Several inequalities of Erdős-Mordell type

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ABSTRACT. The purpose of this paper is to obtain some inequalities between sum or product of power to order, $k \in [0,1]$ relate to distences from an interior point M of triangle ABC to the vertices of triangle, radii of triangle MBC, MCA, MAB and distances from M to the sides BC, CA, AB and using this inequalities and an inversion of pol M and ratio t in conexion with vertices A, B, C to obtain many other new inequalities.

1 INTRODUCTION

Let be M an interior point of triangle ABC, R_a, R_b, R_c the radii of the circumscribe of triangles MBC, MCA, MAB, R_1, R_2, R_3 the distances from M to the vertices A, B, C and r_1, r_2, r_3 the distances from M to the sides BC, CA, AB.
The inequality Erdős-Mordell

$$R_1 + R_2 + R_3 \geq 2\left(r_1 + r_2 + r_3\right) \tag{1.1}$$

is true in any triangle ABC.
Also a generalization of Erdős-Mordell appears in [1] who said that:

$$x^2 R_1 + y^2 R_2 + z^2 R_3 \geq 2\left(yz r_1 + zx r_2 + xy r_3\right) \tag{1.2}$$

is true for any real numbers $x, y, z \geq 0$.
If we take $\lambda_1 = x^2$, $\lambda_2 = y^2$, $\lambda_3 = z^2$ are true inequality

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \geq 2\left(\sqrt{\lambda_2\lambda_3} r_1 + \sqrt{\lambda_3\lambda_1} r_2 + \sqrt{\lambda_1\lambda_2} r_3\right) \tag{1.3}$$

who appears in [2].
Also

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3\left(\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2} + \frac{R_3}{\lambda_3}\right) \tag{1.4}$$

Who are published in [3].
In the following we give a lot of inequalities of this type and many other in connection with them.

Lemma 1. Let be M an interior point of triangle ABC. Then we have:
a). $1 \geq \frac{c}{a} \cdot \frac{R_3}{2R_b} + \frac{b}{a} \cdot \frac{R_2}{2R_c}$
b). $1 \geq \frac{b}{a} \cdot \frac{R_3}{2R_b} + \frac{c}{a} \cdot \frac{R_2}{2R_c}$

Proof.
a). We use the well-known inequality

$$aR_1 \geq cr_2 + br_3$$

and the equality

$$r_1 = \frac{R_2 R_3}{2R_a}$$

who result writing the area of triangle MBC

$$S_{\triangle ABC}=\frac{R_2R_3}{4R_a}=\frac{ar_1}{2}$$

b). It result from the inequality

$$aR_1\geq br_2+cr_3$$

Lemma 2. Let be M an interior point of triangle ABC. Then we have:

$$\frac{R_2}{R_c}+\frac{R_3}{R_b}\leq 4\sin\frac{A}{2}$$

We consider $U=pr_{AB}^M$, $V=pr_{AC}^M$. $u=\mu\left(\measuredangle UAM\right)$, $v=\mu\left(\measuredangle VAM\right)$. We have

$$r_2+r_3=R_1\left(\sin u+\sin v\right)=2R_1\sin\frac{A}{2}\cos\frac{u-v}{2}\leq 2R_1\sin\frac{A}{2}$$

or

$$\frac{R_3R_1}{2R_b}+\frac{R_1R_2}{2R_c}\leq 2R_1\sin\frac{A}{2}$$

Who is equivalent with the inequality from the statement.

Theorem 1. Let be M a interior point of triangle ABC. Then for every real number $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$. We have:

a). $\lambda_1R_a^k+\lambda_2R_b^k+\lambda_3R_c^k\geq\sqrt{\lambda_2\lambda_3}R_1^k+\sqrt{\lambda_3\lambda_1}R_2^k+\sqrt{\lambda_1\lambda_2}R_3^k$

b). $\sqrt{\lambda_2\lambda_3}\frac{R_1^k}{R_b^k+R_c^k}+\sqrt{\lambda_3\lambda_1}\frac{R_2^k}{R_c^k+R_a^k}+\sqrt{\lambda_1\lambda_2}\frac{R_3^k}{R_a^k+R_b^k}\leq$

$\leq\frac{1}{2}\left(\sqrt{\lambda_2\lambda_3}\frac{R_1^k}{\sqrt{R_b^kR_c^k}}+\sqrt{\lambda_3\lambda_1}\frac{R_2^k}{\sqrt{R_c^kR_a^k}}+\sqrt{\lambda_1\lambda_2}\frac{R_3^k}{\sqrt{R_a^kR_b^k}}\right)\leq\frac{\lambda_1+\lambda_2+\lambda_3}{2}$

Proof. a). From the inequality $(x+y)^k\geq 2^{\ k-1}\left(x^k+y^k\right)$, $k\in[0,1]$ it result that:

$$\left(\frac{a}{c}\cdot\frac{R_2}{2R_a}+\frac{b}{c}\cdot\frac{R_1}{2R_b}\right)^k\geq 2^{k-1}\left(\left[\left(\frac{a}{c}\right)^k\frac{R_2^k}{\left(2R_a\right)^k}+\left(\frac{b}{c}\right)^k\left(\frac{R_1}{2R_b}\right)^k\right]\right)$$

Using this inequality and Lemma 1. We obtain

$$1\geq\left(\frac{a}{c}\cdot\frac{R_2}{2R_a}+\frac{b}{c}\cdot\frac{R_1}{2R_b}\right)^k\geq 2^{k-1}\left[\left(\frac{a}{c}\right)^k\frac{R_2^k}{\left(2R_a\right)^k}+\left(\frac{b}{c}\right)^k\left(\frac{R_1^k}{2R_b}\right)^k\right]$$

or

$$2\geq\frac{a^k}{c^k}\cdot\frac{R_2^k}{R_a^k}+\frac{b^k}{c^k}\cdot\frac{R_1^k}{R_b^k}$$

or

$$2\lambda_3R_c^k\geq\lambda_3\frac{a^k}{c^k}\cdot\frac{R_c^k}{R_a^k}\cdot R_2^k+\lambda_3\frac{b^k}{c^k}\cdot\frac{R_c^k}{R_b^k}\cdot R_1^k$$

and the similar inequalities

$$2\lambda_2R_b^k\geq\lambda_2\frac{c^k}{b^k}\cdot\frac{R_b^k}{R_c^k}\cdot R_1^k+\lambda_2\frac{a^k}{b^k}\cdot\frac{R_b^k}{R_a^k}\cdot R_3^k$$

$$2\lambda_1R_a^k\geq\lambda_1\frac{b^k}{s^k}\cdot\frac{R_a^k}{R_b^k}\cdot R_3^k+\lambda_1\frac{c^k}{a^k}\cdot\frac{R_a^k}{R_c^k}\cdot R_2^k$$

Adding this inequalities and apply the A-M inequalities we obtain:

$$2\left(\lambda_1R_a^k+\lambda_2R_b^k+\lambda_3R_c^k\right)\geq R_1^k\left(\lambda_3\cdot\frac{b^k}{c^k}\cdot\frac{R_c^k}{R_b^k}+\lambda_2\cdot\frac{c^k}{b^k}\cdot\frac{R_b^k}{R_c^k}\right)+$$

$$+R_2^k\left(\lambda_3\cdot\frac{a^k}{c^k}\cdot\frac{R_c^k}{R_a^k}+\lambda_1\cdot\frac{c^k}{a^k}\cdot\frac{R_a^k}{R_c^k}\right)+R_3^k\left(\lambda_2\cdot\frac{a^k}{b^k}\cdot\frac{R_b^k}{R_a^k}+\lambda_1\cdot\frac{b^k}{a^k}\cdot\frac{R_a^k}{R_b^k}\right)\geq$$

$$\geq 2\sqrt{\lambda_2\lambda_3}R_1^k+2\sqrt{\lambda_3\lambda_1}R_2^k+2\sqrt{\lambda_2\lambda_1}R_3^k$$

b). Adding the inequalities

$$2\lambda_3\geq\lambda_3\cdot\frac{a^k}{c^k}\cdot\frac{R_2^k}{R_a^k}+\lambda_3\cdot\frac{b^k}{c^k}\cdot\frac{R_1^k}{R_b^k}$$

$$2\lambda_1\geq\lambda_1\cdot\frac{b^k}{a^k}\cdot\frac{R_3^k}{R_b^k}+\lambda_1\cdot\frac{c^k}{a^k}\cdot\frac{R_2^k}{R_c^k}$$

$$2\lambda_3\geq\lambda_2\cdot\frac{c^k}{b^k}\cdot\frac{R_1^k}{Rc}+\lambda_2\cdot\frac{a^k}{b^k}\cdot\frac{R_3^k}{R_a^k}$$

We obtain

$$2\left(\lambda_1+\lambda_2+\lambda_3\right)\geq\left(\lambda_3\cdot\frac{a^k}{c^k}\cdot\frac{R_2^k}{R_a^k}+\lambda_1\cdot\frac{c^k}{a^k}\cdot\frac{R_2^k}{R_c^k}\right)+$$

$$+\left(\lambda_3\cdot\frac{b^k}{c^k}\cdot\frac{R_1^k}{R_b^k}+\lambda_2\cdot\frac{c^k}{b^k}\cdot\frac{R_1^k}{R_c^k}\right)+\left(\lambda_1\cdot\frac{b^k}{a^k}\cdot\frac{R_3^k}{R_b^k}+\lambda_2\cdot\frac{a^k}{b^k}\cdot\frac{R_3^k}{R_a^k}\right)\geq$$

$$\geq 2\sqrt{\lambda_1\lambda_3}\cdot\frac{R_2^k}{\sqrt{R_a^kR_c^k}}+2\sqrt{\lambda_2\lambda_3}\cdot\frac{R_1^k}{\sqrt{R_b^kR_c^k}}+2\sqrt{\lambda_1\lambda_2}\cdot\frac{R_3^k}{\sqrt{R_a^kR_b^k}}\geq$$

$$\geq 4\sqrt{\lambda_1\lambda_3}\cdot\frac{R_2^k}{R_a^k+R_c^k}+4\sqrt{\lambda_2\lambda_3}\cdot\frac{R_1^k}{R_b^k+R_c^k}+4\sqrt{\lambda_1\lambda_2}\cdot\frac{R_3^k}{R_a^k+R_b^k}$$

