About some polynomial functions

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ABSTRACT. In this paper we present some new results about polynomial functions.

Let f and g be two functions defined on C, the set of complex numbers, i.e. $f, g : C \to C$. For any $x \in C$ we denote:

$$u = u(x) = f(x) + g(x), \ v = v(x) = f(x) g(x) \tag{1}$$

and we will evaluate

$$P_n(x) = f^n(x) + g^n(x) \tag{2}$$

only by u and v. If m and n are positive integer such that $m \leq n$, then we have

$$P_n P_m = (f^n(x) + g^n(x))(f^m(x) + g^m(x)) = \left(f^{n+m}(x) + g^{n+m}(x)\right) +$$

$$+ f^m(x) g^m(x) \left(f^{n-m}(x) + g^{n-m}(x)\right) = P_{n+m} + v^m P_{n-m} \tag{3}$$

If $m = 1$, then by (3) we obtain

$$P_{n+1} = P_1 P_n - v P_{n-1} = u P_n - v P_{n-1} \tag{4}$$

Definition 1. We call the integer part of the real number α the integer number denoting by $[\alpha]$ which verifies

$$[\alpha] \leq \alpha < [\alpha] + 1 \tag{5}$$

Definition 2. We call an integer parity a function $p : Z \to \{0, 1\}$, ($Z =$ the set of integer numbers) defined by

$$p(n) = \begin{cases} 0 & \text{if} \quad n = 2k \\ 1 & \text{if} \quad n = 2k + 1 \end{cases} \tag{6}$$

We note that

$$p(n) = \frac{1}{2}(1 - (-1)^n) = 2\left(\frac{n}{2} - \left[\frac{n}{2}\right]\right) \tag{7}$$

Also we introduce the following functions

$$\omega(n) = 1 + (-1)^n = \begin{cases} 2 & \text{if} \quad n = 2k \\ 0 & \text{if} \quad n = 2k + 1 \end{cases} \tag{8}$$

$$w(n) = 2p(n) = 1 - (-1)^n = \begin{cases} 0 & \text{if} \quad n = 2k \\ 2 & \text{if} \quad n = 2k + 1 \end{cases} \tag{9}$$

Theorem 1. For any natural numbers n polynomial P_n satisfies
a) P_n has degree n in u and has degree $\left[\frac{n}{2}\right]$ in v;
b) P_n has the form

$$P_n = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i B_n^i u^{n-2i} v^i \tag{10}$$

[1]Received: 22.07.2013
2010 *Mathematics Subject Classification.* 12E10.
Key words and phrases. Polynomial, function.

c) For any natural number n we have $B_n^0 = 1$
d)

$$B_n^p = B_{n-2}^{p-1} + B_{n-1}^p \tag{11}$$

e) $B_0^0 = 2$ and for any n with $\left[\frac{n}{2}\right] = \frac{n}{2}$, then $B_n^{\left[\frac{n}{2}\right]} = 2$.

Proof. We prove by mathematical induction.
For $n = 0$ we have $P_0 = 2 = B_0^0 u^0$, so $B_0^0 = 2$, the theorem is checked.
For $n = 1$ we have $P_1 = u = B_1^0 u$, so $B_1^0 = 1$, the theorem is checked.
For $n = 2$, then

$$P_2 = f^2(x) + g^2(x) = (f(x) + g(x))^2 - 2f(x)g(x) = u^2 - 2v =$$

$$= B_2^0 u_2 + (-1)^1 B_2^1 u^0 v = B_2^0 u^2 + (-1)^1 B_2^1 v,$$

so $B_2^0 = 1$, $B_2^1 = 2$ which shows that the statement is true.
We suppose that for $n = k$ the theorem is true, i.e.

$$P_k = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i B_k^i u^{k-2i} v^i \tag{12}$$

and show that it is true for $n = k + 1$, i.e.

$$P_{k+1} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i B_{k+1}^i u^{k+1-2i} v^i \tag{13}$$

