SOME INEQUALITIES WITH MEDIANS IN A TRIANGLE

MIHÁLY BENCZE AND MARIUS DRĂGAN

Abstract. In this paper we find the best upper bounds homogenous function depend by R and r (where R is the radius of circumscribed circle and r is the radius of the inscribed circle) of the sums of median of triangle and using this results we prove the Sun Wen Cai inequality (see [2]).

1. Introduction

In [1], Chu Xiao Guang and Yang Xue-Zhi established the inequality

$$(m_a + m_b + m_c)^2 \leq 4s^2 - 16Rr + 5r^2$$

with m_a, m_b, m_c the medians, R circumradius, r inradius and s semiperimeter.

Sun Wen Cai posed a stronger conjecture in a personal communication which is solved by J. Liu in [2] and [3] which say that in any ABC triangle is true the inequality:

$$\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \leq 2 + \frac{r^2}{R^2}\,.$$

In the following we find the best homogenous function $f(R, r)$ such that the inequality $\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \leq f(R, r)$ is true in every ABC triangle.

Theorem 1.1 (Blundon)**.** *In any ABC triangle is true the inequality $s_1 \leq s \leq s_2$ where*

$$s_1 = \sqrt{2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}}$$

$$s_2 = \sqrt{2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}}$$

with the cases of equality for two isosceles triangle with the sides

$$a_1 = \frac{2r(R + r - d)}{\sqrt{(R - d)^2 - r^2}}\,,\ b_1 = c_1 = \frac{(R + r - d)(R - d)}{\sqrt{(R - d)^2 - r^2}}\,,$$

$$a_2 = \frac{2r(R + r + d)}{\sqrt{(R + d)^2 - r^2}}\,,\ b_2 = c_2 = \frac{(R + r + d)(R + d)}{\sqrt{(R + d)^2 - r^2}}$$

where

$$d = \|0I\| = \sqrt{R^2 - 2Rr}\,. \tag{1.1}$$

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Lemma 1.1. *In any triangle ABC are true the following equalities*

$$i) \quad \sum m_a^2 m_b^2 = \frac{9}{16}\left[(s^2+r^2+4Rr)^2 - 16Rrs^2\right] \tag{1.2}$$

$$\begin{aligned} ii) \quad (4m_c m_b m_c)^2 &= s^6 + (33r^2-12Rr)s^4 \\ &- (33r^4+60R^2r^2+120Rr^3) - (r^2+4Rr)^3 \end{aligned} \tag{1.3}$$

Proof. i) We denote $x = a^2+b^2+c^2$

$$\begin{aligned} \sum m_a^2 m_b^2 &= \frac{1}{16}\sum(2x-3a^2)(2x-3b^2) \\ &= \frac{1}{16}\sum\left[4x^2-6x(x-c^2)+9a^2b^2\right] = \frac{9}{16}\sum a^2b^2 \\ &= \frac{9}{16}\left[(s^2+r^2+4Rr)^2-16Rrs^2\right]. \end{aligned}$$

ii) We denote $r^2+4Rr = z$. We have

$$64m_a^2m_b^2m_c^2 = \prod[2(b^2+c^2)-a^2] = \prod(2x-3a^2)$$
$$= 8x^3-12x^3+9\sum a^2b^22x-27a^2b^2c^2 = -4x^3+18x[(\sum ab)^2-4abc\,s]$$
$$-27\cdot 16R^2r^2s^2 = -32(s^2-z)^3+36(s^2-z)[(s^2+z)^2-16Rrs^2]-432R^2r^2s^2$$
$$= -32(s^6-3s^4z+3s^2z^2-z^3)+(36s^2-36z)[s^4+(2z-16Rr)s^2+z^2]$$
$$-432R^2r^2s^2 = -32s^6+96z\,s^4-96z^2s^2+32z^3+36s^6+(72z-576Rr)s^4$$
$$+36z^2s^2-36zs^4-(72z^2-576Rrz)s^2-36z^3-432R^2r^2s^2$$
$$= 4s^6+(96z+72z-576Rr-36z)s^4-(96z^2-36z^2+72z^2-576Rrz$$
$$+432R^2r^2)s^2+32z^3-36z^3 = 4s^6+[132(r^2+4Rr)-576Rr]s^4$$
$$-[132(r^2+4Rr)^2-576Rr(r^2+4Rr)+432R^2r^2]s^2-4z^3$$
$$= 4s^6+(132r^2-48Rr)s^4-(132r^4+480Rr^3+240R^2r^2)s^2-4z^3.$$