We observe that b) $\overset{\text{imply}}{\Rightarrow}$ a)
Indeed putting in b). $\lambda_1\to\lambda_1R_a^k$, $\lambda_2\to\lambda_2R_b^k$, $\lambda_3\to\lambda_3R_c^k$, we obtain

$$\sqrt{R_b^kR_c^k}\cdot\frac{R_1^k\cdot\sqrt{\lambda_2\lambda_3}}{R_b^k+R_c^k}+\sqrt{R_c^kR_a^k}\cdot\frac{R_2^k\cdot\sqrt{\lambda_3\lambda_1}}{R_a^k+R_c^k}+\sqrt{R_a^kR_b^k}\cdot\frac{R_3^k\cdot\sqrt{\lambda_1\lambda_2}}{\sqrt{R_a^k+R_b^k}}\leq$$

$$\frac{1}{2}\left(\lambda_1R_1^k+\lambda_2R_2^k+\lambda_3R_3^k\right)\leq\frac{1}{2}\left(\lambda_1R_a^k+\lambda_2R_b^k+\lambda_3R_c^k\right)$$

Also putting in b). $\lambda_1\to\lambda_1R_b^k$, $\lambda_2\to\lambda_2R_c^k$, $\lambda_3\to\lambda_3R_a^k$ and $\lambda_1\to\lambda_2R_b^k$, $\lambda_2\to\lambda_3R_c^k$, $\lambda_3\to\lambda_1R_a^k$, we obtain:

Theorem 2. Let be M an interior point of triangle ABC. Then for ecery real number $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$, we have:
a).

$$\sqrt{\lambda_2\lambda_3}\cdot\frac{R_1^k\sqrt{R_c^k\cdot R_a^k}}{R_b^k+R_c^k}+\sqrt{\lambda_3\lambda_1}\cdot\frac{R_2^k\sqrt{R_a^k\cdot R_b^k}}{R_a^k+R_c^k}+\sqrt{\lambda_1\lambda_2}\cdot\frac{R_3^k\sqrt{R_b^k\cdot R_c^k}}{R_a^k+R_b^k}\leq$$

$$\leq\frac{1}{2}\left(\sqrt{\lambda_2\lambda_3}R_1^k\sqrt{\frac{R_a^k}{R_b^k}}+\sqrt{\lambda_3\lambda_1}R_2^k\sqrt{\frac{R_b^k}{R_c^k}}+\sqrt{\lambda_1\lambda_2}R_3^k\sqrt{\frac{R_c^k}{R_a^k}}\right)\leq$$

$$\leq\frac{1}{2}\left(\lambda_1R_b^k+\lambda_2R_c^k+\lambda_3R_a^k\right)$$

b).

$$\sqrt{\lambda_3\lambda_1}\cdot\frac{R_1^k\sqrt{R_c^k\cdot R_a^k}}{R_b^k+R_c^k}+\sqrt{\lambda_1\lambda_2}\cdot\frac{R_2^k\sqrt{R_a^k\cdot R_b^k}}{R_c^k+R_a^k}+\sqrt{\lambda_2\lambda_3}\cdot\frac{R_3^k\sqrt{R_b^k\cdot R_c^k}}{R_a^k+R_b^k}\leq$$

$$\leq\frac{1}{2}\left(\sqrt{\lambda_3\lambda_1}R_1^k\sqrt{\frac{R_a^k}{R_b^k}}+\sqrt{\lambda_1\lambda_2}R_2^k\sqrt{\frac{R_b^k}{R_c^k}}+\sqrt{\lambda_2\lambda_3}R_3^k\sqrt{\frac{R_c^k}{R_a^k}}\right)\leq$$

$$\leq\frac{1}{2}\left(\lambda_2R_b^k+\lambda_3R_c^k+\lambda_1R_a^k\right)$$

If in Theorem 1 and 2 we take $\lambda_1=\lambda_2=\lambda_3$ we obtain

Corollary 2.1. a).

$$R_a^k+R_b^k+R_c^k\geq R_1^k+R_2^k+R_3^k$$

b).

$$\frac{R_1^k}{R_b^k+R_c^k}+\frac{R_2^k}{R_c^k+R_a^k}+\frac{R_3^k}{R_a^k+R_b^k}\leq\frac{1}{2}\left(\frac{R_1^k}{\sqrt{R_b^kR_c^k}}+\frac{R_2^k}{\sqrt{R_c^kR_a^k}}+\frac{R_3^k}{\sqrt{R_a^kR_b^k}}\right)\leq\frac{3}{2}$$

c).

$$\frac{R_1^k\sqrt{R_c^kR_a^k}}{R_b^k+R_a^k}+\frac{R_2^k\sqrt{R_a^kR_b^k}}{R_a^k+R_c^k}+\frac{R_3^k\sqrt{R_b^kR_c^k}}{R_a^k+R_b^k}\leq\frac{1}{2}\left(R_1^k\sqrt{\frac{R_a^k}{R_b^k}}+R_2^k\sqrt{\frac{R_b^k}{R_c^k}}+R_3^k\sqrt{\frac{R_c^k}{R_a^k}}\right)\leq$$

$$\leq\frac{1}{2}\left(R_a^k+R_b^k+R_c^k\right)$$

If we take $k=1$ in a). we obtain the following inequality

$$R_a+R_b+R_c\geq R_1+R_2+R_3$$

who appears in [1].
Also if we take $k=1$ in b). we obtain:

$$\frac{R_1}{R_b+R_c}+\frac{R_2}{R_c+R_a}+\frac{R_3}{R_a+R_b}\leq\frac{1}{2}\left(\frac{R_1}{\sqrt{R_bR_c}}+\frac{R_2}{\sqrt{R_aR_b}}+\frac{R_3}{\sqrt{R_aR_b}}\right)$$

who represent a refinement of inequality $\frac{R_1}{R_b+R_c}+\frac{R_2}{R_c+R_a}+\frac{R_3}{R_a+R_b}\leq\frac{3}{2}$ who apperas in [2].
If we take in c). $k=1$ we obtain

$$\frac{R_1\sqrt{R_cR_a}}{R_b+R_c}+\frac{R_2\sqrt{R_aR_b}}{R_a+R_c}+\frac{R_3\sqrt{R_bR_c}}{R_a+R_c}\leq\frac{1}{2}\left(R_1\sqrt{\frac{R_a}{R_b}}+R_2\sqrt{\frac{R_b}{R_c}}+R_3\sqrt{\frac{R_c}{R_a}}\right)\leq$$

$$\leq\frac{1}{2}\left(R_a+R_b+R_c\right)$$

If in Theorem 2) we take $\lambda_1\to\lambda_1^2$, $\lambda_2\to\lambda_2^2$, $\lambda_3\to\lambda_3^2$, we obtain

Corollary 2.2. Let be M an interior point of triangle ABC. Then for every real number $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq0$, we have

$$\lambda_1^2R_a^k+\lambda_2^2R_b^k+\lambda_3^2R_c^2\geq\lambda_1\lambda_2\lambda_3\left(\frac{R_1^k}{\lambda_1}+\frac{R_2^k}{\lambda_2}+\frac{R_3^k}{\lambda_3}\right)$$

If we take $k=1$ we obtain

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3 \left(\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2} + \frac{R_3}{\lambda_3} \right)$$

who represent just inequality (1.4).

Corollary 2.3. Let be M an interior point of triangle ABC. Then for every real number $k \in [0,1]$ we have
a).

$$R_1^k R_a^k + R_2^k R_b^k + R_3^k R_c^k \leq R_a^k R_b^k + R_b^k R_c^k + R_c^k R_a^k$$

b).

$$1 + \frac{R_b^k}{R_c^k} + \frac{R_c^k}{R_b^k} \geq \frac{R_1^k}{\sqrt{R_b^k R_c^k}} + \frac{R_2^k}{\sqrt{R_a^k R_b^k}} + \frac{R_3^k}{\sqrt{R_a^k R_c^k}}$$

and similar inequalities
c).

$$R_1^k R_a^k + R_2^k \sqrt{R_b^k R_c^k} + R_3^k \sqrt{R_b^k R_c^k} \leq R_a^k R_b^k + R_b^k R_c^k + R_c^k R_b^k$$

and similar inequalities
d).

$$\frac{R_a^k}{R_b^k} + \frac{R_b^k}{R_c^k} + \frac{R_c^k}{R_a^k} \geq \frac{R_1^k}{\sqrt{R_c^k R_a^k}} + \frac{R_2^k}{\sqrt{R_b^k R_a^k}} + \frac{R_3^k}{\sqrt{R_b^k R_c^k}}$$

and similar inequalities
e).

$$R_1^k \sqrt{R_a^k R_b^k} + R_2^k \sqrt{R_b^k R_c^k} + R_3^k \sqrt{R_a^k R_c^k} \leq R_a^k R_b^k + R_b^k R_c^k + R_c^k R_b^k$$

Proof. We take in theorem 1

$$(\lambda_1,\lambda_2,\lambda_3) = \left(\frac{1}{R_a^k}, \frac{1}{R_b^k}, \frac{1}{R_c^k} \right), (\lambda_1,\lambda_2,\lambda_3) = \left(\frac{1}{R_a^k}, \frac{1}{R_b^k}, \frac{1}{R_c^k} \right)$$

$$(\lambda_1,\lambda_2,\lambda_3) = \left(\frac{1}{R_b^k}, \frac{1}{R_a^k}, \frac{1}{R_c^k} \right), (\lambda_1,\lambda_2,\lambda_3) = \left(\frac{1}{R_b^k}, \frac{1}{R_c^k}, \frac{1}{R_a^k} \right)$$

$$(\lambda_1,\lambda_2,\lambda_3) = \left(\frac{1}{R_c^k}, \frac{1}{R_a^k}, \frac{1}{R_b^k} \right), (\lambda_1,\lambda_2,\lambda_3) = \left(\frac{1}{R_c^k}, \frac{1}{R_b^k}, \frac{1}{R_a^k} \right)$$

Corollary 2.4. Let be M an interior point of triangle ABC. From every real number $k \in [0,1]$, we have:
a).

$$\frac{R_1^k}{R_b^k} + \frac{R_2^k}{R_c^k} + \frac{R_3^k}{R_a^k} \leq \frac{R_b^k}{R_c^k} + \frac{R_c^k}{R_a^k} + \frac{R_a^k}{R_b^k}$$

b).

$$\frac{R_1^k}{R_b^k}\sqrt{\frac{R_a^k}{R_c^k}} + \frac{R_2^k}{R_c^k}\sqrt{\frac{R_b^k}{R_a^k}} + \frac{R_3^k}{R_a^k}\sqrt{\frac{R_c^k}{R_b^k}} \leq \frac{R_b^k}{R_a^k} + \frac{R_c^k}{R_b^k} + \frac{R_a^k}{R_c^k}$$

c).

$$\frac{R_1^k}{R_b^k}+\frac{R_2^k}{R_c^k}\sqrt{\frac{R_b^k}{R_a^k}}+\frac{R_3^k}{\sqrt{R_b^kR_a^k}}\le\frac{R_b^k}{R_c^k}+\frac{R_c^k}{R_b^k}+1$$

and similar inequalities
d).

$$\frac{R_1^k}{\sqrt{R_b^kR_c^k}}+\frac{R_2^k}{\sqrt{R_c^kR_a^k}}+\frac{R_3^k}{\sqrt{R_a^kR_b^k}}\le 3$$