(i) If $2r = k$,,then $\left[\frac{k}{2}\right] = r$, $\left[\frac{k+1}{2}\right] = r$, $\left[\frac{k-1}{2}\right] = r - 1$, and by (4) we have

$$P_{k+1} = uP_k - vP_{k-1} = u\sum_{i=0}^{r} (-1)^i B_k^i u^{k-2i} v^i - v\sum_{i=0}^{r-1} (-1)^i B_{k-1}^i u^{k-2i-1} v^i =$$

$$= \sum_{i=0}^{r} (-1)^i B_k^i u^{k+1-2i} v^i - \sum_{i=0}^{r-1} (-1)^i B_{k-1}^i u^{k-2i-1} v^{i+1} =$$

$$= B_k^0 u^{k+1} + \sum_{i=1}^{r} (-1)^i B_k^i u^{k+1-2i} v^i - \sum_{j=1}^{r} (-1)^{j-1} B_{k-1}^{j-1} u^{k+1-2j} v^j, \ \ (j = i + 1),$$

which yields

$$P_{k+1} = B_k^0 u^{k+1} + \sum_{i=1}^{r} (-1)^i \left(B_k^i + B_{k-1}^{i-1}\right) u^{k+1-2i} v^i \tag{14}$$

Denoting

$$B_{k+1}^0 = B_1^0, B_{k+1}^1 = B_k^i + B_{k-1}^{i-1} \tag{15}$$

we deduce that the statement is true for $k + 1 = 2r + 1$.

(ii) If $2r + 1 = k$, then $\left[\frac{k}{2}\right] = r$, $\left[\frac{k+1}{2}\right] = r + 1$, $\left[\frac{k-1}{2}\right] = r$ and by (4) we have

$$P_{k+1} = uP_k - vP_{k-1} = u\sum_{i=0}^{r} (-1) B_k^i u^{k-2i} v^i - v\sum_{i=0}^{r} (-1)^i B_{k-1}^i u^{k-2i-1} v^i =$$

$$= \sum_{i=0}^{r} (-1)^i B_k^i u^{k+1-2i} v^i - \sum_{i=0}^{r} (-1)^i B_{k-1}^i u^{k-2i-1} v^{i+1} =$$

$$= B_k^0 u^{k+1} + \sum_{i=0}^{r} (-1)^i B_k^i u^{k+1-2i} v^i - \sum_{j=1}^{r+1} (-1)^{j-1} B_{k-1}^{j-1} u^{k+1-2j} v^j, \quad (j = i+1)$$

so

$$P_{k+1} = B_k^0 u^{k+1} +$$

$$+ \sum_{i=1}^{r} (-1)^i \left(B_k^i + B_{k-1}^{i-1}\right) u^{k+1-2i} v^i + (-1)^r B_{k-1}^r u^{k+1-2r-2} v^{r+1} \tag{16}$$

Denoting

$$B_k^0 = B_{k+1}^0, B_{k+1}^i = B_k^i + B_{k-1}^{i-1}, B_{k+1}^{r+1} = B_{k-1}^r \tag{17}$$

we deduce that the statement is true for $k+1 = 2(r+1)$.
Therefore, by mathematical induction we obtain that the enunciation is true for any natural number n.
So far we have established claims a) and b) from the statement. The relations (15) and (17) shows that

$$B_n^0 = B_{n+1}^0 = B_1^0 = B_2^0 = 1, \ B_{2k}^k = B_{2(k+1)}^{k+1} = B_2^1 = B_0^0 = 2,$$

and

$$B_n^p = B_{n-2}^{p-1} + B_{n-1}^p,$$

i.e. others claims of the statement.
The numbers B_n^p are uniquely determined by the relations:

$$B_n^0 = 1 \ (n > 0), \ B_{2p}^p = 2, \ B_n^p = B_{n-2}^{p-1} + B_{n-1}^p \tag{18}$$

Theorem 2. For any $n \geq 0$ we have

$$B_n^p = n \cdot \frac{(n-p-1)!}{p!\,(n-2p)!} \tag{19}$$

Proof. So we have to check that B_n^p given by (19) verify relations (18).
Indeed

$$B_n^0 = n \cdot \frac{(n-0-1)!}{0!\,(n-2\cdot 0)!} = n \cdot \frac{(n-1)!}{n!} = 1, \text{ for any } n > 0$$

$$B_{2p}^p = 2p \cdot \frac{(2p-p-1)!}{p!\,(2p-2p)!} = 2p \cdot \frac{(p-1)!}{p!0!} = 2 \cdot \frac{p!}{p!} = 2, \text{ for any } p \geq 0$$