So, $(4m_am_bm_c)^2 = s^6+(33r^2-12Rr)s^4-(33r^4+120Rr^3+60R^2r^2)s^2-(r^2+4Rr)^3$. □

2. Main results

Theorem 2.1. *In any triangle ABC is true the inequality*

$$\begin{aligned} \frac{\sqrt{(R-d)^2-r^2}+\sqrt{8r^2+(R-d)^2}}{\sqrt{4r^2+2(R-d)^2}} &\le \frac{m_a+m_b+m_c}{\sqrt{a^2+b^2+c^2}} \\ &\le \frac{\sqrt{(R+d)^2-r^2}+\sqrt{8r^2+(R+d)^2}}{\sqrt{4r^2+2(R+d)^2}}. \end{aligned} \tag{2.1}$$

Proof. We denote $x = a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$ and

$$w = \frac{m_a+m_b+m_c}{\sqrt{a^2+b^2+c^2}} = \sum\sqrt{\frac{m_a^2}{x}} = \sqrt{P(a,b,c)}+\sqrt{P(b,c,a)}+\sqrt{P(c,a,b)}.$$

After squaring we obtain

$$w^2 = \sum \frac{m_a^2}{x} + 2\sqrt{\sum \frac{m_a^2 m_b^2}{x^2} + 2\sqrt{\frac{m_a^2 m_b^2 m_c^2}{x^3}}}$$

or, if we consider $f, g, h : [s_1, s_2] \to \mathbb{R}$

$$f(s) = \sum \frac{m_a^2}{x}\ ,\ g(s) = \sum \frac{m_a^2 m_b^2}{x^2}\ ,\ h(s) = \sqrt{\frac{m_a^2 m_b^2 m_c^2}{x^3}}$$

we have $(w^2 - f(s))^2 = 4(g(s) + 2h(s)w)$, with $w > \sqrt{f(s)}$.

We consider the function $F : \left[\sqrt{f(s)}, +\infty\right) \to \mathbb{R}$

$$F(u) = u^4 - 2f(s)u^2 - 8h(s)u + f^2(s) - 4g(s),\ u \in \left[\sqrt{f(s)}, +\infty\right).$$

We observe that $F(w) = 0$.
Since $F'(u) = 4u^3 - 4f(s)u - 8h(s) = 4u(u^2 - f(s)) - 8h(s)$, it results that F' is increasing on $\left(0, \sqrt{f(s)}\right)$.
But, $F'\left(\sqrt{f(s)}\right) = -8h(s) \leq 0$ and $\lim_{u\to\infty} F'(u) = +\infty$. It follows that F' has only root on $\left[\sqrt{f(s)}, +\infty\right)$.
Since

$$\begin{aligned} F\left(\sqrt{f(s)}\right) &= f^2(s) - 2f^2(s) - 8h(s)\sqrt{f(s)} + f^2(s) - 4g(s) \\ &= -8h(s)\sqrt{f(s)} - 4g(s) \leq 0 \end{aligned}$$

and $\lim_{u\to\infty} F(u) = +\infty$, it results that equation $F(u) = 0$ has w as only solution in $\left[\sqrt{f(s)}, +\infty\right)$ interval.

In conclusion, it exists a only continuous function $u : [s_1, s_2] \to \mathbb{R}$ such that $F(u(s)) = 0$, $(\forall)\, s \in [s_1, s_2]$.
From the implicit theorem it results that the function u is differentiable on interval (s_1, s_2).

