IE675 Game Theory

Lecture Note Set 2

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2 TWO-PERSON GAMES

2.1 Two-Person Zero-Sum Games

2.1.1 Basic ideas

Definition 2.1. *A game (in extensive form) is said to be* **zero-sum** *if and only if, at each terminal vertex, the payoff vector* $(p_1, \ldots, p_n)$ *satisfies* $\sum_{i=1}^{n} p_i = 0$.

Two-person zero sum games in *normal form*. Here's an example...

$$A = \begin{bmatrix} -1 & -3 & -3 & -2 \\ 0 & 1 & -2 & -1 \\ 2 & -2 & 0 & 1 \end{bmatrix}$$

The rows represent the strategies of Player 1. The columns represent the strategies of Player 2. The entries a_{ij} represent the payoff vector $(a_{ij}, -a_{ij})$. That is, if Player 1 chooses row i and Player 2 chooses column j, then Player 1 wins a_{ij} and Player 2 loses a_{ij}. If $a_{ij} < 0$, then Player 1 pays Player 2 $|a_{ij}|$.

Note 2.1. We are using the term *strategy* rather than *action* to describe the player's options. The reasons for this will become evident in the next chapter when we use this formulation to analyze games in extensive form.

Note 2.2. Some authors (in particular, those in the field of control theory) prefer to represent the outcome of a game in terms of *losses* rather than *profits*. During the semester, we will use both conventions.

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How should each player behave? Player 1, for example, might want to place a bound on his profits. Player 1 could ask “For each of my possible strategies, what is the least desirable thing that Player 2 could do to minimize my profits?” For each of Player 1's strategies i, compute

$$\alpha_i = \min_j a_{ij}$$

and then choose that i which produces $\max_i \alpha_i$. Suppose this maximum is achieved for $i = i^*$. In other words, Player 1 is guaranteed to get at least

$$\underline{V}(A) = \min_j a_{i^*j} \geq \min_j a_{ij} \quad i = 1, \ldots, m$$

The value $\underline{V}(A)$ is called the *gain-floor* for the game A.

In this case $\underline{V}(A) = -2$ with $i^* \in \{2, 3\}$.

Player 2 could perform a similar analysis and find that j^* which yields

$$\overline{V}(A) = \max_i a_{ij^*} \leq \max_i a_{ij} \quad j = 1, \ldots, n$$

The value $\overline{V}(A)$ is called the *loss-ceiling* for the game A.

In this case $\overline{V}(A) = 0$ with $j^* = 3$.

Now, consider the joint strategies (i^*, j^*). We immediately get the following:

Theorem 2.1. *For every (finite) matrix game* $A = [a_{ij}]$

1. *The values* $\underline{V}(A)$ *and* $\overline{V}(A)$ *are unique.*
2. *There exists at least one security strategy for each player given by* (i^*, j^*).
3. $\min_j a_{i^*j} = \underline{V}(A) \leq \overline{V}(A) = \max_i a_{ij^*}$

Proof: (1) and (2) are easy. To prove (3) note that for any k and ℓ,

$$\min_j a_{kj} \leq a_{k\ell} \leq \max_i a_{i\ell}$$

and the result follows. ■

2.1.2 Discussion

Let's examine the decision-making philosophy that underlies the choice of (i^*, j^*). For instance, Player 1 appears to be acting as if Player 2 is trying to do as much harm to him as possible. This seems reasonable since this is a zero-sum game. Whatever, Player 1 wins, Player 2 loses.

As we proceed through this presentation, note that this same reasoning is also used in the field of statistical decision theory where Player 1 is the statistician, and Player 2 is "nature." Is it reasonable to assume that "nature" is a malevolent opponent?

2.1.3 Stability

Consider another example

$$A = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -3 \\ -1 & -2 & -1 \end{bmatrix}$$

Player 1 should consider $i^* = 3$ ($\underline{V} = -2$) and Player 2 should consider $j^* = 1$ ($\overline{V} = 0$).

However, Player 2 can continue his analysis as follows

- Player 2 will choose strategy 1
- So Player 1 should choose strategy 2 rather than strategy 3
- But Player 2 would predict that and then prefer strategy 3

and so on.