$$B_{n-1}^p + B_{n-2}^{p-1} = (n-1) \cdot \frac{(n-1-p-1)!}{p!\,(n-1-2p)!} + (n-2) \cdot \frac{(n-2-p+1-1)!}{(p-1)!\,(n-2-2p+2)!} =$$

$$= \frac{(n-p-2)!}{(p-1)!\,(n-2p-1)!} \left(\frac{n-1}{p} + \frac{n-2}{n-2p}\right) = n \cdot \frac{(n-p-1)!}{p!\,(n-2p)!} = B_n^p,$$

and the proof is complete.
We return now to the relation (1). Hence

$$f(x) = \frac{u \pm \sqrt{u^2 - 4v}}{2}, \ g(x) = \frac{u \mp \sqrt{u^2 - 4v}}{2} \tag{20}$$

So

$$2^nP_n = 2^n\left(f^n(x)+g^n(x)\right) = \left(u\pm\sqrt{u^2-4v}\right)^n + \left(u\mp\sqrt{u^2-4v}\right)^n =$$

$$= \sum_{i=0}^{n}(\pm1)^i C_n^i u^{n-i}\sqrt{\left(u^2-4v\right)^i} + \sum_{i=0}^{n}(\mp1)^i C_n^i u^{n-i}\sqrt{\left(u^2-4v\right)^i} =$$

$$= \sum_{i=0}^{n}(-1)^i C_n^i u^{n-i}\sqrt{\left(u^2-4v\right)^i}\omega(i) = 2\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j C_n^{2j}u^{n-2j}\left(u^2-4v\right)^j =$$

$$= 2\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j C_n^{2j}u^{n-2j}\left(\sum_{r=0}^{j}(-1)^r C_j^r u^{2r}2^{2r}v^r\right) =$$

$$= 2\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j 2^{2j}S(n,j)\,u^{n-2j}v^j \tag{21}$$

where

$$S(n,j) = \sum_{r=j}^{n-j} C_{n-j-1}^{r-1}C_r^j \tag{22}$$

Theorem 3. For any $k\le\left[\frac{n}{2}\right]$ we have that

$$S(n,k) = 2^{n-2k-1}B_n^k \tag{23}$$

Proof. Indeed, by (10) and (21) we have

$$2^nB_n^k = 2^{2k+1}S(n,k),$$

so

$$S(n,k) = 2^{n-2k-1}B_n^k,$$

and we are done.

Theorem 4. We have

$$B_{2p+k}^p = B_{2p-2}^{p-1} + B_{2p-1}^{p-1} + \dots + B_{2p+k-2}^{p-1} \tag{24}$$

$$B_{p+2k+2}^k = B_p^0 + B_{p+2}^1 + B_{p+4}^2 + \dots + B_{2p+k-2}^{p-1} \tag{25}$$

Proof. Indeed, we have

$$B_{2p+k-1}^p = B_{2p+k-i-2}^{p-1} + B_{2p+k-i-1}^p,$$

therefore

$$\sum_{i=0}^{k}B_{2p+k-i}^p = \sum_{i=0}^{k}B_{2p+k-i-2}^{p-1} + \sum_{i=0}^{k}B_{2p+k-i-1}^p = \sum_{i=0}^{k}B_{2p+k-i-2}^{p-1} + \sum_{j=1}^{k+1}B_{2p+k-j}^p,$$

so

$$B_{2p+k}^p - B_{2p-2}^{p-1} = \sum_{i=0}^{k}B_{2p+k-i-2}^{p-1} \tag{26}$$

But,

$$B_{2p}^{p} = B_{2p-1}^{p} + B_{2(p-1)}^{p-1},$$

from where

$$B_{2p-1}^{p} = B_{2p}^{p} - B_{2(p-1)}^{p-1} = 2 - 2 = 0,$$

and by (26) we obtain (24).
Also we have that

$$B_{p+2i+1}^{i} = B_{p+2i}^{i} + B_{p+2i-1}^{i-1},$$

so

$$\sum_{i=1}^{k} B_{p+2i+1}^{i} = \sum_{i=1}^{k} B_{p+2i}^{i} + \sum_{i=1}^{k} B_{p+2i-1}^{i-1} = \sum_{i=1}^{k} B_{p+2i}^{i} + \sum_{j=0}^{k-1} B_{p+2j+1}^{j},$$

therefore

$$B_{p+2k+1}^{k} - B_{p+1}^{0} = \sum_{i=1}^{k} B_{p+2i}^{i}.$$

We obtain

$$B_{p+2k+1}^{k} - B_{p+1}^{0} + B_{p}^{0} = \sum_{i=0}^{k} B_{p+2i}^{i}$$

i.e. (25) is proved, and the proof is complete.