Proof. We take in Theorem 2:

$$(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_a^k},\frac{1}{R_b^k},\frac{1}{R_c^k}\right),(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_a^k},\frac{1}{R_b^k},\frac{1}{R_c^k}\right)$$
$$(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_b^k},\frac{1}{R_a^k},\frac{1}{R_c^k}\right),(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_b^k},\frac{1}{R_c^k},\frac{1}{R_a^k}\right)$$
$$(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_c^k},\frac{1}{R_a^k},\frac{1}{R_b^k}\right),(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_c^k},\frac{1}{R_b^k},\frac{1}{R_a^k}\right)$$

Corollary 2.5. Let be M an interior point of triangle ABC. Then for every real number $k\in[0,1]$ we have
a).

$$R_1^{2k}R_c^k+R_2^{2k}R_a^k+R_3^{2k}R_b^k\ge R_1^{2k}R_3^k+R_2^{2k}R_1^k+R_3^{2k}R_2^k$$

b).

$$\frac{R_1^{2k}R_3^k}{\sqrt{R_b^kR_c^k}}+\frac{R_2^{2k}R_1^k}{\sqrt{R_c^kR_a^k}}+\frac{R_3^{2k}R_2^k}{\sqrt{R_a^kR_b^k}}\le R_1^{2k}+R_2^{2k}+R_3^{2k}$$

c).

$$R_1^{2k}R_3^k\sqrt{\frac{R_a^k}{R_b^k}}+R_2^{2k}R_1^k\sqrt{\frac{R_b^k}{R_c^k}}+R_3^{2k}R_2^k\sqrt{\frac{R_c^k}{R_a^k}}\le R_2^{2k}R_b^k+R_3^{2k}R_c^k+R_1^{2k}R_a^k$$

d).

$$R_1^{2k}R_2^k\sqrt{\frac{R_a^k}{R_b^k}}+R_2^{2k}R_3^k\sqrt{\frac{R_b^k}{R_c^k}}+R_3^{2k}R_1\sqrt{\frac{R_c^k}{R_a^k}}\le R_3^{2k}R_b^k+R_1^{2k}R_c^k+R_2^{2k}R_a^k$$

Proof. We take

$$(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{R_2^k}{R_1^kR_3^k},\frac{R_3^k}{R_1^kR_2^k},\frac{R_1^k}{R_2^kR_3^k},\right)$$

in Theorem 1 and 2

Corollary 2.6. Let be M an interior point of triangle ABC. Then for every real number $k\in[0,1]$ we have
a).

$$\frac{R_a^k}{R_2^{2k}}+\frac{R_b^k}{R_3^{2k}}+\frac{R_c^k}{R_1^{2k}}\ge\frac{1}{R_1^k}+\frac{1}{R_2^k}+\frac{1}{R_3^k}$$

b).

$$\frac{1}{R_3^k\sqrt{R_b^kR_c^k}}+\frac{1}{R_1^k\sqrt{R_c^kR_a^k}}+\frac{1}{R_2^k\sqrt{R_a^kR_b^k}}\leq\frac{1}{R_1^{2k}}+\frac{1}{R_2^{2k}}+\frac{1}{R_3^{2k}}$$

Proof. We take in Theorem 1:

$$(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{1}{R_2^{2k}},\frac{1}{R_3^{2k}},\frac{1}{R_1^{2k}}\right)$$

Theorem 3. Let be M a interior point of triangle ABC. Then for every real number $k\in[0,1]$ we have a).

$$\frac{R_aR_bR_c}{R_1R_2R_3}\geq\frac{1}{8^{\frac{1}{k}}}\cdot\frac{\prod\left(b^k+c^k\right)^{1/k}}{abc}\geq 1$$

b).

$$\prod\left(\frac{R_3^k}{R_b^k}+\frac{R_2^k}{R_c^k}\right)\leq\frac{64\left(abc\right)^k}{\prod\left(b^k+c^k\right)}\leq 8$$

Proof. From the inequalities $R_1\geq\frac{c}{a}r_2+\frac{b}{a}r_3$, $R_1\geq\frac{b}{a}r_2+\frac{c}{a}r_3$ and $(x+y)^k\geq 2^{k-1}\left(x^k+y^k\right)$, if $k\in[0,1]$ it result.

$$R_1^k\geq\left(\frac{b}{a}r_2+\frac{c}{a}r_3\right)^k\geq 2^{k-1}\left(\frac{b^k}{a^k}r_2^k+\frac{c^k}{a^k}r_3^k\right)$$

and

$$R_1^k\geq\left(\frac{c}{a}r_2+\frac{b}{a}r_3\right)^k\geq 2^{k-1}\left(\frac{c^k}{a^k}r_2^k+\frac{b^k}{a^k}r_3^k\right)$$

Adding the last two inequalities we obtain

$$2R_1^k\geq 2^{k-1}\left[r_2^k\left(\frac{b^k+c^k}{a^k}\right)+r_3^k\left(\frac{b^k+c^k}{a^k}\right)\right]=\frac{2^{k-1}\left(r_2^k+r_3^k\right)\left(b^k+c^k\right)}{a^k}$$

and the similar inequalities.
By multiplication we obtain:

$$8\left(R_1R_2R_3\right)^k\geq 8^{k-1}\frac{\prod\left(r_2^k+r_3^k\right)\left(b^k+c^k\right)}{a^kb^kc^k}$$

Taking account by equalities $r_2=\frac{R_3R_1}{2R_b}$, $r_3=\frac{R_1R_2}{2R_c}$ we obtain:

$$8\left(R_1R_2R_3\right)^k\geq 8^{k-1}\prod\left(\frac{R_3^kR_1^k}{2^kR_b^k}+\frac{R_1^kR_2^k}{2^kR_c^k}\right)\frac{\prod\left(b^k+c^k\right)}{a^kb^kc^k}$$

or

$$64\geq\prod\left(\frac{R_3^k}{R_b^k}+\frac{R_2^k}{R_c^k}\right)\frac{\prod\left(b^k+c^k\right)}{a^kb^kc^k}\geq 8\prod\left(\frac{R_3^k}{R_b^k}+\frac{R_2^k}{R_c^k}\right)$$

or

$$\prod\left(\frac{R_3^k}{R_b^k}+\frac{R_2^k}{R_c^k}\right)\leq\frac{64a^kb^kc^k}{\prod\left(b^k+c^k\right)}\leq 8$$

b). We have

$$64\geq\prod\left(\frac{R_3^k}{R_b^k}+\frac{R_2^k}{R_c^k}\right)\frac{\prod\left(b^k+c^k\right)}{a^kb^kc^k}\geq 8\frac{\prod\left(b^k+c^k\right)}{a^kb^kc^k}\cdot\frac{\left(R_1R_2R_3\right)^k}{\left(R_aR_bR_c\right)^k}.$$

It result that

$$\frac{R_aR_bR_c}{R_1R_2R_3} \geq \frac{1}{8^{1/k}} \cdot \frac{\prod \left(b^k + c^k\right)^{1/k}}{abc} \geq 1$$

this inequality represent a refinement of inequality

$$R_aR_bR_c \geq R_1R_2R_3$$

Theorem 4. Let be M a interior point of triangle ABC. Then for every real number $k \in [0,1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have
a).

$$2^k \left(\lambda_1 R_a^k \sin^k \frac{A}{2} + \lambda_2 R_b^k \sin^k \frac{B}{2} + \lambda_3 R_c^k \sin^k \frac{C}{2}\right) \geq \sqrt{\lambda_2\lambda_3}R_1^k + \sqrt{\lambda_1\lambda_3}R_2^k + \sqrt{\lambda_1\lambda_2}R_3^k$$

b).

$$2^k \left(\lambda_1 \sin^k \frac{A}{2} + \lambda_2 \sin^k \frac{B}{2} + \lambda_3 \sin^k \frac{C}{2}\right) \geq \sqrt{\lambda_2\lambda_3}\frac{R_1^k}{\sqrt{R_b^kR_c^k}} + \sqrt{\lambda_3\lambda_1}\frac{R_2^k}{\sqrt{R_c^kR_a^k}} +$$

$$+\sqrt{\lambda_1\lambda_2}\frac{R_3^k}{\sqrt{R_a^kR_b^k}} \geq 2\left(\sqrt{\lambda_2\lambda_3}\frac{R_1^k}{R_b^k + R_c^k} + \sqrt{\lambda_3\lambda_1}\frac{R_2^k}{R_c^k + R_a^k} + \sqrt{\lambda_1\lambda_2}\frac{R_3^k}{R_a^k + R_b^k}\right)$$

Proof. From Lemma 2 and inequality $(x+y)^k \geq 2^{k-1}\left(x^k + y^k\right)$, $k \in [0,1]$ it result:

$$4^k \sin^k \frac{A}{2} \geq \left(\frac{R_2}{R_c} + \frac{R_3}{Rb}\right)^k \geq 2^{k-1}\left(\frac{R_2^k}{R_c^k} + \frac{R_3^k}{R_b^k}\right)$$

or

$$2^{k+1} \sin^k \frac{A}{2} \geq \frac{R_2^k}{R_c^k} + \frac{R_3^k}{R_b^k}$$

or

$$2^{k+1}\lambda_1 R_a^k \sin^k \frac{A}{2} \geq \lambda_1 \frac{R_a^k}{R_c^k}R_2^k + \lambda_1 \frac{R_a^k}{R_b^k}R_3^k$$

and the similar inequalities

$$2^{k+1}\lambda_2 R_b^k \sin^k \frac{B}{2} \geq \lambda_2 \frac{R_b^k}{R_a^k}R_3^k + \lambda_2 \frac{R_b^k}{R_c^k}R_1^k$$

$$2^{k+1}\lambda_3 R_c^k \sin^k \frac{C}{2} \geq \lambda_3 \frac{R_c^k}{R_b^k}R_1^k + \lambda_3 \frac{R_c^k}{R_a^k}R_2^k$$