Question 2.1. When do we have a stable choice of strategies?

The answer to the above question gives rise to some of the really important early results in game theory and mathematical programming.

We can see that if $\underline{V}(A) = \overline{V}(A)$, then both Players will settle on (i^*, j^*) with

$$\min_j a_{i^*j} = \underline{V}(A) = \overline{V}(A) = \max_i a_{ij^*}$$

Theorem 2.2. *If $\underline{V}(A) = \overline{V}(A)$ then*

1. *A has a saddle point*

2. *The saddle point corresponds to the security strategies for each player*
3. *The* value *for the game is* $V = \underline{V}(A) = \overline{V}(A)$

Question 2.2. Suppose $\underline{V}(A) < \overline{V}(A)$. What can we do? Can we establish a "spy-proof" mechanism to implement a strategy?

Question 2.3. Is it ever sensible to use expected loss (or profit) as a performance criterion in determining strategies for "one-shot" (non-repeated) decision problems?

2.1.4 Developing Mixed Strategies

Consider the following matrix game…

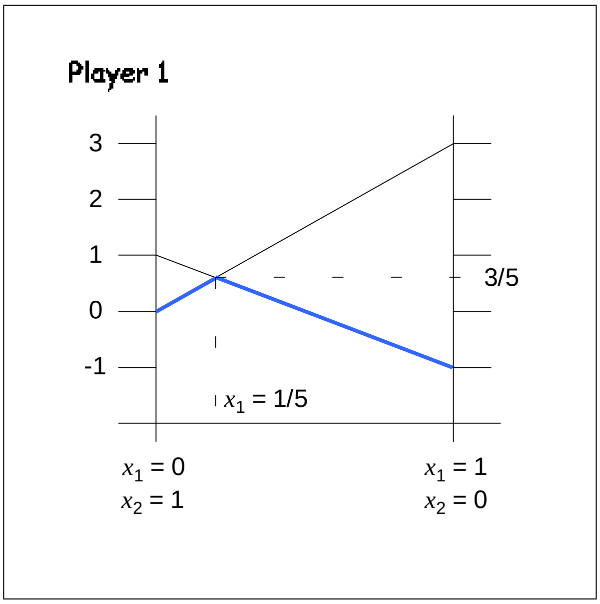
$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

For Player 1, we have $\underline{V}(A) = 0$ and $i^* = 2$. For Player 2, we have $\overline{V}(A) = 1$ and $j^* = 2$. This game does not have a saddle point.

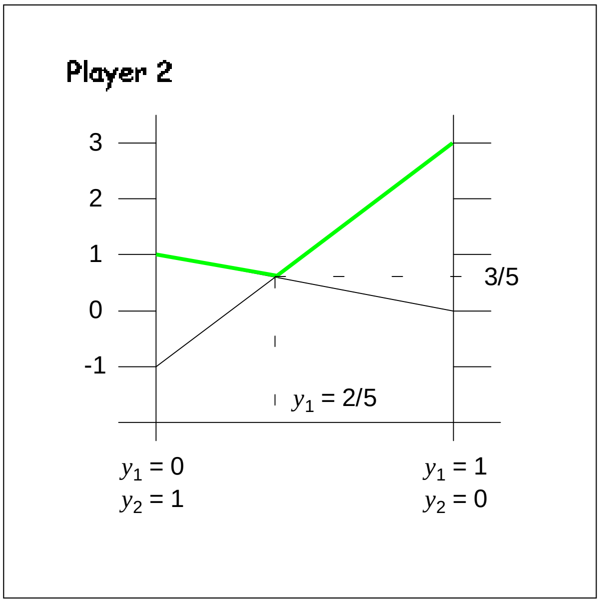
Let's try to create a "spy-proof" strategy. Let Player 1 randomize over his two *pure strategies*. That is Player 1 will pick the vector of probabilities $x = (x_1, x_2)$ where $\sum_i x_i = 1$ and $x_i \geq 0$ for all i. He will then select strategy i with probability x_i.

Note 2.3. When we formalize this, we will call the probability vector x, a *mixed strategy*.