Theorem 5. For any natural numbers n, we have

$$u^{n} P_{n-1} = v^{n} P_{0} + \sum_{i=1}^{n-1} u^{n-(i+1)} v^{i-1} P_{n-i+1} \tag{27}$$

Proof. We have

$$u^{n-i} v^{i-1} P_{n-i} = u^{n-i-1} v^{i-1} \left(u P_{n-i}\right),$$

and if we taking account by (4) we deduce

$$u^{n-i} v^{i-i} P_{n-i} = u^{n-i-1} v^{i-1} \left(P_{n-i+1} + v P_{n-i-1}\right),$$

so,

$$\sum_{i=1}^{n-1} u^{n-i} v^{i-1} P_{n-i} = \sum_{i=1}^{n-1} u^{n-(i+1)} v^{i-1} P_{n-i+1} + \sum_{i=1}^{n-1} u^{n-(i+1)} v\left(i\right) P_{n-(i+1)} =$$

$$= \sum_{i=1}^{n-1} u^{n-(i+1)} v\left(i-1\right) P_{n-i+1} + \sum_{j=2}^{n} u^{n-j} v^{j-1} P_{n-j},$$

hence

$$u^{n} P_{n-1} = v^{n} P_{0} + \sum_{i=1}^{n-1} u^{n-(i+1)} v^{i-1} P_{n-i+1},$$

and we are done.

Theorem 6. For any natural numbers n, we have

$$P_{2n+1} = P_1 \left((-1)^n v^n + \sum_{i=0}^{n-1} (-1)^i v^i P_{2(n-1)} \right) \tag{28}$$

Proof. By (4) we have

$$(-1)^i v^i P_{2(n-i)+1} = (-1)^i v^i \left(uP_{2(n-i)} - vP_{2(n-i-1)+1} \right) =$$

$$= (-1)^i v^i u P_{2(n-i)} + (-1)^{i+1} v^{i+1} P_{2(n-i-1)+1},$$

hence

$$\sum_{i=0}^{n-1} (-1)^i v^i P_{2(n-i)+1} = \sum_{i=0}^{n-1} (-1)^i v^i P_1 P_{2(n-i)} + \sum_{i=0}^{n-1} (-1)^{i+1} v^{i+1} P_{2(n-i-1)+1} =$$

$$= \sum_{i=0}^{n-1} (-1)^i v^i P_1 P_{2(n-i)} + \sum_{j=1}^{n} (-1)^j v^j P_{2(n-j)+1},$$

so

$$P_{2n+1} = (-1)^n v^n P_1 + \sum_{i=0}^{n-1} (-1)^i v^i P_1 P_{2n(n-i)} =$$

$$= P_1 \left((-1)^n v^n + \sum_{i=0}^{n-1} (-1)^i v^i P_{2(n-i)} \right)$$

q.e.d.
The relations (7) and (10) allow us to write

$$P_n = u^{\rho(n)} \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i u^{2\left(\left[\frac{n}{2}\right]-i\right)} v^i B_n^i \tag{29}$$

or

$$P_n = u^{\rho(n)} \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (P_2 + 2v)^{2\left(\left[\frac{n}{2}\right]-i\right)} v^i B_n^i \tag{30}$$

Theorem 7. For any x solution of equation $u(x) = 0$ and for any natural number n we have

$$P_n = (-1)^{\left[\frac{n}{2}\right]} v^{\left[\frac{n}{2}\right]} B_n^{\left[\frac{n}{2}\right]} (1 - p(n)) \tag{31}$$

Proof. Indeed, the relation (29) shows that if $p(n) = 1$, then $P_n(0, v) = 0$; and if $p(n) = 0$then ,$\left[\frac{n}{2}\right] = \frac{n}{2}$ so $P_{2k}(0, v) = (-1)^k v^k B_{2k}^k = 2(-1)^k v^k$, q.e.d.