If we differentiate the equality

$$(u^2(s) - f(s))^2 = 4(g(s) + 2h(s)u(s)),\ (\forall)\, s \in (s_1, s_2),$$

we obtain

$$\begin{aligned} &2(u^2(s) - f(s))(2u(s)u'(s) - f'(s)) \\ &= 4g'(s) + 8h'(s)u(s) + 8h(s)u'(s),\ (\forall)\, s \in (s_1, s_2) \end{aligned}$$

$$\begin{aligned} &(2u^3(s) - 2f(s)u(s) - 4h(s))u'(s) \\ &= f'(s)(u^2(s) - f(s)) + 2g'(s) + 4h'(s)u(s),\ (\forall)\, s \in (s_1, s_2) \end{aligned} \tag{2.2}$$

From AM-GM inequality we have

$$u^2(s) = f(s) + 2\sum \sqrt{P(a,b,c)P(b,c,a)} \geq f(s) + 6\sqrt[3]{h^2(s)} \tag{2.3}$$

or

$$u^2(s) - f(s) > 6\sqrt[3]{u(s)}. \tag{2.4}$$

Also

$$u^2(s) \geq 6\sqrt[3]{h^2(s)} \quad \text{or} \quad u(s) \geq \sqrt{6}\sqrt[3]{u(s)}. \tag{2.5}$$

From (2.4) and (2.5) we obtain that

$$\begin{gathered} 2u^3(s) - 2f(s)u(s) - 4h(s) \\ = 2u(s)(u^2(s) - f(s)) - 4h(s) > 12\sqrt[3]{h^2(s)}u(s) - 4h(s) > 12\sqrt{6}h(s) - 4h(s) \\ = \left(12\sqrt{6} - 4\right)h(s) > 0 \end{gathered}$$

So, we obtain that

$$2u^3(s) - 2f(s)u(s) - 4h(s) > 0. \tag{2.6}$$

We calculate $f(s) = \sum \frac{m_a^2}{x} = \frac{3}{4}$, $(\forall)\, s \in [s_1, s_2]$ and $f'(s) = 0$, $(\forall)\, s \in [s_1, s_2)$.
From (1.2) we have

$$g(s) = \frac{\sum m_a^2 m_b^2}{x^2} = \frac{9}{64}\left[\frac{(s^2 + r^2 + 4Rr)^2 - 16Rrs^2}{s^2 - r^2 - 4Rr}\right]$$

To simply the calculation we denote $t = s^2 - r^2 - 4Rr$. We obtain

$$\begin{aligned} g(s) &= \frac{9\left[(t + 2r^2 + 8Rr)^2 - 16Rr(t + r^2 + 4Rr)\right]}{64t^2} \\ &= \frac{9(t^2 + 4r^2t + 4r^4 + 16Rr^3)}{64t^2} \end{aligned} \tag{2.7}$$

We consider the function $H : [s_1, s_2] \to \mathbb{R}$, $H(s) = h^2(s)$.
From (1.3) we have $H(s) = \frac{1}{32}\frac{s^6 + xs^4 - ys^2 - z^3}{(s^2 - z)^3}$ where we denote
$x = 33r^2 - 12Rr$, $y = 33r^4 + 60R^2r^2 + 120Rr^3$, $z = r^2 + 4Rr$.
We obtain that

$$\begin{aligned} H(s) &= \frac{1}{32}\frac{(t + z)^3 + x(t + z)^2 - y(t + z) - z^3}{t^3} \\ &= \frac{t^3 + (3z + x)t^2 + (3z^2 + 2zx - y)t + z(xz - y)}{32t^3} \\ &= \frac{1}{32}\left[z(xz - y)\left(\frac{1}{s^2 - z}\right)^3 + (3z^2 + 2zx - y)\left(\frac{1}{s^2 - z}\right)^2\right. \\ &\quad \left. + (3z + x)\left(\frac{1}{s^2 - z}\right) + 1\right] = \frac{1}{32}Q\left(\frac{1}{s^2 - z}\right) \end{aligned} \tag{2.8}$$

We define $Q : \left[\frac{1}{\sqrt{4R^2 + 2r^2}}, +\infty\right) \to \mathbb{R}$

$$Q(\alpha) = a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3,\ \beta = \frac{R}{r},$$

where
$a_0 = z(xz - y) = -108\beta^2(1 + 4\beta)r^6 < 0$
$a_1 = 3z^2 + 2zx - y = (-108\beta^2 + 144\beta + 36)r^4 \leq 0$ since $\beta \geq 2$

$a_2 = 3z + x = 36r^2$ and $a_3 = 1$.