To determine the "best" choice of x, Player 1 analyzes the problem, as follows…



Player 2 might do the same thing using probability vector $y = (y_1, y_2)$ where $\sum_i y_i = 1$ and $y_i \geq 0$ for all i.



If Player 1 adopts mixed strategy (x_1, x_2) and Player 2 adopts mixed strategy (y_1, y_2), we obtain an expected payoff of

$$\begin{aligned} V &= 3x_1y_1 + 0(1-x_1)y_1 - x_1(1-y_1) \\ &\quad +(1-x_1)(1-y_1) \\ &= 5x_1y_1 - y_1 - 2x_1 + 1 \end{aligned}$$

Suppose Player 1 uses $x_1^* = \frac{1}{5}$, then

$$V = 5\left(\frac{1}{5}\right)y_1 - y_1 - 2\left(\frac{1}{5}\right) + 1 = \frac{3}{5}$$

which doesn't depend on y! Similarly, suppose Player 2 uses $y_1^* = \frac{2}{5}$, then

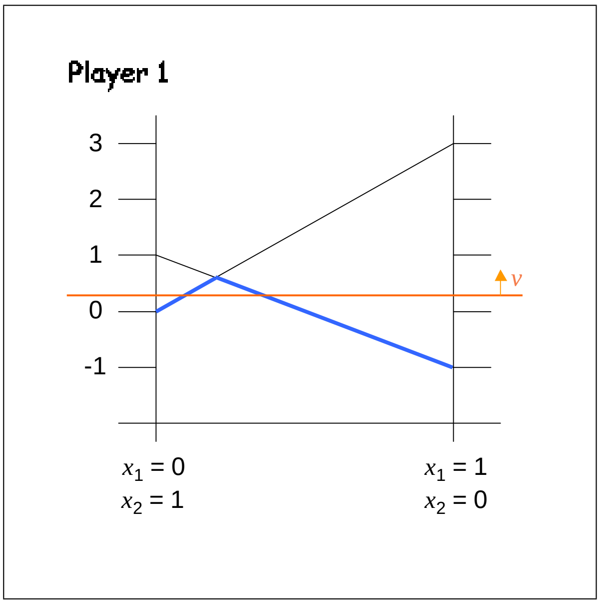
$$V = 5x_1\left(\frac{2}{5}\right) - \left(\frac{2}{5}\right) - 2x_1 + 1 = \frac{3}{5}$$

which doesn't depend on x!

Each player is solving a constrained optimization problem. For Player 1 the problem is

$$\begin{array}{rlcll} & \max\{v\} & & & \\ \text{st:} & +3x_1 + 0x_2 & \geq & v & \\ & -1x_1 + 1x_2 & \geq & v & \\ & x_1 + x_2 & = & 1 & \\ & x_i & \geq & 0 & \forall\, i \end{array}$$

which can be illustrated as follows:



This problem is equivalent to

$$\max_x \min\{(3x_1 + 0x_2), (-x_1 + x_2)\}$$

For Player 2 the problem is

$$\begin{array}{rlcll} & \min\{v\} & & & \\ \text{st:} & +3y_1 - 1y_2 & \leq & v & \\ & +0y_1 + 1y_2 & \leq & v & \\ & y_1 + y_2 & = & 1 & \\ & y_j & \geq & 0 & \forall\, j \end{array}$$

which is equivalent to

$$\min_y \max\{(3y_1 - y_2), (0y_1 + y_2)\}$$

We recognize these as dual linear programming problems.

Question 2.4. We now have a way to compute a "spy-proof" mixed strategy for each player. Modify these two mathematical programming problems to produce the *pure* security strategy for each player.

In general, the players are solving the following pair of dual linear programming problems:

$$\begin{array}{rlcll} & \max\{v\} & & & \\ \text{st:} & \sum_i a_{ij} x_i & \geq & v & \forall\, j \\ & \sum_i x_i & = & 1 & \\ & x_i & \geq & 0 & \forall\, i \end{array}$$

and

$$\begin{array}{rlcll} & \min\{v\} & & & \\ \text{st:} & \sum_j a_{ij} y_j & \leq & v & \forall\, i \\ & \sum_i y_i & = & 1 & \\ & y_i & \geq & 0 & \forall\, j \end{array}$$

