text{if} \quad \frac{R}{r} \geq \sqrt{2}+1.$$

Consider the functions $f_1 : \left[2, \sqrt{2}+1\right] \to \mathbb{R}$ with $f_1(x) = F(s_1, R, r)$ and $f_2 : \left[\sqrt{2}+1, +\infty\right) \to \mathbb{R}$ defined by $f_2(x) = F(s_3, R, r)$.
One can show that $\alpha_1 = \sup\limits_{x\in[2,\sqrt{2}+1]} f_1(x)$ and $\alpha_2 = \sup\limits_{x\in[\sqrt{2}+1,+\infty)} f_2(x)$ are the best constant for inequalities (3)

$$f_1(x) = \left(\frac{1}{x-2}\right)\left(\frac{(4x+1)^3}{2x^2+10x-1-2\sqrt{x(x-2)^3}} - 11 - 8x\right)$$
$$f_2(x) = \left(\frac{1}{x-2}\right)\left(\frac{(4x+1)^3}{(2x+1)^2} - 11 - 8x\right).$$

Using WA we obtain

$$\alpha_1 = \sup_{x\in[2,\sqrt{2}+1]} f_1(x) = f_1\left(\sqrt{2}+1\right) = 10 + 2\sqrt{2}$$
$$\alpha_2 = \sup_{x\in[\sqrt{2}+1,+\infty)} f_2(x) = 10 + 2\sqrt{2}.$$

So the best constant is $10 + 2\sqrt{2}$.

Remark 4. In every acute triangle ABC

$$\frac{bc}{(s-a)^2} + \frac{ca}{(s-b)^2} + \frac{ab}{(s-c)^2} \leq \left(10 + 2\sqrt{2}\right)\frac{R}{r} - 8 - 4\sqrt{2}.$$

VI. Find the best constant k such that the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq k\frac{R-2r}{2R}.$$

it's true in every acute triangle.

From identity $\sum \frac{a}{b+c} = \frac{2s^2-2r^2-2Rr}{s^2+r^2+2Rr}$, the inequality from the statement is equivalent with

$$k \leq \frac{R}{R-2r}\,\frac{s^2-10Rr-7r^2}{s^2+r^2+2Rr} \quad \text{or} \quad k \leq F(R, r, s) = f(s),$$

where $f:(0,+\infty)\to\mathbb{R}$ given by $f(s)=\dfrac{R}{R-2r}\,\dfrac{s^2-10Rr-7r^2}{s^2+r^2+2Rr}$ satisfies $f'(s)=\dfrac{2s^2(12Rr+8r^2)}{s^2+r^2+2Rr}>0$.

Hence f is increasing and $k\le f(s_1)\le f(s)$ if $2\le\dfrac{R}{r}\le\sqrt{2}+1$ and $k\le f(s_3)\le f(s)$ if $\dfrac{R}{r}\ge\sqrt{2}+1$.

So $k\le u(x)=\dfrac{x}{x-2}\,\dfrac{x^2-4-\sqrt{x(x-2)}}{x^2+6x-\sqrt{x(x-2)^3}}$ if $x\in\left[2,\sqrt{2}+1\right]$.

Therefore the best constant is $k_1=\inf\limits_{x\in[2,\sqrt{2}+1]}u(x)\overset{WA}{=}\sqrt{2}-1$.

Also if $\dfrac{R}{r}=x\ge\sqrt{2}+1$, $k\le f(s_3)=v(x)=\dfrac{x}{x-2}\,\dfrac{2x^2-3x-3}{2x^2+3x+1}$.

So the best constant is $k_2=\inf\limits_{x\ge\sqrt{2}+1}\dfrac{x}{x-2}\,\dfrac{2x^2-3x-3}{2x^2+3x+1}=\sqrt{2}-1$.

Therefore, in general, the best constant is $\sqrt{2}-1$.