If we add this inequalities we obtain:

$$2^{k+1}\left(\lambda_1 R_a^k \sin^k \frac{A}{2} + \lambda_2 R_b^k \sin^k \frac{B}{2} + \lambda_3 R_c^k \sin^k \frac{C}{2}\right) \geq R_1^k\left(\lambda_2\frac{R_b^k}{R_c^k} + \lambda_3\frac{R_c^k}{R_b^k}\right) +$$

$$+R_2^k\left(\lambda_1\frac{R_a^k}{R_c^k} + \lambda_3\frac{R_c^k}{R_a^k}\right) + R_3^k\left(\lambda_1\frac{R_a^k}{R_b^k} + \lambda_2\frac{R_c^k}{R_a^k}\right) \geq 2\sqrt{\lambda_2\lambda_3}R_1^k + 2\sqrt{\lambda_1\lambda_3}R_2^k + 2\sqrt{\lambda_1\lambda_2}R_3^k$$

b). Adding the inequalities

$$2^{k+1}\lambda_1 \sin^k \frac{A}{2} \geq \lambda_1\frac{R_2^k}{R_c^k} + \lambda_1\frac{R_3^k}{R_b^k}$$

and

$$2^{k+1}\lambda_2 \sin^k \frac{B}{2} \geq \lambda_2 \frac{R_3^k}{R_a^k} + \lambda_2 \frac{R_1^k}{R_c^k}$$

$$2^{k+1}\lambda_3 \sin^k \frac{C}{2} \geq \lambda_3 \frac{R_1^k}{R_b^k} + \lambda_3 \frac{R_2^k}{R_a^k}$$

we obtain

$$2^{k+1}\left(\lambda_1 \sin^k \frac{A}{2} + \lambda_2 \sin^k \frac{B}{2} + \lambda_3 \sin^k \frac{C}{2}\right) \geq \left(\lambda_3 \frac{R_2^k}{R_a^k} + \lambda_1 \frac{R_2^k}{R_c^k}\right) + \left(\lambda_3 \frac{R_1^k}{R_b^k} + \lambda_2 \frac{R_1^k}{R_c^k}\right) +$$

$$+\left(\lambda_2 \frac{R_3^k}{R_a^k} + \lambda_1 \frac{R_3^k}{R_b^k}\right) \geq 2\sqrt{\lambda_1\lambda_3}\frac{R_2^k}{\sqrt{R_a^k R_c^k}} + 2\sqrt{\lambda_3\lambda_2}\frac{R_1^k}{\sqrt{R_c^k R_b^k}} + 2\sqrt{\lambda_2\lambda_1}\frac{R_3^k}{\sqrt{R_b^k R_a^k}} \geq$$

$$\geq 4\left(\sqrt{\lambda_1\lambda_3}\frac{R_2^k}{R_a^k + R_c^k} + \sqrt{\lambda_3\lambda_2}\frac{R_1^k}{R_c^k + R_b^k} + \sqrt{\lambda_2\lambda_1}\frac{R_3^k}{R_b^k + R_a^k}\right)$$

Corollary 4.1 Let be M an interior point of triangle ABC. Then for every real number $k \in [0,1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. We have:
a).

$$\sqrt{\lambda_2\lambda_3}\frac{R_1^k}{\sqrt{R_b^k R_c^k}} + \sqrt{\lambda_3\lambda_1}\frac{R_2^k}{\sqrt{R_c^k R_a^k}} + \sqrt{\lambda_1\lambda_2}\frac{R_3^k}{\sqrt{R_a^k R_b^k}} \leq 2^k \sin^k \frac{\lambda_1 A + \lambda_2 B + \lambda_3 C}{2}$$

b).

$$\frac{R_1^k}{\sqrt{R_b^k R_c^k}} + \frac{R_2^k}{\sqrt{R_c^k R_a^k}} + \frac{R_3^k}{\sqrt{R_a^k R_b^k}} \leq 3\left(\frac{8R+2r}{9R}\right)^{\frac{k}{2}} \leq 3$$

Proof. We consider the function $f:(0,\pi) \to R$, $f(x) = \sin^k \frac{x}{2}$ with $f''(x) = \frac{k}{4}\sin^{k-2}\frac{x}{2}\left(k\cos^2\frac{x}{2} - 1\right)$. But $k\cos^2\frac{x}{2} - 1 \leq \cos^2\frac{x}{2} - 1 = -\sin^2\frac{x}{2} < 0$ it result that f is a concave function. From Jensen inequality and Theorem 4 a it result a).
b). From Jensen inequality we have

$$\sin^k \frac{A}{2} + \sin^k \frac{B}{2} + \sin^k \frac{C}{2} \leq \frac{1}{3^{k-1}}\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)^k$$

In [6] is proved the inequality

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \leq \sqrt{\frac{4R+r}{2R}}$$

It result that

$$\sin^k \frac{A}{2} + \sin^k \frac{B}{2} + \sin^k \frac{C}{2} \leq \left(\frac{4R+r}{2R}\right)^{k/2}\frac{1}{3^{k-1}}$$

From Theorem 4 b). we have for $\lambda_1 = \lambda_2 = \lambda_3$

$$\frac{R_1^k}{\sqrt{R_b^k R_c^k}} + \frac{R_2^k}{\sqrt{R_c^k R_a^k}} + \frac{R_3^k}{\sqrt{R_a^k R_b^k}} \leq \frac{2^k}{3^{k-1}}\left(\sin^k \frac{A}{2} + \sin^k \frac{B}{2} + \sin^k \frac{C}{2}\right) \leq$$

$$\leq \left(\frac{4R+r}{2R}\right)^{\frac{k}{2}}\frac{1}{3^{k-1}} \cdot 2^k$$

Corollary 4.2

$$2^k\left(\lambda_1^2R_a^k\sin^k\frac{A}{2}+\lambda_2^2R_b^k\sin^k\frac{B}{2}+\lambda_3^2R_c^k\sin^k\frac{C}{2}\right)\geq\lambda_1\lambda_2\lambda_3\left(\frac{R_1^k}{\lambda_1}+\frac{R_2^k}{\lambda_2}+\frac{R_3^k}{\lambda_3}\right)$$

Proof. It result from Theorem 4a). taking $\lambda_1\to\lambda_1^2$, $\lambda_2\to\lambda_2^2$, $\lambda_3\to\lambda_3^2$.

Theorem 5. Let be M a interior point of triangle ABC. Then for every real number $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$, we have
a).

$$2^k\left(\lambda_1R_b^k\sin^k\frac{A}{2}+\lambda_2R_c^k\sin^k\frac{B}{2}+\lambda_3R_a^k\sin^k\frac{C}{2}\right)\geq$$

$$\sqrt{\lambda_2\lambda_3}R_1^k\sqrt{\frac{R_a^k}{R_b^k}}+\sqrt{\lambda_3\lambda_1}R_2^k\sqrt{\frac{R_b^k}{R_c^k}}+\sqrt{\lambda_1\lambda_2}R_3^k\sqrt{\frac{R_c^k}{R_a^k}}$$

b).

$$2^k\left(\lambda_2R_b^k\sin^k\frac{A}{2}+\lambda_3R_c^k\sin^k\frac{B}{2}+\lambda_1R_a^k\sin^k\frac{C}{2}\right)\geq$$

$$\geq\sqrt{\lambda_3\lambda_1}R_1^k\sqrt{\frac{R_a^k}{R_b^k}}+\sqrt{\lambda_1\lambda_2}R_2^k\sqrt{\frac{R_b^k}{R_c^k}}+\sqrt{\lambda_2\lambda_3}R_3^k\sqrt{\frac{R_c^k}{R_a^k}}$$

Proof. a). We take in Theorem 4 $\lambda_1\to\lambda_1R_b^k$, $\lambda_2\to\lambda_2R_c^k$, $\lambda_3\to\lambda_3R_a^k$
b). We take in Theorem 4 $\lambda_1\to\lambda_2R_b^k$, $\lambda_2\to\lambda_3R_c^k$, $\lambda_3\to\lambda_1R_a^k$

Corollary 5.1. Let be M a interior point of triangle ABC. Then for every real number $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$, we have
a).

$$2^k\left(R_a^k\sin^k\frac{A}{2}+R_b^k\sin^k\frac{B}{2}+R_c^k\sin^k\frac{C}{2}\right)\geq R_1^k+R_2^k+R_3^k$$

b).

$$2^\lambda\left(\sin^k\frac{A}{2}+\sin^k\frac{B}{2}+\sin^k\frac{C}{2}\right)\geq\frac{R_1^k}{\sqrt{R_b^kR_c^k}}+\frac{R_2^k}{\sqrt{R_c^kR_a^k}}+\frac{R_3^k}{\sqrt{R_a^kR_b^k}}\geq$$

$$\geq 2\left(\frac{R_1^k}{R_b^k+R_c^k}+\frac{R_2^k}{R_c^k+R_a^k}+\frac{R_3^k}{R_a^k+R_b^k}\right)$$

c).

$$2^k\left(R_b^k\sin^k\frac{A}{2}+R_c^k\sin^k\frac{B}{2}+R_a^k\sin^k\frac{C}{2}\right)\geq R_1^k\sqrt{\frac{R_a^k}{R_b^k}}+R_2^k\sqrt{\frac{R_b^k}{R_c^k}}+R_3^k\sqrt{\frac{R_c^k}{R_a^k}}$$

Proof. It result from Theorem 4 and 5 taking $\lambda_1=\lambda_2=\lambda_3$. If we take in theorem 5 $\lambda_1=\frac{1}{R_c^k}$, $\lambda_2=\frac{1}{R_a^k}$, $\lambda_3=\frac{1}{R_b^k}$, we obtain the following Corollary.

Corollary 5.2. Let be M an interior point of triangle ABC. Then we have:
a).

$$2^k\left(\frac{R_b^k}{R_c^k}\sin^k\frac{A}{2}+\frac{R_c^k}{R_a^k}\sin^k\frac{B}{2}+\frac{R_a^k}{R_b^k}\sin^k\frac{C}{2}\right)\geq\frac{R_1^k}{R_b^k}+\frac{R_2^k}{R_c^k}+\frac{R_3^k}{R_a^k}$$

b).

$$2^k\left(\frac{R_b^k}{R_a^k}\sin^k\frac{A}{2}+\frac{R_c^k}{R_b^k}\sin^k\frac{B}{2}+\frac{R_a^k}{R_c^k}\sin^k\frac{C}{2}\right)\geq$$

$$\geq\frac{R_1^k}{R_b^k}\sqrt{\frac{R_a^k}{R_c^k}}+\frac{R_2^k}{R_c^k}\sqrt{\frac{R_b^k}{R_a^k}}+\frac{R_3^k}{R_a^k}\sqrt{\frac{R_c^k}{R_b^k}}$$

Theorem 6. Let be M a interior point of triangle ABC. Then for every $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$, we have

$$4\left(R_1^kR_2^k\sin^k\frac{A}{2}\sin^k\frac{B}{2}+R_2^kR_3^k\sin^k\frac{B}{2}\sin^k\frac{C}{2}+R_3^kR_1^k\sin^k\frac{C}{2}\sin^k\frac{A}{2}\right)\geq$$

$$\geq r_1^{2k}+r_2^{2k}+r_3^{2k}+3\left(r_1^kr_2^k+r_2^kr_3^k+r_3^kr_1^k\right)$$

Proof. From Lemma 2 and inequality $(x+y)^k\geq 2^{k-1}\left(x^k+y^k\right)$, for $k\in[0,1]$ it result

$$2^kR_1^k\sin^k\frac{A}{2}\geq(r_2+r_3)^k\geq 2^{k-1}\left(r_2^k+r_3^k\right)$$

or

$$2R_1^k\sin^k\frac{A}{2}\geq r_2^k+r_3^k$$

Also we have

$$2R_2^k\sin^k\frac{B}{2}\geq r_3^k+r_1^k$$

It result that

$$4R_1^kR_2^k\sin^k\frac{A}{2}\sin^k\frac{B}{2}\geq\left(r_2^k+r_3^k\right)\left(r_3^k+r_1^k\right)$$

and the similar inequalities.
By adding we obtain the inequality from the statement.