Note 2.4. Consider, once again, the example game

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

If Player 1 (the maximizer) uses mixed strategy $(x_1, (1 - x_1))$, and if Player 2 (the minimizer) uses mixed strategy $(y_1, (1 - y_1))$ we get

$$E(x, y) = 5x_1 y_1 - y_1 - 2x_1 + 1$$

and letting $x^* = \frac{1}{5}$ and $y^* = \frac{2}{5}$ we get $E(x^*, y) = E(x, y^*) = \frac{3}{5}$ for any x and y. These choices for x^* and y^* make the expected value independent of the opposing strategy. So, if Player 1 becomes a minimizer (or if Player 2 becomes a maximizer) the resulting mixed strategies would be the same!

Note 2.5. Consider the game

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

By "factoring" the expression for $E(x, y)$, we can write

$$\begin{aligned} E(x, y) &= x_1 y_1 + 3x_1(1 - y_1) + 4(1 - x_1)y + 2(1 - x_1)(1 - y_1) \\ &= -4x_1 y_1 + x_1 + 2y_1 + 2 \\ &= -4(x_1 y_1 - \frac{x_1}{4} - \frac{y_1}{2} + \frac{1}{8}) + 2 + \frac{1}{2} \\ &= -4(x_1 - \frac{1}{2})(y_1 - \frac{1}{4}) + \frac{5}{2} \end{aligned}$$

It's now easy to see that $x_1^* = \frac{1}{2}$, $y_1^* = \frac{1}{4}$ and $v = \frac{5}{2}$.

2.1.5 A more formal statement of the problem

Suppose we are given a matrix game $A_{(m \times n)} \equiv [a_{ij}]$. Each row of A is a pure strategy for Player 1. Each column of A is a pure strategy for Player 2. The value of a_{ij} is the payoff from Player 1 to Player 2 (it may be negative).

For Player 1 let

$$\underline{V}(A) = \max_i \min_j a_{ij}$$

For Player 2 let

$$\overline{V}(A) = \min_j \max_i a_{ij}$$

$\{$**Case 1**$\}$ (Saddle Point Case where $\underline{V}(A) = \overline{V}(A) = V$)
Player 1 can assure himself of getting at least V from Player 2 by playing his maximin strategy.

$\{$**Case 2**$\}$ (Mixed Strategy Case where $\underline{V}(A) < \overline{V}(A)$)
Player 1 uses probability vector

$$x = (x_1, \dots, x_m) \quad \sum_i x_i = 1 \quad x_i \geq 0$$

Player 2 uses probability vector

$$y = (y_1, \dots, y_n) \quad \sum_j y_j = 1 \quad y_j \geq 0$$

If Player 1 uses x and Player 2 uses strategy j, the expected payoff is

$$E(x, j) = \sum_i x_i a_{ij} = x A_j$$

where A_j is column j from matrix A.

If Player 2 uses y and Player 1 uses strategy i, the expected payoff is

$$E(i, y) = \sum_j a_{ij} y_j = A^i y^{\mathrm{T}}$$

where A^i is row i from matrix A.

Combined, if Player 1 uses x and Player 2 uses y, the expected payoff is

$$E(x, y) = \sum_i \sum_j x_i a_{ij} y_j = x A y^{\mathrm{T}}$$

The players are solving the following pair of dual linear programming problems:

$$\begin{array}{llll} & \max\{v\} & & \\ \text{st:} & \sum_i a_{ij} x_i & \geq & v \quad \forall\, j \\ & \sum_i x_i & = & 1 \\ & x_i & \geq & 0 \quad \forall\, i \end{array}$$

and

$$\begin{array}{llll} & \min\{v\} & & \\ \text{st:} & \sum_j a_{ij} y_j & \leq & v \quad \forall\, i \\ & \sum_i y_i & = & 1 \\ & y_i & \geq & 0 \quad \forall\, j \end{array}$$

The *Minimax Theorem* (von Neumann, 1928) states that there exists mixed strategies x^* and y^* for Players 1 and 2 which solve each of the above problems with equal objective function values.