Theorem 8. For any x solution of equation $v(x) = 0$ and for any natural number n, we have

$$P_n(u, 0) = u^n \tag{32}$$

Proof. If in (10) we take $v = 0$ then we obtain $P_n(u, 0) = B_n^0 u^n = u^n$.
Since $v = 0$ we have the following cases
- $f(x) = 0, g(x) \neq 0$, and then $P_n(u, 0) = g^n(x)$;
- $f(x) = 0, g(x) = 0$, and then $P_n(u, 0) = P_n(0, 0) = 0$;

- $f(x) \neq 0, g(x) = 0$, and then $P_n(u, 0) = f^n(x)$.
The proof is complete.

Theorem 9. For any natural number n, we have

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i 2^{n-2i} B_n^i = 2 \tag{33}$$

Proof. Indeed, if in (10) we take $f(x) = at(x)$, $g(x) = bt(x)$, we deduce

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (a+b)^{n-2i} (ab)^i B_n^i t^n = (a^n + b^n) t^n,$$

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (a+b)^{n-2i} (ab)^i B_n^i = a^n + b^n,$$

and for $a = b = 1$ we obtain the result.

Theorem 10. For any natural number n, we have

$$\sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^i B_n^i = \begin{cases} 1 & if \quad n = 6k \pm 1 \\ -1 & if \quad n = 6k \pm 2 \\ -2 & if \quad n = 6k + 3 \\ 2 & if \quad n = 6k \end{cases} \tag{34}$$

Proof. We give two demonstrations.
(i) We note that LHS of (34) results by (10) for $u = v = 1$. In other words if we denote $f(x) = \xi$, $g(x) = \eta$, LHS is obtain for x which verify

$$\xi + \eta = 1, \ \xi\eta = 1 \tag{35}$$

By (35) we deduce

$$\xi^2 - \xi + 1 = \eta^2 - \eta + 1 = 0 \tag{36}$$

so

$$\xi^2 = (1-\eta)^2 = 1 - 2\eta + \eta^2 = 1 - \eta + \eta^2 - \eta = -\eta;$$

$$\eta^2 = (1-\xi)^2 = 2\xi + \xi^2 = 1 - \xi + \xi^2 - \xi = -\xi \tag{37}$$

$$\xi^3 = \xi\xi^2 = \xi(-\eta) = -\xi\eta = -1, \ \eta^3 = \eta\eta^2 = \eta(-\xi) = -\xi\eta = -1 \tag{38}$$

Making these substitutions in (10) we obtain

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i B_n^i = \xi^n + \eta^n \tag{39}$$

If
- $-n = 6k$, then $\xi^n + \eta^n = \left(\xi^3\right)^{2k} + \left(\eta^3\right)^{2k} = 2$;
- $n = 6k + 1$, then

$$\xi^n + \eta^n = \left(\xi^3\right)^{2k} \xi^{\pm 1} + \left(\eta^3\right)^{2k} \eta^{\pm 1} = \xi^{\pm} + \eta^{\pm} = \begin{cases} \xi + \eta = 1 \\ \frac{1}{\xi} + \frac{1}{\eta} = \frac{\xi + \eta}{\xi\eta} = 1 \end{cases}$$

- $n=6k\pm 2$, then

$$\xi^n+\eta^n=\left(\xi^3\right)^{2k}\left(\xi^2\right)^{\pm 1}+\left(\eta^3\right)^{2k}\left(\eta^2\right)^{\pm 1}=\left(-\eta\right)^{\pm 1}+\left(-\xi\right)^{\pm 1}=-1$$

- $n=6k+3$, then

$$\xi^n+\eta^n=\left(\xi^3\right)^{2k+1}+\left(\eta^3\right)^{2k+1}=\left(-1\right)^{2k+1}+\left(-1\right)^{2k+1}=-2$$

