Sphere covering by minimal number of caps and short closed sets [*]

A. B. Németh

October 31, 2015

Abstract

A subset of the sphere is said short if it is contained in an open hemisphere. A short closed set which is geodesically convex is called a cap. The following theorem holds: 1. The minimal number of short closed sets covering the n-sphere is $n + 2$. 2. If $n + 2$ short closed sets cover the n-sphere then (i) their intersection is empty; (ii) the intersection of any proper subfamily of them is non-empty. In the case of caps (i) and (ii) are also sufficient for the family to be a covering of the sphere.

1   Introduction and the main result

Denote by $\mathbb{R}^{n+1}$ the $n+1$-dimensional Euclidean space endowed with a Cartesian reference system, with the scalar product $\langle \cdot, \cdot \rangle$ and with the topology it generates.

Denote by S^n the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

A subset of the sphere S^n is said *short* if it is contained in an open hemisphere.

The subset $C \subset S^n$ is called *geodesically convex* if together with any two of its points it contains the arc of minimal length of the principal circle on S^n through these points. S^n itself is a geodesically convex set.

A short closed set which is geodesically convex is called a *cap*.

We use the notation $\operatorname{co} A$ for the convex hull of A and the notation $\operatorname{sco} A$ for the geodesical convex hull of $A \subset S^n$ (the union of the geodesical lines with endpoints in A). Further $\operatorname{dist}(\cdot, \cdot)$ will denote the geodesical distance of

---

points. Besides the standard notion of simplex we also use the notion of the spherical simplex Δ placed in the north hemisphere S^+ of S^n such that their vertices are on the equator of S^n. In this case $\|\Delta\| = S^+$.

Our main result is:

Theorem 1   *1. The minimal number of short closed sets covering S^n is $n + 2$.*

   2. If a family $F_1, ...F_{n+2}$ of short closed sets covers S^n, then:

   (i) $\cap_{i=1}^{n+2} F_i = \emptyset$;

   (ii) $\cap_{i \neq j} F_i \neq \emptyset$, $\forall\ j = 1, ..., n + 2$;

   (iii) if $a_j \in \cap_{i \neq j} F_i$, then the vectors $a_1, ..., a_{n+2}$ are the vertices of an $n + 1$-simplex containing 0 in its interior.
   If the sets F_i are caps, then (i) and (ii) are also sufficient for the family to be a cover of S^n.

Let Δ be an $n + 1$-dimensional simplex with vertices in S^n containing the origin in its interior. Then the radial projection from 0 of the closed n-dimensional faces of Δ into S^n furnishes $n+2$ caps covering S^n and satisfying (i) and (ii).

A first version for caps of the above theorem is the content of the unpublished note [6].

Remark 1 *We mention the formal relation in case of caps with those in the Nerve Theorem ([2] Corollary 4G3). If we consider "open caps" in place of caps, then the conclusion (ii) can be deduced from the mentioned theorem. Moreover, the conclusion holds for a "good" open cover of the sphere too, i. e., an open cover with contractible members and contractible finite intersections. In our theorem the covering with caps has the properties of a "good" covering in this theorem: the members of the covering together with their nonempty intersections are contractible, but their members are closed, circumstance which seems to be rather sophisticated to be surmounted. (Thanks are due to Imre Bárány, who mentioned me this possible connection.)*

We shall use in the proofs the following (spherical) variant of Sperner's lemma (considered for simplices by Ky Fan [1]):

Lemma 1 *If a collection of closed sets $F_1, ..., F_{n+1}$ in S^n covers the spherical simplex engendered by the points $a_1, ...a_{n+1} \in S^n$ and $\mathrm{sco}\{a_1, ..., a_{i-1}, a_{i+1}, ..., a_{n+1}\} \subset F_i$, $i = 1, ..., n + 1$ then $\cap_{i=1}^{n+1} F_i \neq \emptyset$.*

Our first goal is to present the proof for caps. (We mention that using the methods in [3] and [4] the proof can be carried out in a purely geometric way in contrast with the proof in [6], where we refer to the Sperner lemma.)

Using the variant for caps of the theorem and the Sperner lemma we prove then the variant for short closed sets.

Except the usage of Lemma 1, our methods are elementary: they use repeatedly the induction with respect to the dimension.