2.1.6 Proof of the Minimax Theorem

Note 2.6. (From Başar and Olsder [2]) The theory of finite zero-sum games dates back to Borel in the early 1920's whose work on the subject was later translated into English (Borel, 1953). Borel introduced the notion of a conflicting decision situation that involves more than one decision maker, and the concepts of pure and mixed strategies, but he did not really develop a complete theory of zero-sum games. Borel even conjectured that the Minimax Theorem was false.

It was von Neumann who first came up with a proof of the Minimax Theorem, and laid down the foundations of game theory as we know it today (von Neumann 1928, 1937).

We will provide two proofs of this important theorem. The first proof (Theorem 2.4) uses only the Separating Hyperplane Theorem. The second proof (Theorem 2.5) uses the similar, but more powerful, tool of duality from the theory linear programming.

Our first, and direct, proof of the Minimax Theorem is based on the proof by von Neumann and Morgenstern [7]. It also appears in the book by Başar and Olsder [2]. It depends on the Separating Hyperplane Theorem:[1]

Theorem 2.3. (From [1]) **Separating Hyperplane Theorem.** *Let S and T be two non-empty, convex sets in $\mathbb{R}^n$ with no interior point in common. Then there exists a pair (p, c) with $p \in \mathbb{R}^n \neq 0$ and $c \in \mathbb{R}$ such that*

$$\begin{array}{rcll} px & \geq & c & \forall x \in S \\ py & \leq & c & \forall y \in T \end{array}$$

i.e., there is a hyperplane $H(p, c) = \{x \in \mathbb{R}^n \mid px = c\}$ that separates S and T.

Proof: Define $S - T = \{x - y \in \mathbb{R}^n \mid x \in S, y \in T\}$. $S - T$ is convex. Then $0 \notin \text{int}(S - T)$ (if it was, i.e., if $0 \in \text{int}(S - T)$, then there is an $x \in \text{int}(S)$ and $y \in \text{int}(T)$ such that $x - y = 0$, or simply $x = y$, which would be a common interior point). Thus, we can "separate" 0 from $S - T$, i.e., there exists $p \in \mathbb{R}^n$ where $p \neq 0$ and $c \in \mathbb{R}$ such that $p \cdot (x - y) \geq c$ and $p \cdot 0 \leq c$. But, this implies that

$$p \cdot 0 = 0 \leq c \leq p \cdot (x - y)$$

which implies $p \cdot (x - y) \geq 0$. Hence, $px \geq py$ for all $x \in S$ and for all $y \in T$. That is, there must be a $c \in \mathbb{R}$ such that

$$py \leq c \leq px \quad \forall x \in S \text{ and } \forall y \in T$$

■

A version of Theorem 2.3 also appears in a paper by Gale [5] and a text by Boot [3].

Theorem 2.3 can be used to produce the following corollary that we will use to prove the Minimax Theorem:

Corollary 2.1. *Let A be an arbitrary $(m \times n)$-dimensional matrix. Then either*

(i) there exists a nonzero vector $x \in \mathbb{R}^m$, $x \geq 0$ such that $xA \geq 0$, or

(ii) there exists a nonzero vector $y \in \mathbb{R}^n$, $y \geq 0$ such that $Ay^{\mathrm{T}} \leq 0$.

Theorem 2.4. **Minimax Theorem.** *Let $A = [a_{ij}]$ be an $m \times n$ matrix of real numbers. Let Ξ^r denote the set of all r-dimensional probability vectors, that is,*

$$\Xi^r = \{x \in \mathbb{R}^r \mid \textstyle\sum_{i=1}^r x_i = 1 \text{ and } x_i \geq 0\}$$

[1] I must thank Yong Bao for his help in finding several errors in a previous version of these notes.

We sometimes call Ξ^r the **probability simplex.**

Let $x \in \Xi^m$ and $y \in \Xi^n$. Define

$$\begin{aligned}\underline{V}_m(A) &\equiv \max_x \min_y xAy^{\mathrm{T}} \\ \overline{V}_m(A) &\equiv \min_y \max_x xAy^{\mathrm{T}}\end{aligned}$$

Then $\underline{V}_m(A) = \overline{V}_m(A)$.