Theorem 7. Let be M a interior point of triangle ABC. Then for every $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$, we have

$$\frac{R_1R_2R_3}{r_1r_2r_3}\geq\frac{\prod\left(r_2^k+r_3^k\right)^{1/k}}{2^{\frac{3-2k}{k}}r_1r_2r_3}\cdot\frac{R}{r}\geq 1$$

Proof. If we multiply the inequality $2R_1^k\sin^k\frac{A}{2}\geq r_2^k+r_3^k$ and the similar inequalities we obtain:

$$8R_1^kR_2^kR_3^k\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^k\geq\prod\left(r_2^k+r_3^k\right)\geq 8\prod\left(r_1r_2r_3\right)^k$$

or

$$\left(\frac{R_1R_2R_3}{r_1r_2r_3}\right)^k\geq\frac{\prod\left(r_2^k+r_3^k\right)}{8\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^k\left(r_1r_2r_3\right)^k}\geq 1$$

or

$$\frac{R_1R_2R_3}{r_1r_2r_3}\geq\frac{\prod\left(r_2^k+r_3^k\right)^{1/k}}{8^{\frac{1}{k}}r_1r_2r_3}\cdot\frac{4R}{R}\geq 1$$

Corollary 7.1. Let M be an interior point of triangle ABC, k a real number $k\in[0,1]$. Then we have:

$$\frac{R_1^k}{r_2^k+r_3^k}+\frac{R_2^k}{r_3^k+r_1^k}+\frac{R_3^k}{r_1^k+r_2^k}\geq 3\cdot 2^{\frac{2k-3}{3}}\left(\frac{R}{r}\right)^{\frac{k}{3}}\geq 3\cdot 2^{k-1}$$

Proof. Using the A-M inequality and Theorem 7, it result that:

$$\sum\frac{R_1^k}{r_2^k+r_3^k}\geq 3\sqrt[3]{\frac{(R_1R_2R_3)^k}{\prod\left(r_2^k+r_3^k\right)}}\geq 3\sqrt[3]{\left(\frac{4R}{r}\right)^k\cdot\frac{1}{8}}\geq 3\sqrt[3]{8^{k-1}}$$

Theorem 8. Let M be an interior point of triangle ABC, $\lambda_1,\lambda_2\lambda_3\geq 0$. Then we have

$$(aR_1)^{2\lambda_1}(bR_2)^{2\lambda_2}(cR_3)^{2\lambda_3}\geq 4^{\lambda_1+\lambda_2+\lambda_3}(ar_1)^{\lambda_2+\lambda_3}(br_2)^{\lambda_3+\lambda_1}(cr_3)^{\lambda_1+\lambda_2}$$

Proof. As

$$R_1^{\lambda_1}\geq 2^{\lambda_1-1}\left[\left(\frac{b}{a}\right)^{\lambda_1}r_2^{\lambda_1}+\left(\frac{c}{a}\right)^{\lambda_1}r_3^{\lambda_2}\right]$$

we obtain

$$(aR_1)^{2\lambda_1}\geq 4^{\lambda_1-1}\left(b^{\lambda_1}r_2^{\lambda_1}+c^{\lambda_1}r_3^{\lambda_1}\right)^2\geq 4^{\lambda_1}(br_2)^{\lambda_1}(cr_3)^{\lambda_1}$$

After multiplication with other two cyclic inequalities we obtain:

$$(aR_1)^{2\lambda_1}\cdot(bR_2)^{2\lambda_2}\cdot(cR_3)^{2\lambda_3}\geq 4^{\lambda_1+\lambda_2+\lambda_3}(ar_1)^{\lambda_2+\lambda_3}(br_2)^{\lambda_3+\lambda_1}(cr_3)^{\lambda_1+\lambda_2}$$

Corollary 8. Let be M a interior point of triangle ABC. Then for every $k\in[0,1]$, $\lambda_1,\lambda_2,\lambda_3\geq 0$, we have

$$\lambda_1R_1^k+\lambda_2R_2^k+\lambda_3R_3^k\geq 2^k\left(\sqrt{\lambda_2\lambda_3}r_1^k+\sqrt{\lambda_3\lambda_1}r_2^k+\sqrt{\lambda_1\lambda_2}r_3^k\right)$$

Proof. We have

$$R_1^k\geq\left(\frac{c}{a}r_2+\frac{b}{a}r_3\right)^k\geq 2^{k-1}\left(\frac{c^k}{a^k}r_2^k+\frac{b^k}{a^k}r_3^k\right)$$

and the like one. Adding this inequality we obtain:

$$\lambda_1R_1^k+\lambda_2R_2^k+\lambda_3R_3^k\geq$$

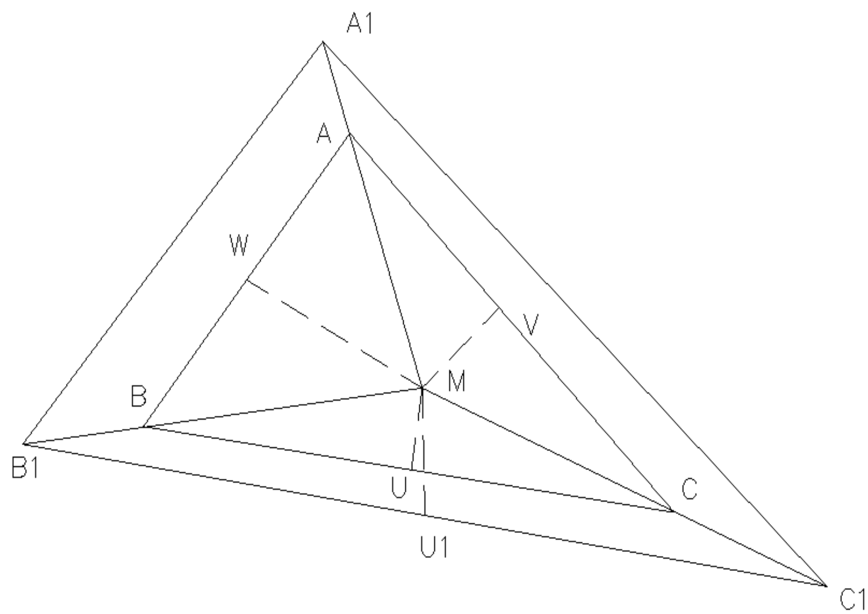
$$\geq 2^{k-1}\left(\lambda_1\frac{c^k}{a^k}r_2^k+\lambda_1\frac{b^k}{a^k}r_3^k+\lambda_2\frac{a^k}{b^k}r_3^k+\lambda_2\frac{c^k}{b^k}r_1^k+\lambda_3\frac{b^k}{c^k}r_1^k+\lambda_3\frac{a^k}{c^k}r_2^k\right)=$$

$$=2^{k-1}\left[r_1^k\left(\lambda_2\frac{c^k}{b^k}+\lambda_3\frac{b^k}{c^k}\right)+r_2^k\left(\lambda_1\frac{c^k}{a^k}+\lambda_3\frac{a^k}{c^k}\right)+r_3^k\left(\lambda_1\frac{b^k}{a^k}+\lambda_2\frac{a^k}{b^k}\right)\right]\geq$$

$$\geq 2^{k-1}\left(\sqrt{\lambda_2\lambda_3}r_1^k+\sqrt{\lambda_3\lambda_1}r_2^k+\sqrt{\lambda_1\lambda_2}r_3^k\right)$$

In the following we consider the inversion of pol M and ratio t, $i_M^t:\pi\to\pi$.
Let be $A_1=i_M^t(A),B_1=i_M^t(B),C_1=i_M^t(C);$ and U,V,W,U_1 the projection of the point M on the sides BC,AC,AB,B_1C_1.

We denote $A_1M = R_1^{'}$



As

$$MA \cdot MA_1 = MB \cdot MB_1 = MC \cdot MC_1 = t$$

it result that

$$R_1' = \frac{t}{R_1} \tag{1}$$

We have $\triangle BMU \sim \triangle MU_1C_1$. Then $\frac{MU}{MU_1} = \frac{BM}{MC_1}$ or

$$\frac{r_1}{r_1'} = \frac{R_2}{R_3'} \text{ or } \frac{r_1}{r_1'} = \frac{R_2R3}{t} \text{ or } r_1' = \frac{tr_1}{R_2R3} \tag{2}$$

$$R_a' = \frac{R_2^{'}R_3'}{2r_1'} = \frac{\frac{t^2}{R_2R_3}}{\frac{2tr_1}{R_2R_3}} = \frac{t}{2r_1} \text{ or } R_a' = \frac{t}{2r_1} \tag{3}$$

Also $\triangle BMC \sim \triangle B_1MC_1$. It result that

$$\frac{BC}{B_1C_1} = \frac{BM}{MC_1}$$

so we have:

$$\frac{a}{a'} = \frac{R_2}{R_3'} \text{ or } a' = \frac{aR_3'}{R_2} = \frac{at}{R_3R_2}$$

Then

$$a' = \frac{at}{R_2R_3} \tag{4}$$

We calculate the area of the triangle $A_1B_1C_1$

$$S' = S_{\triangle B_1MC_1} + S_{\triangle B_1MA_1} + S_{\triangle A_1MC_1} = \sum \frac{a'r_1'}{2} = \frac{1}{2}\sum \frac{at}{R_2R_3} \cdot \frac{kr_1}{R_2R_3} =$$

$$= \frac{1}{2}t^2 \sum \frac{ar_1}{R_2^2R_3^2} = \frac{1}{2} \cdot \frac{t^2}{R_1^2R_2^2R_3^2} \sum aR_1^2r_1 \tag{5}$$