We have $\dfrac{1}{s^2 - z} \geq \dfrac{1}{4R^2 + 2r^2}$. Since $Q''(\alpha) = 6a_0\alpha + 2a_1 < 0$, it follows that

$$Q'(\alpha) \leq Q'\left(\frac{1}{4R^2 + 2r^2}\right). \tag{2.9}$$

We prove that

$$Q'\left(\frac{1}{4R^2 + 2r^2}\right) \leq 0 \tag{2.10}$$

or

$$3a_0 + 2a_1(4R^2 + 2r^2) + a_2(4R^2 + 2r^2)^2 \leq 0$$

or

$$-324(\beta^2 + 4\beta^3) + (8\beta^2 + 4)(-108\beta^2 + 144\beta + 36) + 36(16\beta^4 + 16\beta^2 + 4) \leq 0$$

or

$$-324\beta^2 - 1296\beta^3 - 864\beta^4 + 1152\beta^3 + 288\beta^2 - 432\beta^2$$
$$+576\beta + 144 + 576\beta^4 + 576\beta^2 + 144 \leq 0$$

or

$$288\beta^4 + 144\beta^3 - 108\beta^2 - 576\beta - 288 \geq 0,\ (\forall)\,\beta \geq 2,$$

inequality which is true.

From (2.9) and (2.10) it results that $Q'(\alpha) \leq 0$ which implies that $H'(s) \geq 0$, $(\forall)\, s \in [s_1, s_2]$.

From AM-GM inequality we have that

$$H(s) = \frac{m_a^2 m_b^2 m_c^2}{\left(\sum a^2\right)^3} = \frac{27}{4^3}\,\frac{m_a^2 m_b^2 m_c^2}{\left(\sum m_a^2\right)^3} \leq \frac{1}{4^3}$$

or $\dfrac{1}{H(s)} \geq 4^3$.

From (2.3) we have

$$u^2(s) \geq \frac{3}{4} + 6\sqrt[3]{H(s)} \quad \text{or} \quad u(s) \geq \sqrt{\frac{3}{4} + 6\sqrt[3]{H(s)}}$$

we have since $H'(s) \geq 0$ that

$$g'(s) + 2h'(s)\cdot u(s)$$
$$= g'(s) + \frac{1}{\sqrt{H(s)}}H'(s)u(s) \geq g'(s) + \frac{1}{\sqrt{H(s)}}\cdot\sqrt{\frac{3}{4} + 6\sqrt[3]{H(s)}}H'(s)$$
$$= g'(s) + \sqrt{\frac{3}{4}\,\frac{1}{H(s)} + 6\sqrt[3]{\left(\frac{1}{H(s)}\right)^2}}\,H'(s)$$
$$\geq g'(s) + \sqrt{\frac{3}{4}\cdot 4^3 + 6\sqrt[3]{4^6}}H'(s) = g'(s) + \sqrt{3\cdot 4^2 + 6\cdot 4^2}H'(s)$$
$$= g'(s) + 12H'(s) \tag{2.11}$$

We prove that the function $G : [s_1, s_2] \to \mathbb{R}$ is increasing $G(s) = g(s) + 12H(s)$ is an increasing function

$$G(s) = \frac{9}{64}\,\frac{t^2+4r^2t+4r^2z}{t^2}$$
$$+\frac{12}{32}\,\frac{[t^3+(3z+x)t^2+(3z^2+2zx-y)t+z(xz-y)]}{t^3}$$
$$= \frac{9t^3+36r^2t^2+36r^2zt+24t^3+24(3z+x)t^2+24(32^2+2zx-y)t+24z(xz-y)}{64t^3}$$
$$= \frac{33t^3+[36r^2+24(3z+x)t^2+(36r^2z+24(3z^2+2zx-y)]t+24z(xz-y)}{64+3}$$
$$= \frac{1}{64}\,24z(xz-y)\left(\frac{1}{s^2-z}\right)^3 + \left[36r^2z+24(3z^2+2zx-y)\left(\frac{1}{s^2-z}\right)^2\right]$$
$$+\left[36r^2+24(3z+x)\left(\frac{1}{s^2-z}\right)+33\right]$$
$$= \frac{1}{64}\left[a_0\left(\frac{1}{s^2-z}\right)^3 + a_1\left(\frac{1}{s^2-z}\right)^2 + a_2\left(\frac{1}{s^2-z}\right) + a_3\right].$$