Proof: First we will prove that

(1) $$\underline{V}_m(A) \leq \overline{V}_m(A)$$

To do so, note that xAy^{T}, $\max_x xAy^{\mathrm{T}}$ and $\min_y xAy^{\mathrm{T}}$ are all continuous functions of (x, y), x and y, respectively. Any continuous, real-valued function on a compact set has an extermum. Therefore, there exists x^0 and y^0 such that

$$\begin{aligned}\underline{V}_m(A) &= \min_y x^0 Ay^{\mathrm{T}} \\ \overline{V}_m(A) &= \max_x xAy^{0\mathrm{T}}\end{aligned}$$

It is clear that

(2) $$\underline{V}_m(A) \leq x^0 Ay^{0\mathrm{T}} \leq \overline{V}_m(A)$$

Thus relation (1) is true.

Now we will show that one of the following must be true:

(3) $$\overline{V}_m(A) \leq 0 \qquad \text{or} \qquad \underline{V}_m(A) \geq 0$$

Corollary 2.1 provides that, for any matrix A, one of the two conditions (i) or (ii) in the corollary must be true. Suppose that condition (ii) is true. Then there exists $y^0 \in \Xi^n$ such that[2]

$$\begin{aligned} Ay^{0\mathrm{T}} &\leq 0 \\ \Rightarrow \quad xAy^{0\mathrm{T}} &\leq 0 \quad \forall x \in \Xi^m \\ \Rightarrow \quad \max_x xAy^{0\mathrm{T}} &\leq 0\end{aligned}$$

Hence

$$\overline{V}_m(A) = \min_y \max_x xAy^{\mathrm{T}} \leq 0$$

[2]Corollary 2.1 says that there must exist such a $y^0 \in \mathbb{R}^n$. Why doesn't it make a difference when we use Ξ^n rather than $\mathbb{R}^n$?

Alternatively, if (i) is true then we can similarly show that

$$\underline{V}_m(A) = \max_x \min_y xAy^{\mathrm{T}} \geq 0$$

Define the $(m \times n)$ matrix $B = [b_{ij}]$ where $b_{ij} = a_{ij} - c$ for all (i, j) and where c is a constant. Note that

$$\overline{V}_m(B) = \overline{V}_m(A) - c \quad \text{and} \quad \underline{V}_m(B) = \underline{V}_m(A) - c$$

Since A was an arbitrary matrix, the previous results also hold for B. Hence either

$$\begin{aligned} \overline{V}_m(B) = \overline{V}_m(A) - c &\leq 0 \quad \text{or} \\ \underline{V}_m(B) = \underline{V}_m(A) - c &\geq 0 \end{aligned}$$

Thus, for any constant c, either

$$\begin{aligned} \overline{V}_m(A) &\leq c \quad \text{or} \\ \underline{V}_m(A) &\geq c \end{aligned}$$

Relation (1) guarantees that

$$\underline{V}_m(A) \leq \overline{V}_m(A)$$

Therefore, there exists a $\Delta \geq 0$ such that

$$\underline{V}_m(A) + \Delta = \overline{V}_m(A).$$

Suppose $\Delta > 0$. Choose $c = \Delta/2$ and we have found a c such that both

$$\begin{aligned} \overline{V}_m(A) &\geq c \quad \text{and} \\ \underline{V}_m(A) &\leq c \end{aligned}$$

are true. This contradicts our previous result. Hence $\Delta = 0$ and $\underline{V}_m(A) = \overline{V}_m(A)$.

■

2.1.7 The Minimax Theorem and duality

The next version of the Minimax Theorem uses duality and provides several fundamental links between game theory and the theory of linear programming.[3]

Theorem 2.5. *Consider the matrix game A with mixed strategies x and y for Player 1 and Player 2, respectively. Then*

[3]This theorem and proof is from my own notebook from a Game Theory course taught at Cornell in the summer of 1972. The course was taught by Professors William Lucas and Louis Billera. I believe, but I cannot be sure, that this particular proof is from Professor Billera.

1. **minimax statement**

$$\max_x \min_y E(x,y) = \min_y \max_x E(x,y)$$

2. **saddle point statement (mixed strategies)** *There exists x^* and y^* such that*

$$E(x,y^*) \leq E(x^*,y^*) \leq E(x^*,y)$$

for all x and y.