2   The proof of the theorem for caps

1. Consider $n+1$ caps $C^1, ..., C^{n+1}$ on S^n. C^i being a cap, can be separated strictly by a hyperplane

$$H_i = \{x \in \mathbb{R}^{n+1} : \langle a_i, x \rangle + \alpha_i = 0\}$$

from the origin. We can suppose without loss of generality, that the normals a_i are linearly independent, since by slightly moving them we can achieve this, without affecting the geometrical picture. If the normals a_i are considered oriented toward 0, this strict separation means that $\alpha_i > 0$, $i = 1, ..., n$. The vectors a_i, $i = 1, ..., n+1$ engender a reference system in $\mathbb{R}^{n+1}$. Let x be a nonzero element of the positive orthant of this reference system. Then, for $t \geq 0$, one has $\langle a_i, tx \rangle \geq 0$, $\forall i = 1, ..., n+1$.

Hence, for each $t \geq 0$, tx will be a solution of the system

$$\langle a_i, y \rangle + \alpha_i > 0, \ i = 1, ..., n+1.$$

and thus

$$(*) \qquad tx \in \cap_{i=1}^{n+1} H_i^+, \ \forall t \geq 0$$

with

$$H_i^+ = \{y \in R^{n+1}; \ \langle a_i, y \rangle + \alpha_i > 0\}.$$

Now, if $C^1, ..., C^{n+1}$ covers S^n, then so does the union $\cup_{i=1}^{n+1} H_i^-$ of halfspaces

$$H_i^- = \{y \in R^{n+1}; \ \langle a_i, y \rangle + \alpha_i \leq 0\}.$$

Since H_i^+ is the complementary set of H_i^- and $S^n \subset \cup_{i=1}^{n+1} H_i^-$, the set $\cap_{i=1}^{n+1} H_i^+$ must be inside S^n and hence bounded. But $(*)$ shows that tx with $x \neq 0$ is in this set for any $t \geq 0$. The obtained contradiction shows that the family $C^1, ..., C^{n+1}$ cannot cover S^n.

Remark 2 *The proof of this item is also consequence of the Lusternik-Schnirelmann theorem [5] which asserts that if S^n is covered by the closed sets $F_1, ..., F_k$ with $F_i \cap (-F_i) = \emptyset$, $i = 1, ..., k$, then $k \geq n + 2$.*

2. Let $C^1, ..., C^{n+2}$ be caps covering S^n.

(i) Then they cannot have a common point x, since this case $-x$ cannot be covered by any C^i. (No cap can contain diametrically opposite points of S^n.)

Hence, condition (i) must hold.

(ii) To prove that $\cap_{j \neq i} C^j \neq \emptyset$, $\forall\, i = 1, ..., n+2$ we proceed by induction.

For S^1, the circle, C^i is an arc (containing its endpoints) of length $< \pi$, $i = 1, 2, 3$. The arcs C^1, C^2, C^3 cover S^1. Hence, they cannot have common points, and the endpoint of each arc must be contained in exactly one of the other two arcs. Hence, C^i meets C^j for every $j \neq i$. If $c_i \in C^j \cap C^k$, $j \neq i \neq k \neq j$, then c_1, c_2, c_3 are tree pairwise different points on the circle, hence they are in general position and 0 is an interior point of the triangle they span.

Suppose the assertions (ii) and (iii) hold for $n - 1$ and let us prove them for n.

Take C^{n+2} and let H be a hyperplane through 0 which does not meet C^{n+2}. Then, H determines the closed hemispheres S^- and S^+. Suppose that C^{n+2} is placed inside S^- (in the interior of S^- with respect the topology of S^n). Hence, $C^1, ..., C^{n+1}$ must cover S^+ and denoting by S^{n-1} the $n-1$-dimensional sphere $S^n \cap H$, these sets cover S^{n-1}. Now, $D^i = C^i \cap S^{n-1}$, $i = 1, ..., n+1$ are caps in S^{n-1} which cover this sphere. Thus, the induction hypothesis works for these sets.

Take the points $d_i \in \cap_{j \neq i} D^j$. Then, $d_1, ..., d_{n+1}$ will be in general position and 0 is an interior point of the simplex they span. By their definition, it follows that $d_k \in D^j$, $\forall k \neq j$ and hence $d_1, ..., d_{j-1}, d_{j+1}, ..., d_{n+1} \in D^j$, $j = 1, 2, ... n+1$.

Consider the closed hemisphere S^+ to be endowed with a spherical simplex structure Δ whose vertices are the points $d_1, ..., d_{n+1}$.

Since $C^1, ..., C^{n+1}$ cover S^+, and $d_1, ..., d_{j-1}, d_{j+1}, ..., d_{n+1} \in D^j \subset C^j \cap S^+$, $j = 1, 2, ... n+1$, Lemma 1 can be applied to the spherical simplex Δ, yielding

$$\cap_{j=1}^{n+1} C^j \supset \cap_{j=1}^{n+1} (S^+ \cap C^j) \neq \emptyset.$$

This shows that each collection of $n+1$ sets C^j have nonempty intersection and proves (ii) for n.