We consider the function $U : \left[\frac{1}{4R^2+2r^2}, +\infty\right) \to \mathbb{R}$, $U(\alpha) = a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3$

$$G(s) = \frac{1}{64}\,U\left(\frac{1}{s^2-z}\right) \tag{2.12}$$

where

$$\begin{aligned}
a_0 &= 24z(xz-y)\\
&= 24r^6(1+4\beta)(33+132\beta-12\beta-48\beta^2-33-60\beta^2-120\beta)\\
&= -24\cdot 108r^6\beta^2(1+4\beta) < 0
\end{aligned}$$

$$\begin{aligned}
a_1 &= 36r^2z+24(3z^2+2zx-y)\\
&= r^4\Big\{36(1+4\beta)+24\Big[3(1+16\beta^2+8\beta)+(2+8\beta)(33-12\beta)\\
&\quad -33-60\beta^2-120\beta\Big]\Big\} = r^4\Big[36+144\beta+24(3+48\beta^2+24\beta\\
&\quad +66-24\beta+264\beta-96\beta^2-33-60\beta^2-120\beta)\Big]\\
&= r^4\left[36+144\beta+24(-108\beta^2+144\beta+36)\right]\\
&= r^4(-2592\beta^2+360\beta+900) \le 0 \quad \text{for } \beta \ge 2
\end{aligned}$$

$$\begin{aligned}
a_2 &= 36r^2+24(3z+x) = r^2\left[36+24(3+12\beta+33-12\beta)\right]\\
&= r^2(36+24\cdot 36) = 900r^2
\end{aligned}$$

But $\alpha = \frac{1}{s^2-z} \ge \frac{1}{4R^2+2r^2}$. We have $U'(\alpha) = 3a_0\alpha^2 + 2a_1\alpha + a_2$. We prove that

$$U'(\alpha) \le 0 \tag{2.13}$$

for each $\alpha \geq \dfrac{1}{4R^2+2r^2}$.

Since $U''(\alpha) = 6a_0\alpha + 2a_1 \leq 0$ it results that $U'(\alpha) \leq U'\left(\dfrac{1}{4R^2+2r^2}\right)$.

To prove (2.13) it will be sufficient to prove that

$$U'\left(\frac{1}{4R^2+2r^2}\right) \leq 0$$

or

$$3a_0 + 2a_1(4R^2+2r^2) + a_2(4R^2+2r^2)^2 \leq 0$$

or

$$-72\cdot 108\beta^2(1+4\beta) + (8\beta^2+4)(-2592\beta^2+360\beta+900) + (4\beta^2+2)^2 900 \leq 0$$

or

$$-18\cdot 108(\beta^2+4\beta^3) + (2\beta^2+1)(-2592\beta^2+360\beta+900)$$
$$+225(16\beta^4+16\beta^2+4) \leq 0$$

or

$$-1994\beta^2 - 7776\beta^3 - 5184\beta^4 + 720\beta^3 + 1800\beta^2$$
$$-2592\beta^2 + 360\beta + 900 + 3600\beta^4 + 3600\beta^2 + 900 \leq 0$$

or

$$1584\beta^4 + 7056\beta^3 - 814\beta^2 - 360\beta - 1800 \geq 0$$

which is true since $\beta \geq 2$. So, we have U is decreasing function. From (2.12) it follows that G is an increasing function. So, we obtain

$$G'(s) = 9'(s) + 12H'(s) \geq 0 \tag{2.14}$$

From (2.11) and (2.14) it results that

$$2g'(s) + 2h'(s)u(s) \geq 0,\ (\forall)\, s \in [s_1, s_2]. \tag{2.15}$$

From (2.2), (2.6) and (2.15) it results that $u'(s) \geq 0$, $(\forall)\, s \in [s_1, s_2]$. So, u is an increasing function. It results that $u(s_1) \leq u(s) \leq u(s_2)$.

