About Brocard point in a triangle

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In this paper we present some geometrical inequalities related to the Brocard point in a triangle.

Definition 1. *In any triangle ABC there exists a unique point P for each $\sphericalangle PAB = \sphericalangle PBC = \sphericalangle PCA = \omega$. The point P is the Brocard point, they are named after Henri Brocard (1845-1922), a French mathematician.*

This point P is called the first Brocard point, and ω the Brocard angle.

Theorem 1. *(Brocard). In any triangle ABC holds the identity:*

$$\operatorname{ctg}\omega = \operatorname{ctg}A + \operatorname{ctg}B + \operatorname{ctg}C.$$

Proof. The given relation is equivalent to $\sin(A-\omega)\cdot\sin(B-\omega)\cdot\sin(C-\omega) = \sin^3\omega$. But we have $\frac{\sin(A-\omega)}{\sin\omega} = \frac{CP}{AP}$, $\frac{\sin(B-\omega)}{\sin\omega} = \frac{AP}{BP}$, $\frac{\sin(C-\omega)}{\sin\omega} = \frac{BP}{CP}$ and after multiplication holds the identity. □

Remark 1. We have the following relations

$$\operatorname{ctg}\omega = \sum \operatorname{ctg}A = \frac{\sum a^2}{4sr} = \frac{1+\prod\cos A}{\prod \sin A}$$

$$= \frac{\sum \sin^2 A}{2\prod \sin A} = \frac{\sum a\sin A}{\sum a\cos A},$$

$$\csc^2\omega = \csc^2 A + \csc^2 B + \csc^2 C,$$

$$\sin\omega = \frac{2sr}{\sqrt{\sum a^2b^2}}.$$

Corollary 1. *Denote P the Brocard point in the triangle ABC, then*

$$\frac{AP}{c} + \frac{BP}{a} + \frac{CP}{b} = 2\cos\omega.$$

Proof. In the triangle APC we have $\sphericalangle CAP = A-\omega$, $\sphericalangle CPA = \pi - A$ and using the sine rule we can write

$$\frac{\sin(A-\omega)}{CP} = \frac{\sin A}{b}.$$

Similarly hold

$$\frac{\sin(B-\omega)}{AP} = \frac{\sin B}{c}, \frac{\sin(C-\omega)}{PB} = \frac{\sin C}{a}.$$

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Expanding the expressions $\sin(A-\omega)$, $\sin(B-\omega)$, $\sin(C-\omega)$ we get

$$\frac{CP}{b} = \cos\omega - \sin\omega \operatorname{ctg} A,$$

$$\frac{PB}{a} = \cos\omega - \sin\omega \operatorname{ctg} C,$$

$$\frac{AP}{c} = \cos\omega - \sin\omega \operatorname{ctg} B.$$

Adding these we get

$$\frac{AP}{c} + \frac{PB}{a} + \frac{CP}{b} = 3\cos\omega - \sin\omega \sum \operatorname{ctg} A = 2\cos\omega.$$

□

Corollary 2. *In any triangle ABC, we have*

$$\cos^3\omega \geq \frac{27 \cdot AP \cdot BP \cdot CP}{8abc}.$$

Proof. By AM-GM Inequality,

$$2\cos\omega = \sum \frac{AP}{c} \geq 3\sqrt[3]{\prod \frac{AP}{c}}.$$

□

Corollary 3. *In any triangle ABC, we have*

$$AP^2 + BP^2 + CP^2 \geq \frac{4a^2b^2c^2\cos^2\omega}{\sum a^2b^2}.$$

Proof. By Cauchy-Schwarz Inequality,

$$2\cos\omega = \sum \frac{AP}{c} \leq \sqrt{\left(\sum AP^2\right)\left(\sum \frac{1}{a^2}\right)}.$$

□

Corollary 4. *In all triangle ABC, we have*

$$\sqrt{AP} + \sqrt{BP} + \sqrt{CP} \leq \sqrt{2(a+b+c)\cos\omega}.$$

Proof. By Cauchy-Schwarz Inequality,

$$2\cos\omega = \sum \frac{AP}{c} \geq \frac{(\sum \sqrt{AP})^2}{\sum c}.$$

□

Corollary 5. *In all triangle ABC, we have*

$$\left(\frac{2\cos\omega}{\sum a}\right)^{\sum a} \geq \left(\frac{AP}{c^2}\right)^c \left(\frac{BP}{a^2}\right)^a \left(\frac{CP}{b^2}\right)^b.$$

Proof. By weighted AM-GM Inequality, we get

$$\frac{2\cos\omega}{\sum a} = \frac{\sum c \cdot \frac{AP}{c^2}}{\sum c} \geq \left(\prod \left(\frac{AP}{c^2}\right)^c\right)^{\frac{1}{\sum c}}.$$

□

Corollary 6. *In any triangle ABC, we have*

$$\frac{3(2R-r)}{s} \leq \operatorname{ctg}\omega \leq \frac{2R^2+r^2}{sr}.$$

Proof. We have the follows

$$\operatorname{ctg}\omega = \frac{\sum a^2}{4sr} = \frac{s^2-r^2-4Rr}{2sr}.$$

Using the Gerretsen's Inequalities

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$

we obtain the desired results. □

Remark 2. We have the following inequality

$$\operatorname{ctg}\omega \geq \frac{s}{3r}.$$

Proof. We have the follows

$$\operatorname{ctg}\omega = \frac{\sum a^2}{4sr} \geq \frac{(\sum a)^2}{12sr} = \frac{4s^2}{12sr} = \frac{s}{3r}.$$

□

Corollary 7. *In any triangle ABC, we have*

$$\left(\frac{5R-r}{s}\right)^2 - \frac{R}{r} \leq \left(\frac{1}{2\sin\omega}\right)^2 \leq \left(\frac{(R+r)^2}{sr}\right)^2 - \frac{R}{r}.$$

Proof. We have the follows

$$\frac{1}{\sin^2\omega} = 4R^2\sum\frac{1}{a^2} = \left(\frac{s^2+r^2+4Rr}{2sr}\right)^2 - \frac{4R}{r}.$$

Using the Gerretsen's Inequalities holds the desired results. □

Remark 3. We have the following inequality

$$\frac{1}{\sin^2\omega} \geq \frac{2R}{r}.$$

Proof. We have the follows

$$\frac{1}{\sin^2\omega} = \frac{\sum a^2b^2}{4s^2r^2} \geq \frac{abc\sum a}{4s^2r^2} = \frac{2R}{r}.$$

□

Open Question. Prove or disprove

$$\left(\frac{R}{r}\right)^2 \geq \frac{1}{\sin^2\omega} \geq \frac{2R}{r}.$$

when give a new refinement for Euler's $R \geq 2r$ Inequality.

References

[1] Octogon Mathematical Magazine (1993-2021).

[2] MATINF (2018-2021).