So the theorem is proved.
(ii) Substituting in (10) on $f\left(x\right)$ and $g\left(x\right)$ with μ and υ where $\mu+\upsilon=-1$, $\mu\upsilon=1$ and

$$\mu^2+\mu+1=\upsilon^2+\upsilon+1=0 \tag{40}$$

We have

$$\mu^2=\left(-1-\upsilon\right)^2=1+2\upsilon+\upsilon^2=\upsilon,\ \ \upsilon^2=\left(-1-\mu\right)^2=1+2\mu+\mu^2=\mu \tag{41}$$

and

$$\mu^3=\mu\mu^2=\mu\upsilon=1,\ \ \upsilon^3=\upsilon\upsilon^2=\upsilon\mu=1 \tag{42}$$

Therefore

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^{n-i}B_n^i=\mu^n+\upsilon^n,\ \ \sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^{i}B_n^i=\left(-1\right)^n\left(\mu^n+\upsilon^n\right) \tag{43}$$

If
- $n=6k$, then $\left(-1\right)^n\left(\mu^n+\upsilon^n\right)=\mu^{6k}++\upsilon^{6k}=2$;
- $n=6k\pm 1$, then

$$\left(-1\right)^n\left(\mu^n+\upsilon^n\right)=-\left(\left(\mu^3\right)^{2k}\mu^{\pm 1}+\left(\upsilon^3\right)^{2k}\upsilon^{\pm 1}\right)=-\left(\mu^{\pm 1}+\upsilon^{\pm 1}\right)=1;$$

because $\mu+\upsilon=-1$ and $\frac{1}{\mu}+\frac{1}{\upsilon}=\frac{\mu+\upsilon}{\mu\upsilon}=-1$.
- $n=6k\pm 2$, then

$$\left(-1\right)^n\left(\mu^n+\upsilon^n\right)=\left(\left(\mu^3\right)^{2k}\mu^{\pm 2}+\left(\upsilon^3\right)^{2k}\upsilon^{\pm 2}\right)=\mu^{\pm 2}+\upsilon^{\pm 2}=-1$$

- $n=6k+3$, then

$$\left(-1\right)^n\left(\mu^n+\upsilon^n\right)=-\left(\left(\mu^3\right)^{2k+1}+\left(\upsilon^3\right)^{2k+1}\right)=-2$$

The proof is complete.

Particular cases
1. If $f\left(x\right)=\sin^2 x,\ g\left(x\right)=\cos^2 x$, then

$$P_n\left(u,v\right)=\sin^{2n}x+\cos^{2n}x=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{-n-2i}B_n^i\sin^{2i}2x \tag{44}$$

2. If $f\left(x\right)=\sin x,\ g\left(x\right)=\cos x$, then

$$P_n\left(u,v\right)=\sin^n x+\cos^n x=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{-i}B_n^i\left(\sin x+\cos x\right)^{n-2i}\sin^i 2x \tag{45}$$

3. If $f\left(x\right)=e^{i\alpha\left(x\right)},\ g\left(x\right)=e^{-i\alpha\left(x\right)}$, then

$$P_n\left(u,v\right)=2\cos n\alpha\left(x\right)=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{n-2i}B_n^i\cos^{n-2i}\alpha\left(x\right)$$

so

$$\cos n\alpha\left(x\right)=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{n-2i-1}B_n^i\cos^{n-2i}\alpha\left(x\right)=$$

$$=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i S\left(n,i\right)\cos^{n-2i}\alpha\left(x\right) \tag{46}$$

4. If $if\left(x\right)=e^{i\alpha(x)}$, $ig\left(x\right)=-e^{-i\alpha(x)}$, then (10) becomes

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{n-2i}B_n^i\sin^{n-2i}\alpha\left(x\right)=\frac{e^{in\alpha(x)}+\left(-1\right)^n e^{-in\alpha(x)}}{i^n} \tag{47}$$

and if we taking account by (8) and (9) the relation (47) becomes

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{n-2i}B_n^i\sin^{n-2i}\alpha\left(x\right)=\left(-1\right)^{\left[\frac{n}{2}\right]}\left(\omega\left(n\right)\cos n\alpha\left(x\right)+w\left(n\right)\sin n\alpha\left(x\right)\right)$$

So

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^{i-\left[\frac{n}{2}\right]} 2^{n-2i}B_n^i\sin^{n-2i}\alpha\left(x\right)=\omega\left(n\right)\cos n\alpha\left(x\right)+w\left(n\right)\sin n\alpha\left(x\right) \tag{48}$$