From (1.1) we obtain

$$\begin{aligned} u(s_1) &= \frac{1}{\sqrt{a_1^2+2b_1^2}}\left[\sqrt{\frac{4b_1^2-a_1^2}{4}} + 2\sqrt{\frac{2a_1^2+b_1^2}{4}}\right] \\ &= \sqrt{\frac{R-r-d}{(R+r-d)[4r^2+2(R-d)^2]}}\,(R+r-d)\left[1+\frac{\sqrt{8r^2+(R-d)^2}}{\sqrt{(R-d)^2-r^2}}\right] \\ &= \frac{\sqrt{(R-d)^2-r^2}+\sqrt{8r^2+(R-d)^2}}{\sqrt{4r^2+2(R-d)^2}} \end{aligned}$$

$$\begin{aligned} u(s_2) &= \frac{1}{\sqrt{a_2^2+2b_2^2}}\left[\sqrt{\frac{4b_2^2-a_2^2}{4}} + 2\sqrt{\frac{2a_1^2+b_1^2}{4}}\right] \\ &= \frac{\sqrt{(R+d)^2-r^2}+\sqrt{8r^2+(R+d)^2}}{\sqrt{4r^2+2(R+d)^2}}. \end{aligned}$$

$\square$

In [1] is proved the following inequality.

Theorem 2.2. *In any triangle ABC is true the inequality*

$$\frac{(m_a+m_b+m_c)^2}{a^2+b^2+c^2} \le 2+\frac{r^2}{R^2} \tag{2.16}$$

Proof. We denote $R/r = x$, $d_x = \sqrt{x(x-2)}$. From (2.1) it results that to prove (2.16) it will be sufficient to prove that

$$\sqrt{(x+d_x)^2-1}+\sqrt{(x+d_x)^2+8} \le \sqrt{\left(2+\frac{1}{x^2}\right)\left[4+2(x+dx)^2\right]}$$

and after squaring

$$2x^2(x+dx)^2+7x^2+2x^2\sqrt{\left[(x+d_x)^2-1\right]\left[(x+d_x)^2+8\right]}$$
$$\le 8x^2+4+(4x^2+2)(x+d_x)^2$$

or

$$2x^2\sqrt{\left[(x+d_x)^2-1\right]\left[(x+d_x)^2+8\right]} \le x^2+4+(2x^2+2)(x+d_x)^2$$

or

$$2x^2\sqrt{\left[(x+d_x)^2-1\right]\left[(x+d_x)^2+8\right]}$$
$$\le 4x^4-4x^3+5x^2-4x+4+(4x^3+4x)d_x. \tag{2.17}$$

After squaring inequality (2.17) may be written as

$$4x^4(2x^2-2x-1)(2x^2-2x+8)+16x^6(x^2-2x)+8x^5(4x^2-4x+7)d_x$$
$$\le (4x^4-4x^3+5x^2-4x+4)^2+(4x^3+4x)^2(x^2-2x)$$
$$+(8x^3+8x)(4x^4-4x^3+5x^2-4x+4)d_x$$

or

$$16x^8-16x^7+64x^6-16x^7+16x^6-64x^5-8x^6+8x^5-32x^4$$
$$+16x^6(x^2-2x)+(32x^7-32x^6+56x^5)d_x$$
$$\le 16x^8+16x^6+25x^4+16x^2+16x^2-32x^7+40x^6-32x^5$$
$$+32x^4-40x^5+32x^4-32x^3+(16x^6+32x^4+16x^2)(x^2-2x)$$
$$+(32x^7-32x^6+40x^5-32x^4+32x^3+32x^5-32x^4+40x^3-32x^2+32x)d_x$$
$$-40x^3+40x^2-32x$$

or

$$16x^6+16x^5-121x^4+72x^3-56x^2+32x-16-(32x^5+16x^3)(x-2)$$
$$\le (16x^5-64x^4+72x^3-32x^2+32x)d_x$$

or

$$(x-2)(16x^5+48x^4-25x^3+22x^2-12x+8)-(x-2)(32x^5+16x^3)$$
$$\le 8x(x-2)^2(2x^2+1)d_x$$

or

$$(x-2)^2\left[8x(2x^2+1)d_x+16x^4-16x^3+9x^2-4x+4\right] \ge 0$$

which is true since $x \ge 2$.