We calculate the semiperimeter p', the radius of incircle r' and the radius of circumscrible R'

$$p' = \frac{1}{2}t \sum \frac{a}{R_2R_3} \text{ or } p' = \frac{1}{2} \cdot \frac{t}{R_1R_2R_3} \sum ar_1 \tag{6}$$

$$r' = \frac{s'}{p'} = \frac{t}{R_1R_2R_3} \cdot \frac{\sum aR_1^2r_1}{\sum aR_1} \tag{7}$$

$$R' = \frac{a'b'c'}{4s} = \frac{abct}{2\sum aR_1^2r_1} \tag{8}$$

If α, β, γ represent the baricentric coordinates of point M in triangle ABC and α', β', γ' represent the baricentric coordinates of point M in triangle $A_1B_1C_1$, then

$$\alpha' = \frac{a'r_1'}{2s'} = \frac{ar_1R_1^2}{\sum aR_1^2r_1} \tag{9}$$

Also

$$\sin A' = \frac{a'}{2R'} = \frac{\sum aR_1^2r_1}{bcR_2R_3} \tag{10}$$

Theorem 9. Let be M a interior point of triangle ABC. Then for every $k \in [0,1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have

$$\lambda_1R_2^kR_3^k + \lambda_2R_3^kR_1^k + \lambda_3R_1^kR_2^k \geq 2^t\left(\sqrt{\lambda_2\lambda_3}R_1^kr_1^k + \sqrt{\lambda_3\lambda_1}R_2^kr_2^k + \sqrt{\lambda_1\lambda_2}R_3^kr_3^k\right)$$

Proof. From Corollary 8 it result that:

$$\lambda_1R_1'^k + \lambda_2R_2'^k + \lambda_3R_3'^k \geq 2^k\left(\sqrt{\lambda_2\lambda_3}r_1'^k + \sqrt{\lambda_3\lambda_1}r_2'^k + \sqrt{\lambda_1\lambda_2}r_3'^k\right)$$

or

$$\lambda_1\frac{t^k}{R_1^k} + \lambda_2\frac{t^k}{R_2^k} + \lambda_3\frac{t^k}{R_3^k} \geq 2^k\left(\sqrt{\lambda_2\lambda_3}\frac{t^kr_1^k}{R_2^kR_3^k} + \sqrt{\lambda_3\lambda_1}\frac{t^kr_2^k}{R_3^kR_1^k} + \sqrt{\lambda_1\lambda_2}\frac{t^kr_3^k}{R_1^kR_2^k}\right)$$

or

$$\lambda_1R_2^kR_2^k + \lambda_2R_3^kR_1^k + \lambda_3R_1^kR_2^k \geq 2^k\left(\sqrt{\lambda_2\lambda_3}R_1^kr_1^k + \sqrt{\lambda_3\lambda_1}R_2^kr_2^k + \sqrt{\lambda_1\lambda_2}R_3^kr_3^k\right)$$

Theorem 10. Let be M a interior point of triangle ABC. Then for every $k \in [0,1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have

$$\lambda_1R_1^kr_1^k + \lambda_2R_2^kr_2^k + \lambda_3R_3^kr_3^k \geq 2^k\left(\sqrt{\lambda_2\lambda_1}r_2^kr_1^k + \sqrt{\lambda_1\lambda_3}r_1^kr_3^k + \sqrt{\lambda_3\lambda_2}r_3^kr_2^k\right)$$

Proof. We have:

$$R_1^k \geq \left(\frac{b}{a}r_2 + \frac{c}{a}r_3\right)^k \geq 2^{k-1}\left(\frac{b^k}{a^k}r_2^k + \frac{c^k}{a^k}r_3^k\right)$$

or

$$\lambda_1R_1^kr_1^k \geq 2^{k-1}\left(\lambda_1\frac{b^k}{a^k}r_2^kr_1^k + \lambda_1\frac{c^k}{a^k}r_3^kr_1^k\right)$$

and the line one.
Adding this inequalities we obtain

$$\lambda_1 R_1^k r_1^k + \lambda_2 R_2^k r_2^k + \lambda_3 R_3^k r_3^k \geq$$

$$\geq 2^{k-1}\left[\left(\lambda_1 \frac{b^k}{a^k} + \lambda_2 \frac{a^k}{b^k}\right) r_2^k r_1^k + \left(\lambda_1 \frac{c^k}{a^k} + \lambda_3 \frac{a^k}{c^k}\right) r_1^k r_3^k + \left(\lambda_2 \frac{c^k}{b^k} + \lambda_3 \frac{b^k}{c^k}\right) r_3^k r_2^k\right] \geq$$

$$\geq 2^k \left(\sqrt{\lambda_2\lambda_1} r_2^k r_1^k + \sqrt{\lambda_1\lambda_3} r_1^k r_3^k + \sqrt{\lambda_3\lambda_2} r_3^k r_2^k\right)$$

Theorem 11.

$$\lambda_1 R_2^k R_3^k + \lambda_2 R_3^k R_1^k + \lambda_3 R_1^k R_2^k \geq 4^k \sqrt[4]{\lambda_1\lambda_2\lambda_3}\left(\sqrt{\lambda_1} r_2^k r_3^k + \sqrt{\lambda_2} r_3^k r_1^k + \sqrt{\lambda_3} r_1^k r_2^k\right)$$

Proof. Taking in Theorem 10 $\lambda_1 \to \sqrt{\lambda_2\lambda_3}, \lambda_2 \to \sqrt{\lambda_3\lambda_1}, \lambda_3 \to \sqrt{\lambda_1\lambda_2}$, we obtain:

$$2^k\left(\sqrt{\lambda_2\lambda_3} R_1^k r_1^k + \sqrt{\lambda_3\lambda_1} R_2^k r_2^k + \sqrt{\lambda_1\lambda_2} R_3^k r_3^k\right) \geq$$

$$\geq 4^k \left(\sqrt{\lambda_3\sqrt{\lambda_1\lambda_2}} r_2^k r_1^k + \sqrt{\lambda_2\sqrt{\lambda_1\lambda_3}} r_1^k r_3^k + \sqrt{\lambda_1\sqrt{\lambda_2\lambda_3}} r_3^k r_2^k\right)$$

From this inequality and Theorem 9 it result the inequality from the statement.
Corollary 11.1. Let be M an interior point of triangle ABC. Then we have:
a).

$$\left(R_1^k + R_2^k + R_3^k\right)\sqrt[4]{R_1^k R_2^k R_3^k} \geq 4^k \left(\frac{r_2^k r_3^k}{\sqrt{R_2^k}} + \frac{r_3^k r_1^k}{\sqrt{R_3^k}} + \frac{r_1^k r_2^k}{\sqrt{R_1^k}}\right)$$

b).

$$\left(\frac{R_3^k}{R_2^k} + \frac{R_1^k}{R_3^k} + \frac{R_2^k}{R_1^k}\right)\sqrt{R_1^k R_2^k R_3^k} \geq 4^k \left(\frac{r_2^k r_3^k}{R_2^k} + \frac{r_3^k r_1^k}{R_3^k} + \frac{r_1^k r_2^k}{R_1^k}\right)$$

Proof. We take in Theorem 11

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{R_1^k}, \frac{1}{R_2^k}, \frac{1}{R_3^k}\right) \text{ and } (\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{R_1^{2k}}, \frac{1}{R_2^{2k}}, \frac{1}{R_3^{2k}}\right)$$

Theorem 12. Let be M a interior point of triangle ABC. Then for every $k \in [0,1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have
a).

$$\frac{\lambda_1}{r_1^k} + \frac{\lambda_2}{r_2^k} + \frac{\lambda_3}{r_3^k} \geq 2^k \left(\frac{\sqrt{\lambda_2\lambda_3}}{R_1^k} + \frac{\sqrt{\lambda_3\lambda_1}}{R_2^k} + \frac{\sqrt{\lambda_1\lambda_2}}{R_2^k}\right)$$

b).

$$2^{k+1}\left(\sqrt{\lambda_2\lambda_3}\frac{r_2^k r_3^k}{R_1^k\left(r_2^k + r_3^k\right)} + \sqrt{\lambda_3\lambda_1}\frac{r_3^k r_1^k}{R_2^k\left(r_3^k + r_1^k\right)} + \sqrt{\lambda_1\lambda_2}\frac{r_1^k r_2^k}{R_3^k\left(r_1^k + r_2^k\right)}\right) \leq$$

$$\leq \lambda_1 + \lambda_2 + \lambda_3$$

Proof. a). From Theorem 1a). it result that:

$$\lambda_1 R_a'^k + \lambda_2 R_b'^k + \lambda_3 R_c'^k \geq \sqrt{\lambda_2\lambda_3} R_1'^k + \sqrt{\lambda_3\lambda_1} R_2'^k + \sqrt{\lambda_1\lambda_2} R_3'^k$$

or

$$\lambda_1 \frac{t^k}{2^k r_1^k} + \lambda_2 \frac{t^k}{2^k r_2^k} + \lambda_3 \frac{t^k}{2^k r_3^k} \geq \sqrt{\lambda_2\lambda_3}\frac{t^k}{R_1^k} + \sqrt{\lambda_3\lambda_1}\frac{t^k}{R_2^k} + \sqrt{\lambda_1\lambda_2}\frac{t^k}{R_3^k}$$

b). From Theorem 1b) and (1) (2) (3) it result the inequality from the statement.

Theorem 13.

$$\sqrt{\lambda_2\lambda_3}\frac{r_2^k}{R_1^k\left(r_3^k+r_2^k\right)}\sqrt{\frac{r_3^k}{r_1^k}} + \sqrt{\lambda_3\lambda_1}\frac{r_3^k}{R_2^k\left(r_1^k+r_3^k\right)}\sqrt{\frac{r_1^k}{r_2^k}} + \sqrt{\lambda_1\lambda_2}\frac{r_1^k}{R_3^k\left(r_2^k+r_1^k\right)}\sqrt{\frac{r_2^k}{r_3^k}} \leq$$

$$\leq \frac{1}{2}\left(\frac{\sqrt{\lambda_2\lambda_3}}{R_1^k}\cdot\sqrt{\frac{r_2^k}{r_1^k}} + \frac{\sqrt{\lambda_3\lambda_1}}{R_2^k}\cdot\sqrt{\frac{r_3^k}{r_2^k}} + \frac{\sqrt{\lambda_1\lambda_2}}{R_3^k}\cdot\sqrt{\frac{r_1^k}{r_3^k}}\right) \leq \frac{1}{2^{k+1}}\left(\frac{\lambda_1}{r_2^k}+\frac{\lambda_2}{r_3^k}+\frac{\lambda_3}{r_1^k}\right)$$

Proof. From Theorem 2a). and (1) (3) it result the inequality from the statement.

Theorem 13. Let be M a interior point of triangle ABC. Then for every $k \in [0,1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have

$$\sqrt{\lambda_3\lambda_1}\frac{r_2^k}{R_1^k\left(r_2^k+r_3^k\right)}\sqrt{\frac{r_3^k}{r_1^k}} + \sqrt{\lambda_1\lambda_2}\frac{r_3^k}{R_2^k\left(r_3^k+r_1^k\right)}\sqrt{\frac{r_1^k}{r_2^k}} + \sqrt{\lambda_2\lambda_3}\frac{r_1^k}{R_3^k\left(r_1^k+r_2^k\right)}\sqrt{\frac{r_2^k}{r_3^k}} \leq$$

$$\leq \frac{1}{2}\left(\frac{\sqrt{\lambda_3\lambda_1}}{R_1^k}\cdot\sqrt{\frac{r_2^k}{r_1^k}} + \frac{\sqrt{\lambda_1\lambda_2}}{R_2^k}\cdot\sqrt{\frac{r_3^k}{r_2^k}} + \frac{\sqrt{\lambda_2\lambda_3}}{R_3^k}\cdot\sqrt{\frac{r_1^k}{r_3^k}}\right) \leq \frac{1}{2^{k+1}}\left(\frac{\lambda_1}{r_1^k}+\frac{\lambda_2}{r_2^k}+\frac{\lambda_3}{r_3^k}\right)$$

Proof. It result from (1), (3) and Theorem 2b).