(If we prefer a purely geometric proof of this item, we can refer to the spherical analogue of the results in [4].)

From the geometric picture is obvious that two caps meet if and only if their convex hulls meet. Hence, from the conditions (i) and (ii) for the caps C^i, it follows that these conditions hold also for $A^i = \mathrm{co}\, C^i$, $i = 1, ..., n+2$.

Take
$$a_i \in \cap_{j \neq i} A^j, \ i = 1, ..., n + 2.$$

Let us show that for an arbitrary $k \in N$,
$$a_k \notin \text{aff}\{a_1, ..., a_{k-1}, a_{k+1}, ..., a_{n+2}\}.$$

Assume the contrary. Denote
$$H = \text{aff}\{a_1, ..., a_{k-1}, a_{k+1}, ..., a_{n+2}\}.$$

Thus, $\dim H \leq n$. The points a_i are all in the manifold H. Denote
$$B^i = H \cap A^i.$$

Since $a_i \in \cap_{j \neq i} A^j$ and $a_i \in H$, it follows that
$$a_i \in \cap_{j \neq i} A^j \cap H = \cap_{j \neq i} B^j, \ \forall i.$$

This means that the family of convex compact sets $\{B^j : j = 1, ..., n + 2\}$ in H possesses the property that every $n + 1$ of its elements have nonempty intersection. Then, by Helly's theorem, they have a common point. But this would be a point of $\cap_{i=1}^{n+2} A^i$ too, which contradicts (i) for the sets A^i.

Hence, every $n + 2$ points $c_i \in \cap_{j \neq i} C^j \subset \cap_{j \neq i} A^j, \ i = 1, ..., n + 2$ are in general position. Since $c_1, ..., c_{i-1}, c_{i+1}, ..., c_{n+2} \in C^i$, it follows that the open halfspace determined by the hyperplane they engender containing 0 contains also the point c_i.

This proves (iii).

Suppose that the caps $C^1, ..., C^{n+2}$ posses the properties in (i) and (ii). Then, the method in the above proof yields that the points
$$c_i \in \cap_{j \neq i} C^j, \ i = 1, ..., n + 2$$

engender an $n + 1$-simplex with 0 in its interior and
$$c_1, ..., c_{i-1}, c_{i+1}, ..., c_{n+2} \in C^i, \ i = 1, ..., n + 2.$$

The radial projections of the n-faces of this simplex into S^n obviously cover S^n. The union of these projections are contained in $\cup_{i=1}^{n+2} C^i$.

This completes the proof for cups.

3 The proof of the theorem for short closed sets

We carry out the proof by induction.

Consider $n = 1$ and suppose F_1, F_2, F_3 are short closed sets covering S^1.

If $a \in \cap_{i=1}^3 F_i$, then by the above hypothesis $-a \notin \cup_{i=1}^3 F_i$, which is impossible. Hence,

$$\cap_{i=1}^3 F_i = \emptyset$$

must hold.

Denote $C_3 = \text{sco } F_3$, then $C = \text{cl } S^1 \setminus C_3$ is a connected arc of S^1 covered by F_1, F_2. One must have $C \cap F_i \neq \emptyset, i = 1, 2$, since if for instance $C \cap F_2 = \emptyset$, then it would follow that the closed sets F_1 and C_3, both of geodesical diameter $< \pi$ cover S^1, which is impossible. Since C is connected and $C \cap F_i$, $i = 1, 2$ are closed sets in C covering this set, $F_1 \cap F_2 \supset (C \cap F_1) \cap (C \cap F_2) \neq \emptyset$ must hold.

The geodesically convex sets $C_i = \text{sco } F_i$, $i = 1, 2, 3$ cover S^1, hence applying the theorem for caps to $a_j \in \cap_{i \neq j} F_i \subset \cap_{i \neq j} C_i$, $j = 1, 2, 3$, we conclude that these points are in general position and the simplex engendered by them must contain 0 as an interior point.

Suppose that the assertions hold for $n - 1$ and prove them for n.

Suppose that

$$S^n \subset \cup_{i=1}^{n+2} F_i, \ F_i \text{ short, closed, } i = 1, ..., n + 2.$$

The assertion (i) is a consequence of the theorem for caps applied to $C_i = \text{sco } F_i$, $i = 1, ..., n + 2$ (or a consequence of the Lusternik Schnirelmann theorem).