In the following we find the best constant $\alpha, \beta \in \mathbb{R}$, $n \geq 0$, such that the inequality

$$\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \leq \alpha + \beta \left(\frac{r}{R}\right)^n \tag{2.18}$$

is true in every ABC triangle.
We prove that the inequality (2.16) is the best which we search.

We suppose that it exists $\alpha, \beta \in \mathbb{R}$, $n \geq 0$ better than in theorem (2.16). From Theorem 2.2 we obtain

$$\begin{aligned}\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} &\leq \frac{\left(\sqrt{(R+d)^2 - r^2} + \sqrt{8r^2 + (R+d)^2}\right)^2}{4r^2 + 2(R+d)^2} \\ &\leq \alpha + \beta \left(\frac{r}{R}\right)^n \leq 2 + \left(\frac{r}{R}\right)^2 \end{aligned}\tag{2.19}$$

is true in every ABC triangle.

In the case of isosceles triangle with $a = 0$, $b = c = 1$, $m_a = 1$, $m_b = m_c = \frac{1}{2}$, $R = \frac{1}{2}$, $r = 0$, from (2.19) we obtain $\alpha = 2$.

In the case of equilateral triangle we obtain $\alpha + \frac{\beta}{2^n} = 2 + \frac{1}{4}$. So, $\beta = 2^{n-2}$. Also, from (2.19) we have that $\left(\frac{x}{2}\right)^{n-2} \geq 1$ or $n \geq 2$. We consider $n > 2$

$$\left(\sqrt{(x+d_x)^2 - 1} + \sqrt{(x+d_x)^2 + 8}\right)^2 \leq \left(2 + \frac{2^{n-2}}{x^n}\right)\left[4 + 2(x+d_x)^2\right]$$

or after perform some calculation

$$\begin{aligned}&2x^n\sqrt{[(x+d_x)^2 - 1]\,[(x+d_x)^2 + 8]} \\ &\leq 4x^{n+2} - 4x^{n+1} + x^n + 2^n x^2 - 2^n x + 2^n + (4x^{n+1} + 2^n x)d_x\end{aligned}$$

or

$$\begin{aligned}&4x^{2n}(2x^2 - 2x - 1 + 2xd_x)(2x^2 - 2x + 8 + 2xd_x) \\ &\leq (4x^{n+2} - 4x^{n+1} + x^n + 2^n x^2 - 2^n x + 2^n)^2 + (4x^{n+1} + 2^n x)^2(x^2 - 2x) \\ &\quad + (8x^{n+1} + 2^{n+1}x)(4x^{n+2} - 4x^{n+1} + x^n + 2^n x^2 - 2^n x + 2^n)d_x\end{aligned}$$

or

$$\begin{aligned}&48x^{2n+2} + a_1x^{2n+1} + a_2x^{2n} + a_3x^{n+4} + a_4x^{n+3} + a_5x^{n+2} \\ &\quad + a_6x^{n+1} + a_7x^n + a_8x^4 + a_9x^3 + a_{10}x^2 + a_{11}x + a_{12} \\ &\leq d_x(-48x^{2n+1} + b_1x^{n+3} + b_2x^{n+2} + b_3x^{n+1} + b_4x^3 + b_5x^2 + b_6x.\end{aligned}\tag{2.20}$$

Since $n > 2$, we have $2n+2 \notin \{2n+1, 2n, n+4, n+3, n+2, n+1, n, 4, 3, 2, 1\}$ and $2n+1 \notin \{n+3, n+2, n+1, 3, 2, 1\}$.
Dividing (2.20) by x^{2n+2} and taking $x \to \infty$, we obtain $48 \leq -48$ which represent a contradiction. So, remain $n = 2$. It results that inequality (2.16) is the best of type (2.18). $\square$

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