5. If $f\left(x\right)=e^{\beta(x)}$, $g\left(x\right)=e^{-\beta(x)}$, then $u=2ch\beta\left(x\right)$ and $v=1$; by (10) we deduce

$$2chn\beta\left(x\right)=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{n-2i}B_n^i ch^{n-2i}\beta\left(x\right)$$

or

$$chn\beta\left(x\right)=\sum_{i=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^i 2^{n-2i-1}B_n^i ch^{n-2i}\beta\left(x\right) \tag{49}$$

6. If $f\left(x\right)=e^{\beta(x)}$, $g\left(x\right)=-e^{-\beta(x)}$ then $u=2sh\beta\left(x\right)$ and $v=-1$; by (10) we obtain

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}2^{n-2i}B_n^i sh^{n-2i}\beta\left(x\right)=e^{n\beta(x)}+\left(-1\right)^n e^{-n\beta(x)}$$

hence

$$\sum_{i=0}^{\left[\frac{n}{2}\right]}2^{n-2i}B_n^i sh^{n-2i}\beta\left(x\right)=\omega\left(n\right)chn\beta\left(x\right)+w\left(n\right)shn\beta\left(x\right) \tag{50}$$

7. If in (10) we take $f\left(x\right)=e^{i\arccos x}$, $g\left(x\right)=e^{-i\arccos x}$ then ,$v=1$

$$u=2\cos\left(\arccos x\right)=2x=2T_1\left(x\right),$$

and

$$P_n(x) = e^{in\arccos x} + e^{-in\arccos x} = 2\cos(n\arccos x) = 2T_n(x),$$

where T_n is Chebyshev polynomial with degree n. In this case the relation (3) becomes

$$T_{n+m}(x) = 2T_n(x)T_m(x) - T_{n-m}(x) \tag{51}$$

and (4) becomes

$$T_{n+1}(x) = 2T_n(x)T_1(x) - T_{n-1}(x) = 2xT_n(x) - T_{n-1}(x) =$$

$$= 2xT_n(x) - T_{n-1}(x) \tag{52}$$

By (10) we obtain that

$$T_n(x) = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i 2^{n-2i-1} B_n^i x^{n-2i} \tag{53}$$

So the coefficient of x^{n-2i} from Chebyshev polynomial is

$$(-1)^i 2^{n-2i-1} B_n^i = (-1)^i 2^{n-2i-1} \cdot \frac{n}{i} \cdot C_{n-i-1}^{i-1} = (-1)^i S(n,i) \tag{54}$$

Theorem 11. For any f and g and for any $n \in N$, then $P_n(u,v)$ is solution of the following differential equation

$$\left(4v - u^2\right) \cdot \frac{\theta^2 z(u,v)}{\theta u^2} - u \cdot \frac{\theta z(u,v)}{\theta u} + n^2 z(u,v) = 0 \tag{55}$$

Proof. We have to show that replacing in (55) with $P_n(u,v)$ the equation is verify. By (10) we have

$$\frac{\theta P_n(u,v)}{\theta u} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (n-2i) B_n^i u^{n-2i-i} v^i \tag{56}$$

and

$$\frac{\theta^2 P_n(u,v)}{\theta u^2} = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (n-2i)(n-2i-1) B_n^i u^{n-2i-2} v^i \tag{57}$$

By (56) and (57) LHS of (55) becomes

$$4\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (n-2i)(n-2i-1) B_n^i u^{n-2i-2} v^{i+1} -$$

$$-\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (n-2i)(n-2i-1) B_n^i u^{n-2i} v^i -$$

$$-\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (n-2i) B_n^i u^{n-2i} v^i + \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i n^2 B_n^i u^{n-2i} v^i =$$

$$= 4\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i (n-2i)(n-2i-1) B_n^i u^{n-2i-2} v^{i+1} -$$

$$-\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \left((n-2i)(n-2i-1) + (n-2i) - n^2\right) B_n^i u^{n-2i} v^i =$$

$$= 4\sum_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i(n-2i)(n-2i-1)B_n^i u^{n-2i-2}v^{i+1}+$$

$$+4\sum_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i(n-i)\,iB_n^i u^{n-2i}v^i=$$