Suppose that F_{n+2} is contained in the interior (with respect to the topology of S^n) of the south hemisphere S^- and denote by S^{n-1} the equator of S^n.

Now $S^{n-1} \subset \cup_{i=1}^{n+1} F_i'$ with $F_i' = (S^{n-1} \cap F_i)$, $i = 1, ..., n + 1$, and we can apply the induction hypothesis for S^{n-1} and the closed sets F_i', $i = 1, ..., n+1$. Since $C_i' = \text{sco } F_i'$, $i = 1, ..., n + 1$ cover S^{n-1}, and they are caps, the theorem for caps applies and hence the points

$$a_j \in \cap_{i=1, i \neq j}^{n+1} C_i', \ j = 1, ..., n + 1$$

are in general position.

The closed sets

$$A_i = C_i' \cup (F_i \cap S^+) = C_i' \cup (F_i \cap \text{int } S^+), \ i = 1, ..., n + 1$$

cover S^+, the north hemisphere considered as a spherical simplex Δ engendered by $a_1, ..., a_{n+1}$ ($\|\Delta\| = S^+$). (Here int S^+ is the interior of S^+ in the space S^n.) Further,

$$\text{sco}\{a_1, ..., a_{k-1}, a_{k+1}, ..., a_{n+1}\} \subset A_k, \ k = 1, ..., n+1.$$

Hence, we can apply Lemma 1 to conclude that there exists a point a in $\cap_{i=1}^{n+1} A_i \neq \emptyset$.

Since

$$C_i' \cap (F_j \cap \text{int } S^+) = \emptyset, \ \forall \ i, \ j,$$

it follows that

$$a \in \cap_{i=1}^{n+1} A_i = \cap_{i=1}^{n+1} C_i' \cup \cap_{i+1}^{n+1} F_i \cap \text{int } S^+ = \cap_{i+1}^{n+1} F_i \cap \text{int } S^+,$$

because $\cap_{i=1}^{n+1} C_i' = \emptyset$ by the induction hypothesis and the theorem for caps. Thus,

$$a \in \cap_{i=1}^{n+1} F_i,$$

and we have condition (ii) fulfilled for n.

The condition (iii) follows from the theorem for caps applied to

$$C^i = \text{sco } F^i, \ i = 1, ..., n+2.$$

Remark 3 *If S^1 is covered by the closed sets F_1, F_2, F_3 with the property $F_i \cap (-F_i) = \emptyset$, $i = 1, 2, 3$, then $F_i \cap F_j \neq \emptyset \ \forall \ i, j$.*

Indeed, assume that $F_1 \cap F_2 = \emptyset$. Then $\text{dist}(F_1, F_2) = \varepsilon > 0$. If $a_i \in F_i$ are the points in F_i, $i = 1, 2$ with $\text{dist}(a_1, a_2) = \varepsilon$, then the closed arc $C \subset S^1$ with the endpoints a_1, a_2 must be contained in F_3, and hence $-C \cap F_3 = \emptyset$, and then $-C$ must be covered by $F_1 \cup F_2$. Since $-a_1 \in -C$ cannot be in F_1, it must be in F_2, and $-a_2 \in F_1$. Thus, $F_1 \cap -C \neq \emptyset$ and $F_2 \cap -C \neq \emptyset$, while the last two sets cover $-C$. Since $-C$ is connected and the respective sets are closed, they must have a common point, contradicting the hypothesis $F_1 \cap F_2 = \emptyset$.

This way, we obtain (ii) fulfilled for $n = 1$ for this more general case. We claim that the conditions also hold for n, that is, if the closed sets $F_1, ..., F_{n+2}$ with $F_i \cap (-F_i) = \emptyset$, $i = 1, ..., n+2$ cover S^n, then condition (ii) holds. (Condition (i) is a consequence of the definition of the sets F_i.)

References

[1] Ky Fan, A covering property of simplexes, Math. Scand. 22 (1968), 17-20.

[2] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.

[3] H. Kramer and A.B. Németh, Supporting sphere for a special family of compact convex sets in the Euclidean space, Math. Pannon. 19 (2008), 3-12.

[4] H. Kramer and A.B. Németh, Family of closed convex sets covering the faces of a simplex, arXiv, 1373445, 2015.

[5] L. A. Lusternik, L. G. Schnirelman, Topologiceskie metody v variacionnom iscislenie, Uspehi Mat. Nauk, 2/1 (1947), 166-217.

[6] Németh Sándor, Gömblefedés geodetikusan konvex halmazokkal, Days of Hungarian Science in Transylvania , November 26, 2006, Miercurea Ciuc (Csíkszereda).