Corollary 2.7. Let M an interior point of triangle ABC. Then for every real number $k \in [0,1]$ we have:

a).

$$2^k\left(\frac{1}{R_1^k r_1^k}+\frac{1}{R_2^k r_2^k}+\frac{1}{R_3^k r_3^k}\right) \leq \frac{1}{r_1^k r_2^k}+\frac{1}{r_2^k r_3^k}+\frac{1}{r_3^k r_1^k}$$

b).

$$1+\frac{r_3^k}{r_2^k}+\frac{r_2^k}{r_3^k} \geq 2^k\left(\frac{\sqrt{r_2^k r_3^k}}{R_1^k}+\frac{\sqrt{r_1^k r_2^k}}{R_2^k}+\frac{\sqrt{r_1^k r_3^k}}{R_3^k}\right)$$

c).

$$2^k\left(\frac{1}{R_1^k r_1^k}+\frac{1}{R_2^k\sqrt{r_2^k r_3^k}}+\frac{1}{R_3^k\sqrt{r_2^k r_3^k}}\right) \leq \frac{1}{r_1^k r_2^k}+\frac{1}{r_2^k r_3^k}+\frac{1}{r_3^k r_1^k}$$

d).

$$\frac{r_2^k}{r_1^k}+\frac{r_3^k}{r_2^k}+\frac{r_1^k}{r_3^k} \geq 2^k\left(\frac{\sqrt{r_3^k r_1^k}}{R_1^k}+\frac{\sqrt{r_1^k r_2^k}}{R_2^k}+\frac{\sqrt{r_2^k r_3^k}}{R_3^k}\right)$$

e).

$$2^k\left(\frac{1}{R_1^k\sqrt{r_1^k r_2^k}}+\frac{1}{R_2^k\sqrt{r_2^k r_3^k}}+\frac{1}{R_3^k\sqrt{r_3^k r_1^k}}\right) \leq \frac{1}{r_1^k r_2^k}+\frac{1}{r_2^k r_3^k}+\frac{1}{r_3^k r_2^k}$$

Proof. We use the Corollary 2.3. in triangle $A_1B_1C_1$ and the equalities (1) and (3).

Corollary 2.8. Let be M an interior point of triangle ABC. For every real number $k \in [0,1]$ we have:

a).

$$2^k\left(\frac{r_2^k}{R_1^k}+\frac{r_3^k}{R_2^k}+\frac{r_1^k}{R_3^k}\right)\leq\left(\frac{r_3^k}{r_2^k}+\frac{r_1^k}{r_3^k}+\frac{r_2^k}{r_1^k}\right)$$

b).

$$2^k\left(\frac{r_2^k}{R_1^k}\cdot\sqrt{\frac{r_3^k}{r_1^k}}+\frac{r_3^k}{R_2^k}\cdot\sqrt{\frac{r_1^k}{r_2^k}}+\frac{r_1^k}{R_3^k}\cdot\sqrt{\frac{r_2^k}{r_3^k}}\right)\leq\frac{r_1^k}{r_2^k}+\frac{r_2^k}{r_3^k}+\frac{r_3^k}{r_1^k}$$

c).

$$2^k\left(\frac{r_2^k}{R_1^k}+\frac{r_3^k}{R_2^k}\cdot\sqrt{\frac{r_1^k}{r_2^k}}+\frac{\sqrt{r_2^k r_1^k}}{R_3^k}\right)\leq\frac{r_3^k}{r_2^k}+\frac{r_2^k}{r_3^k}+1$$

d).

$$\frac{\sqrt{r_2^k r_3^k}}{R_1^k}+\frac{\sqrt{r_3^k r_1^k}}{R_2^k}+\frac{\sqrt{r_1^k r_2^k}}{R_3^k}\leq\frac{3}{2^k}$$

Proof. We use the Corollary 2.4. in triangle $A_1B_1C_1$ and the equalities (1) and (3).

Corollary 2.9. Let be M an interior point of triangle ABC. For every real number $k\in[0,1]$ we have:
a).

$$\frac{1}{R_1^{2k}r_3^k}+\frac{1}{R_2^{2k}r_1^k}+\frac{1}{R_3^{2k}r_2^k}\geq 2^k\left(\frac{1}{R_1^{2k}R_3^k}+\frac{1}{R_2^{2k}R_1^k}+\frac{1}{R_3^{2k}R_2^k}\right)$$

b).

$$2^k\left(\frac{\sqrt{r_3^k r_2^k}}{R_1^{2k}R_3^k}+\frac{\sqrt{r_3^k r_1^k}}{R_2^{2k}R_1^k}+\frac{\sqrt{r_1^k r_2^k}}{R_3^{2k}R_2^k}\right)\leq\frac{1}{R_1^{2k}}+\frac{1}{R_2^{2k}}+\frac{1}{R_3^{2k}}$$

c).

$$2^k\left(\frac{1}{R_1^{2k}R_3^k}\sqrt{\frac{r_2^k}{r_1^k}}+\frac{1}{R_2^{2k}R_1^k}\sqrt{\frac{r_3^k}{r_2^k}}+\frac{1}{R_3^{2k}R_2^k}\sqrt{\frac{r_1^k}{r_3^k}}\right)\leq\frac{1}{R_1^{2k}r_1^{2k}}+\frac{1}{R_2^{2k}r_2^{2k}}+\frac{1}{R_3^{2k}r_3^{2k}}$$

d).

$$2^k\left(\frac{1}{R_1^{2k}R_2^k}\sqrt{\frac{r_2^k}{r_1^k}}+\frac{1}{R_2^{2k}R_3^k}\sqrt{\frac{r_3^k}{r_2^k}}+\frac{1}{R_3^{2k}R_1^k}\sqrt{\frac{r_1^k}{r_3^k}}\right)\leq\frac{1}{R_3^{2k}r_2^k}+\frac{1}{R_1^{2k}r_3^k}+\frac{1}{R_2^{2k}r_1^k}$$

e).

$$\frac{R_2^{2k}}{r_1^k}+\frac{R_3^{2k}}{r_2^k}+\frac{R_1^{2k}}{r_3^k}\geq 2^k\left(R_1^k+R_2^k+R_3^k\right)$$

f).

$$2^k\left(R_3^k\sqrt{r_2^k r_3^k}+R_1^k\sqrt{r_3^k r_1^k}+R_2^k\sqrt{r_1^k r_2^k}\right)\leq R_1^{2k}+R_2^{2k}+R_3^{2k}$$

Proof. We use the Corollary 2.5. and 2.6 in triangle $A_1B_1C_1$ and the equalities (1) and (2).

Theorem 15. Let be M an interior point of triangle ABC. Then for every real number $k\in[0,1]$ we have:
a).

$$\frac{abc}{r_1r_2r_3} \geq 8^{\frac{k-1}{k}} \prod \left(\frac{b^k}{R_2^k} + \frac{c^k}{R_3^k} \right) \geq 1$$

b).

$$8^{k-1} \prod \left(\frac{r_2^k}{R_1^k R_3^k} + \frac{r_3^k}{R_1^k R_2^k} \right) \leq \frac{8a^k b^k c^k}{\prod \left(R_1^k b^k + R_2^k c^k \right)} \leq 1$$

Proof. We use the Theorem 3 in triangle $A_1B_1C_1$ and the equalities (1) (2) (3) and (4).

Theorem 16. Let be M a interior point of triangle ABC. Then for every $k \in [0, 1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have
a).

$$\lambda_1 r_1^k + \lambda_2 r_2^k + \lambda_3 r_3^k \geq 2^k \left(\sqrt{\lambda_3 \lambda_1} \frac{r_2^k r_1^k}{R_3^k} + \sqrt{\lambda_1 \lambda_3} \frac{r_1^k r_3^k}{R_2^k} + \sqrt{\lambda_3 \lambda_2} \frac{r_3^k r_2^k}{R_1^k} \right)$$

b).

$$\lambda_1 R_1^k + \lambda_2 R_2^k + \lambda_3 R_3^k \geq 4^k \sqrt[4]{\lambda_1 \lambda_2 \lambda_3} \left(\sqrt{\lambda_1} \frac{r_2^k r_3^k}{R_1^k} + \sqrt{\lambda_2} \frac{r_3^k r_1^k}{R_2^k} + \sqrt{\lambda_3} \frac{r_1^k r_2^k}{R_3^k} \right)$$

Proof. We use Theorem 10 and 11 in triangle $A_1B_1C_1$ and the equalities (1) and (2).

Lemma 3. Let be M an interior point of triangle ABC. Then we have
a).

$$R_1 \geq \frac{b}{a} \cdot \frac{R_2}{R_3} \cdot r_2 + \frac{c}{a} \cdot \frac{R_3}{R_2} \cdot r_3$$

b).

$$\frac{R_2 b}{2R_b} + \frac{R_3 c}{2R_c} \leq a$$

Proof. It result from the inequality
a).

$$R_1' \geq \frac{c'}{a'} r_3' + \frac{b'}{a'} r_2'$$

Theorem 17. Let be M a interior point of triangle ABC. Then for every $k \in [0, 1]$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$, we have
b). We take $r_2 = \frac{R_3 R_1}{2R_b}$ and $r_3 = \frac{R_1 R_2}{2R_c}$.

$$\lambda_1 R_1^k r_1^k + \lambda_2 R_2^k r_2^k + \lambda_3 R_3^k r_3^k \geq 2^k \left(\sqrt{\lambda_1 \lambda_2} \frac{\sqrt{R_1^k R_2^k}}{R_3^k} + \sqrt{\lambda_2 \lambda_3} \frac{\sqrt{R_3^k R_2^k}}{R_1^k} + \sqrt{\lambda_3 \lambda_1} \frac{\sqrt{R_3^k R_1^k}}{R_2^k} \right)$$