Remark 5. In every acute triangle holds

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\left(\sqrt{2}-1\right)\frac{r}{R}\ge\frac{2+\sqrt{2}}{2}.$$

VII. Find the best constant α such that

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}\ge\alpha+\frac{3-4\alpha}{2}\,\frac{r}{R}$$

is true in every acute triangle.

We see at problem **I** that the function f is increasing.
So according Blundon inequality in acute triangle, we have since $\alpha\le f(s)$ that $\alpha\le f(s_1^2)\le f(s^2)$ if $2\le\dfrac{R}{r}\le\sqrt{2}+1$ or

$$\alpha\le u(x)=\frac{29x^2+26x+8-(7x+2)\sqrt{x(x-2)}}{2(3x+2)^2}.$$

So the best constant is $\alpha_2 = \inf\limits_{2\le x\le\sqrt{2}+1} u(x) \overset{WA}{=} \dfrac{4-\sqrt{2}}{2}$.

Also if $\dfrac{R}{r} \ge \sqrt{2}+1$, we have $\alpha \le f(s_3^2) \le f(s^2)$ or

$$\alpha \le v(x) = \frac{16x^5+4x^4-46x^3-55x^2-22x-3}{2(x+1)^2(2x+1)^2(x-2)}.$$

So the best constant is $\alpha_3 = \inf\limits_{x\ge\sqrt{2}+1} v(x) \overset{WA}{=} \dfrac{4-\sqrt{2}}{2}$.

Therefore the best constant is $\dfrac{4-\sqrt{2}}{2}$.

Remark 6. In every acute triangle it's true the inequality

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2} \ge \frac{4-\sqrt{2}}{2}\,\frac{5-2\sqrt{2}}{2}\,\frac{r}{R}.$$

VIII. Find the best constant k such that the inequality

$$\cos^4 A+\cos^4 B+\cos^4 C \ge k$$

is true in every acute triangle.

Let $x=a^2$, $y=b^2$, $z=c^2$. Since a,b,c are the sides of an acute triangle, then x,y,z is the sides of an triangle with R,r the circumradius and inradius and s semiperimeter. From the cosine law, for all $s\in(0,+\infty)$ we have

$$\sum\cos^4 A = \sum\frac{(y+z-x)^4}{16y^2z^2} = \frac{1}{16x^2y^2z^2}\sum x^2(y+z-x)^4$$
$$= \frac{1}{16\sigma_3^2}\left[48\sigma_3^2+\sigma_1^6-10\sigma_1^4+8\sigma_3\sigma_1^3+32\sigma_2^2\sigma_1^2-32\sigma_1\sigma_2\sigma_3-32\sigma_2^2\right]$$
$$= \frac{(256R^2r^2+32r^4)s^2-(2048R^3r^3+1536R^2r^2+384Rr^5+32r^6)}{256R^2r^2s^2}$$
$$= \frac{256R^2r^2+32r^4}{256R^2r^2}-\frac{2084R^3r^3+1536R^2r^4+384Rr^5+32r^6}{256R^2r^2s^2} = f(s).$$

So f is increasing (we use $\sigma_1 = 2s$, $\sigma_2 = s^2+r^2+4Rr$, $\sigma_3 = 4Rrs$).
We obtain $f(s) = F(R,r,s) \geq k$ or $f(s) \geq f(s_1) \geq k$ or

$$k \leq f(s_1) = u(x) = \frac{256x^2+32}{256x^2} - \frac{2048x^3+1536x^2+384x+32}{256x^2(2x^2+10x-1)+2\sqrt{x(x-2)^3}}.$$