$$= 4\sum_{j=1}^{\left[\frac{n}{2}\right]+1}(-1)^{j-1}(n-2j+2)(n-2j+1)B_n^{j-1}u^{n-2j}v^j+$$

$$+4\sum_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i(n-i)\,iB_n^i u^{n-2i}v^i=$$

$$= 4\sum_{j=1}^{\left[\frac{n}{2}\right]}(-1)^j\left[(n-j)\,jB_n^j-(n-2j+2)(n-2j+1)B_n^{j-1}\right]u^{n-2j}v^j+$$

$$+4(-1)^{\left[\frac{n}{2}\right]}\left(n-2\left[\frac{n}{2}\right]\right)\left(n-2\left[\frac{n}{2}\right]-1\right)B_n^{\left[\frac{n}{2}\right]}u^{n-2\left[\frac{n}{2}\right]+2}v^{\left[\frac{n}{2}\right]+1}+$$

$$+(-1)^0 n\cdot 0\cdot B_n^0 u^n v^0 \tag{58}$$

If $n=2k$, then $\left[\frac{n}{2}\right]=1$, and if $n=2k+1$, then $\left[\frac{n}{2}\right]=\frac{n-1}{2}$, so one or other of the numbers $n-2\left[\frac{n}{2}\right]$, $n-2\left[\frac{n}{2}\right]-1$ is null. Therefore LHS of (55) becomes

$$\sum_{j=1}^{\left[\frac{n}{2}\right]}(-1)^j\left[(n-j)\,jB_n^j-(n-2j+2)(n-2j+1)B_n^{j-1}\right]u^{n-2j}v^j \tag{59}$$

Also we have

$$(n-j)\,jB_n^j-(n-2j+2)(n-2j+1)B_n^{j-1}=(n-j)\,jn\cdot\frac{(n-j-1)!}{j!\,(n-2j)!}-$$

$$-(n-2j+2)(n-2j+1)\,n\cdot\frac{(n-j)!}{(j-1)!\,(n-2j+2)!}=$$

$$=n\cdot\frac{(n-j)!}{(j-1)!\,(n-2j)!}-n\cdot\frac{(n-j)!}{(j-1)!\,(n-2j)!}=0 \tag{60}$$

The relations (59) and (60) proves the statement.

Remarks.
1. *If $f(x)$ and $g(x)$ have the property that for any $x\in R$, $f(x)\,g(x)=a=$ constant, then $P_n(u,a)=P_n(u)$ verify the equation*

$$\left(4a-u^2\right)\cdot\frac{d^2z(u)}{du^2}-u\cdot\frac{dz(u)}{du}+n^2z(u)=0 \tag{61}$$

2. *If $a=1$, then $P_n(u,1)=P_n(u)$ verify the equation*

$$\left(4-u^2\right)\cdot\frac{d^2z}{du^2}-u\cdot\frac{dz}{du}+n^2z=0 \tag{62}$$

3. *By (62) if $f(x)=e^{i\arccos x}$, $g(x)=e^{-i\arccos x}$, then*

$$P_n(u)=P_n(2x)=2T_n(x)\,,\ u=2x$$

and (62) becomes

$$4\left(1-x^2\right)\cdot 2\cdot\frac{d^2T_n\left(2x\right)}{d\left(2x\right)^2}-4x\cdot\frac{dT_n\left(2x\right)}{d\left(2x\right)}+n^2T_n\left(2x\right)=0 \tag{63}$$

But

$$\frac{dT_n\left(2x\right)}{d\left(2x\right)}=\frac{dT_n\left(x\right)}{dx}\cdot\frac{dx}{d\left(2x\right)}=\frac{1}{2}\cdot\frac{dT_n\left(x\right)}{dx};$$

$$\frac{d^2T_n\left(2x\right)}{d\left(2x\right)^2}=\frac{1}{4}\cdot\frac{d^2T_n\left(x\right)}{dx^2},$$

so (63) becomes

$$\left(1-x^2\right)\cdot\frac{d^2T_n\left(x\right)}{dx^2}-x\cdot\frac{dT_n\left(x\right)}{dx}+n^2T_n\left(x\right)=0 \tag{64}$$

which shows that Chebyshev verifies the following differential equation

$$\left(1-x^2\right)y''-xy'+n^2y=0 \tag{65}$$

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