Proof. From Lemma 3 and inequality $(x+y)^k \geq 2^{k-1}\left(x^k + y^k\right)$, $k \in [0, 1]$ it result that

$$R_1 r_1 \geq \frac{b}{a} \cdot \frac{R_2}{R_3} \cdot r_1 r_2 + \frac{c}{a} \cdot \frac{R_3}{R_2} \cdot r_3 r_1$$

or

$$\lambda_1 R_1^k r_1^k \geq 2^{k-1} \left(\lambda_1 \frac{b^k}{a^k} \cdot \frac{R_2^k}{R_3^k} \cdot r_1^k r_2^k + \lambda_1 \frac{c^k}{a^k} \cdot \frac{R_3^k}{R_2^k} \cdot r_3^k r_1^k \right)$$

and the like one. Adding this inequalities we obtain

$$\lambda_1 R_1^k r_1^k + \lambda_2 R_2^k r_2^k + \lambda_3 R_3^k r_3^k \geq 2^{k-1}\left[r_1^k r_2^k\left(\lambda_1 \frac{b^k}{a^k}\cdot\frac{R_2^k}{R_3^k} + \lambda_2 \frac{a^k}{b^k}\cdot\frac{R_1^k}{R_3^k}\right) + \right.$$

$$\left. + r_2^k r_3^k\left(\lambda_2 \frac{c^k}{b^k}\cdot\frac{R_3^k}{R_1^k} + \lambda_3 \frac{b^k}{c^k}\cdot\frac{R_2^k}{R_1^k}\right) + r_3^k r_1^k\left(\lambda_1 \frac{c^k}{a^k}\cdot\frac{R_3^k}{R_2^k} + \lambda_3 \frac{a^k}{c^k}\cdot\frac{R_1^k}{R_2^k}\right)\right] \geq$$

$$\geq 2^k\left(r_1^k r_2^k \cdot \frac{\sqrt{R_2^k R_1^k}}{R_3^k}\cdot\sqrt{\lambda_2\lambda_1} + r_2^k r_3^k \cdot \frac{\sqrt{R_3^k R_2^k}}{R_1^k}\cdot\sqrt{\lambda_3\lambda_2} + r_3^k r_1^k \cdot \frac{\sqrt{R_3^k R_1^k}}{R_2^k}\cdot\sqrt{\lambda_1\lambda_3}\right)$$

Theorem 18. Let be M an interior point of triangle ABC. Then for every real number $k \in [0,1]$, we have:

$$\frac{R_1^k a^k}{R_a^k} + \frac{R_2^k b^k}{R_b^k} + \frac{R_3^k c^k}{R_c^k} \leq a^k + b^k + c^k \leq 3\left(\frac{2}{3}\right)^k p^k$$

Proof. From Lemma 3b) we have:

$$a^k \geq \left(\frac{R_2 b}{2R_b} + \frac{R_3 c}{2R_c}\right)^k \geq 2^{k-1}\left(\frac{R_2^k b^k}{2^k R_b^k} + \frac{R_3^k c^k}{2^k R_c^k}\right) = \frac{1}{2}\left(\frac{R_2^k b^k}{R_b^k} + \frac{R_3^k c^k}{R_c^k}\right)$$

and the like one. Adding this inequality we obtain:

$$\frac{R_1^k a^k}{R_a^k} + \frac{R_1^k a^k}{R_b^k} + \frac{R_3^k c^k}{R_c^k} \leq a^k + b^k + c^k$$

The part two of inequality it result from Jensen inequality.

Theorem 18. Let be M an interior point of triangle ABC. Then for every number $k \in [0,1]$, we have

$$2^k\left(a^k r_1^k + b^k r_2^k + c^k r_3^k\right) \leq a^k R_1^k + b^k R_2^k + c^k R_3^k$$

Proof. From inequality Theorem 18 and equalitites (1) (3) and (4) we have

$$\frac{R_1'^k a^k}{R_a'^k} + \frac{R_2'^k b'^k}{R_b'^k} + \frac{R_3'^k c'^k}{R_c'^k} \leq a'^k + b'^k + c'^k$$

who is equivalent with the inequality from the statement.

2 APPLICATIONS

Let be M an interior point of triangle ABC. Then we have

2.1. $aR_1 + bR_2 > cR_3$

2.2. $a^2R_1^2 + b^2R_2^2 + c^2R_3^2 \geq 2\sqrt{3}\left(aR_1^2 r_1 + bR_2^2 r_2 + cR_3^2 r_3\right)$

2.3. $a^2R_1^2 + b^2R_2^2 + c^2R_3^2 \geq 4\sqrt{3}R\left(ar_2r_3 + br_3r_1 + cr_1r_2\right)$

2.4. $a^2R_1^2 + b^2R_2^2 + c^2R_3^2 \geq 4\sqrt{3}R\left(ar_2r_3 + br_3r_1 + cr_1r_2\right) + \left(aR_1 - bR_2\right)^2 + \left(bR_2 - cR_3\right)^2 + \left(cR_3 - aR_1\right)^2$

2.5. $a^2R_1^2 + b^2R_2^2 + c^2R_3^2 \leq$

$\leq 4\sqrt{3}R\left(ar_2r_3 + br_3r_1 + cr_1r_2\right) + 3\left[\left(aR_1 - bR_2\right)^2 + \left(bR_2 - cR_3\right)^2 + \left(cR_3 - aR_1\right)^2\right]$

2.6. $abcR_1R_2R_3\left(aR_1 + bR_2 + cR_3\right) \geq 16R^2\left(ar_2r_3 + br_3r_1 + cr_1r_2\right)^2$

2.7. $4R\left(aR_1 + bR_2 + cR_3\right)\left(ar_2r_3 + br_3r_1 + cr_1r_2\right) \leq 3\sqrt{3}abcR_1R_2R_3$

2.8. $ar_2r_3 + br_3r_1 + cr_1r_2 \leq RS$

2.9. $abR_1R_2 + bcR_2R_3 + caR_3R_1 \geq 4\sqrt{3}R\left(ar_2r_3 + br_3r_1 + cr_1r_2\right)$

2.10. $R_1^2 + R_2^2 + R_3^2 \leq \max\left\{\frac{a^2b^2}{R_3^2}, \frac{b^2c^2}{R_1^2}, \frac{c^2a^2}{R_2^2}\right\}$

2.11. $4R^2\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right)\left(ar_2r_3 + br_3r_1 + cr_1r_2\right)^2 \leq 3\left(R_1^2r_1^2 + R_2^2r_2^2 + R_3^2r_3^2\right)$

2.12. $\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} \geq \frac{\left(\sum ar_1\right)^2}{2R\sum ar_2r_3}$

Proof.

2.1. As $a'+b'>c'$ it result that

$$\frac{at}{R_2R_3}+\frac{bt}{R_3R_1}>\frac{ct}{R_1R_2} \text{ or } aR_1+bR_2>cR_3$$

2.2. From inequality Weisenbock applied in triangle $A_1B_1C_1$ we obtain

$$\sum a'^2\geq 4\sqrt{3}S' \text{ or } t^2\sum\frac{a^2}{R_2^2R_3^2}\geq 4\sqrt{3}\frac{1}{2}t^2\frac{1}{R_1^1R_2^2R_3^2}\sum aR_1^2r_1 \text{ or } \sum a^2R_1^2\geq 2\sqrt{3}\sum aR_1^2r_1$$

2.3. We prove that $\sum ar_1R_1^2=2R\sum ar_2r_3$. Let be α,β,γ the boricentric coordinates of interior point M. Then

$$R_1^2=-\beta\gamma a^2-(\alpha-1)\gamma b^2-(\alpha-1)\beta c^2$$

and $\alpha=\frac{ar_1}{2s}$, $\beta=\frac{br_2}{2s}$, $\gamma=\frac{cr_3}{2s}$, $r_3=\frac{2\gamma s}{c}$. It result that

$$R_1^2=-\frac{bcr_2r_3}{4s^2}a^2-\frac{(ar_1-2s)\,cr_3}{4s^2}b^2-\frac{(ar_1-2s)\,br_2}{4s^2}c^2$$

or

$$R_1^2=\frac{bc}{4s^2}\left[-r_2r_3a^2-(ar_1-2s)\,br_3-(ar_1-2s)\,cr_2\right]=$$

$$=\frac{bc}{4s^2}\left[-a^2r_2r_3-abr_1r_3-acr_1r_2+2s\,(br_3+cr_2)\right]=$$

$$=\frac{abc}{4s^2}\left[-ar_2r_3-br_1r_3-cr_1r_2+2s\left(\frac{b}{a}r_3+\frac{c}{a}r_2\right)\right]$$

or

$$R_1^2=-\frac{R}{s}\sum ar_2r_3+2R\left(\frac{b}{a}r_3+\frac{c}{a}r_2\right) \text{ or } ar_1R_1^2=-\frac{Rar_1}{s}\sum ar_2r_3+2R\,(br_1r_3+cr_1r_2)$$

or

$$\sum ar_1R_1^2=-\frac{R}{s}\sum ar_1\sum ar_2r_3+4R\sum ar_2r_3=-2R\sum ar_2r_3+4R\sum ar_2r_3=2R\sum ar_2r_3$$

According 2.2. it result the inequality from the statement.
2.4. From inequality Hadwiger-Finsler applied in triangle $A_1B_1C_1$ it result

$$\sum a'^2\geq 4\sqrt{3}s'+\sum(b'-c')^2$$

Replacing (4) and (5) we obtain the inequality from the statement.
2.5. It result from reverse of Hadwiger-Finsler inequality in triangle $A_1B_1C_1$:

$$\sum a'^2\leq 4\sqrt{3}s'+3\sum(b'-c')^2$$

and equality (4) and (5).
2.6. It result from Euler inequality $R'\geq 2r'$ applied in triangle $A'B'C'$ and the equalities (7) and (8).
2.7. It result from inequality $p'\leq\frac{3\sqrt{3}}{2}R'$ and (6) and (8).
2.8. From inequality Stevin-Bottema in triangle $A_1B_1C_1$ it result

$$\alpha'\beta'c'^2+\beta'\gamma'a'^2+\gamma'\alpha'b'^2\leq R'^2$$

Using (4) (8) and (9) it result the inequality from the statement.
2.9. It result from inequality $\sum b'c'\geq 4\sqrt{3}s'$ and equalities (4) and (5).
2.10. We apply the inequality

$$\left(R_1'R_2'\right)^2+\left(R_2'R_3'\right)^2+\left(R_3'R_1'\right)^2\le\max\left\{b'^2c'^2,c'^2a'^2,a'^2b'^2\right\}$$

(Jian Liu) and the equalities (4) and (8).
2.11. We consider the inequality

$$R_1'^2\sin^2A'+R_2'^2\sin^2B'+R_3'^2\sin^2C'\le3\left(r_1'^2+r_2'^2+r_3'^2\right)$$

(L. Carlitz, American Mathematical Monthly) and the equalities (3) (8) and (10).
2.11. From [5] and (2) (4) (6) and (7) it result that

$$\frac{a'}{r_1'}+\frac{b'}{r_2'}+\frac{c'}{r_3'}\ge\frac{2p'}{r'}$$

who is equivalent with the inequlaity from the statment.

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