Therefore the best constant is

$$k_0 = \inf_{x\geq 2} u(x) \overset{WA}{=} \frac{73}{384}.$$

Remark 7. In every acute triangle the following inequality holds

$$\cos^4 A + \cos^4 B + \cos^4 C \geq \frac{73}{384}.$$

IX. Find the best constant k such that the inequality

$$\cos^3 A + \cos^3 B + \cos^3 C \leq k.$$

holds in every acute triangle. It's known the identity

$$\sum \cos^3 A = \frac{4R^3+12R^2r+6Rr^2+r^3-3rs^2}{4R^3} = F(s,R,r) = f(s),$$

f is decreasing in s. So we search k such that $f(s) \leq k$.
In the case $2 \leq \dfrac{R}{r} \leq \sqrt{2}+1$, according Blundon theorem in acute triangle we have $f(s) \leq f(s_1) \leq k$ or

$$u(x) = f(s_1) \leq k,\ \forall\, x \in \left[2, \sqrt{2}+1\right)$$
$$u(x) = \frac{4x^3+6x^2-24x+4+6\sqrt{x(x-2)^3}}{4x^3},\ \forall\, x \in \left[2, \sqrt{2}+1\right]$$

So the best constant in this case is $k = \sup\limits_{x\in[2,\sqrt{2}+1]} u(x) \overset{WA}{=} \dfrac{1}{\sqrt{2}}$.

If $\dfrac{R}{r} \geq \sqrt{2}+1$ we have $f(s) \leq f(s_3) \leq k$ or $v(s) = f(s_3) \leq k$, $\forall\, x \in \left[\sqrt{2}+1, +\infty\right)$, where $v(x) = \dfrac{4x^3-6x-2}{4x^3}$ $\forall\, x \geq \sqrt{2}+1$.

So the best constant is $k_2 = \sup\limits_{x\in[\sqrt{2}+1,+\infty)} v(x) \stackrel{WA}{=} \dfrac{1}{\sqrt{2}}.$

Therefore the best constant is $\dfrac{1}{\sqrt{2}}$.

Remark 8. 1) In every acute triangle one has

$$\cos^3 A + \cos^3 B + \cos^3 C \le \frac{1}{\sqrt{2}} \simeq 0,707.$$

2) In [**5**] Yu-Dong Wu and Nu-Chun Hu find that $k_0 \simeq 1,225\ldots$ is the best constant for which $\sum \cos^3 A \le k$ holds in every ABC triangle. In [**6**] we give an easier solution for this inequality.

X. Find the best constant k such that the inequality

$$\cos^3 A + \cos^3 B + \cos^3 C \ge k$$

holds in every ABC triangle.

By Problem IX, function f is decreasing or $f(s) \ge f(s_2) \ge k$.
So $v(x) = f(s_2) \ge k$, $\forall\, x \ge 2$, where

$$v(x) = \frac{4x^3 + 6x^2 - 24x + 4 - 6\sqrt{x(x-2)^3}}{4x^3}, \ \forall\, x \ge 2$$

So the best constant is

$$k_2 = \inf_{x\ge 2} v(x) \stackrel{WA}{=} \frac{3}{8} \simeq 0,375.$$

References

[1] Drăgan, M., Haivas, M. and Maftei, I,V., *O demonstraţie geometrică a inegalităţii lui Blundon*, Recreaţii Matematice, No 1(2012), 20–22

[2] Drăgan, M. and Stanciu, N., *A new proof of Blundon inequality*, Recreaţii Matematice, No 2(2017), 100–105

[3] Drăgan, M. and Stanciu, N., *On geometric interpretation of Blundon's inequality and some consequences*, GMB no 1(2019), 4–12

[4] Bencse, M. and Drăgan, M., *The Blundon theorem in acute triangle and some consequences*, Forum Geometricorum (2018), 85–194

[5] Yu-Dong Wu and Nu-Chen Hu, *The maximum of* $\sum \cos^3 A$, Octogon Mathematical Magazine, vol. 19, No 1, April 2011, 114–119

[6] Drăgan, M. and Stanciu, N., *Cea mai buna constantă pentru unele inegalităţi geometrice*, GMB no 12(2018), 571–578

Mihály Bencze
Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Braşov, Romania
benczemihaly@yahoo.com

Marius Drăgan
61311 bd. Timişoara Nr. 35, Bl. 0D6, Sc. E, et. 7, Ap. 176, Sect. 6, Bucureşti, Romania
marius.dragan2005@yahoo.com