2a. **saddle point statement (pure strategies)** *Let $E(i,y)$ denote the expected value for the game if Player 1 uses pure strategy i and Player 2 uses mixed strategy y. Let $E(x,j)$ denote the expected value for the game if Player 1 uses mixed strategy x and Player 2 uses pure strategy j. There exists x^* and y^* such that*

$$E(i,y^*) \leq E(x^*,y^*) \leq E(x^*,j)$$

for all i and j.

3. **LP feasibility statement** *There exists x^*, y^*, and $v' = v''$ such that*

$$\boxed{\begin{array}{lcll} \sum_i a_{ij}x_i^* & \geq & v' & \forall\, j \\ \sum_i x_i^* & = & 1 & \\ x_i^* & \geq & 0 & \forall\, i \end{array}} \qquad \boxed{\begin{array}{lcll} \sum_j a_{ij}y_j^* & \leq & v'' & \forall\, i \\ \sum_j y_j^* & = & 1 & \\ y_j^* & \geq & 0 & \forall\, j \end{array}}$$

4. **LP duality statement** *The objective function values are the same for the following two linear programming problems:*

$$\boxed{\begin{array}{llcll} & \max\{v\} & & & \\ \text{st:} & \sum_i a_{ij}x_i^* & \geq & v & \forall\, j \\ & \sum_i x_i^* & = & 1 & \\ & x_i^* & \geq & 0 & \forall\, i \end{array}} \qquad \boxed{\begin{array}{llcll} & \min\{v\} & & & \\ \text{st:} & \sum_j a_{ij}y_j^* & \leq & v & \forall\, i \\ & \sum_i y_j^* & = & 1 & \\ & y_j^* & \geq & 0 & \forall\, j \end{array}}$$

Proof: We will sketch the proof for the above results by showing that

$$(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (4)$$

and

$$(2) \Leftrightarrow (2a)$$

.

$\{(4) \Rightarrow (3)\}$ (3) is just a special case of (4).

$\{(3) \Rightarrow (2)\}$ Let 1_n denote a column vector of n ones. Then (3) implies that there exists x^*, y^*, and $v' = v''$ such that

$$\begin{aligned} x^*A &\geq v'1_n \\ x^*Ay^{\mathrm{T}} &\geq v'(1_n y^{\mathrm{T}}) = v' \qquad \forall\, y \end{aligned}$$

and

$$\begin{aligned} Ay^{*\mathrm{T}} &\leq v''1_m \\ xAy^{*\mathrm{T}} &\leq xv''1_m = v''(x1_m) = v'' \qquad \forall\, x \end{aligned}$$

Hence,

$$E(x^*, y) \geq v' = v'' \geq E(x, y^*) \quad \forall\, x, y$$

and

$$E(x^*, y^*) = v' = v'' = E(x^*, y^*)$$

$\{(2) \Rightarrow (2a)\}$ (2a) is just a special case of (2) using mixed strategies x with $x_i = 1$ and $x_k = 0$ for $k \neq i$.

$\{(2a) \Rightarrow (2)\}$ For each i, consider all convex combinations of vectors x with $x_i = 1$ and $x_k = 0$ for $k \neq i$. Since $E(i, y^*) \leq v$, we must have $E(x^*, y^*) \leq v$.

$\{(2) \Rightarrow (1)\}$

- $\{$**Case** $\geq\}$

$$\begin{aligned} E(x, y^*) &\leq E(x^*, y) \qquad \forall\, x, y \\ \max_x E(x, y^*) &\leq E(x^*, y) \qquad \forall\, y \\ \max_x E(x, y^*) &\leq \min_y E(x^*, y) \\ \min_y \max_x E(x, y) \leq \max_x E(x, y^*) &\leq \min_y E(x^*, y) \leq \max_x \min_y E(x, y) \end{aligned}$$

- $\{$**Case** $\leq\}$

$$\begin{aligned} \min_y E(x, y) &\leq E(x, y) \qquad \forall\, x, y \\ \max_x \left[\min_y E(x, y)\right] &\leq \max_x E(x, y) \qquad \forall\, y \\ \max_x \left[\min_y E(x, y)\right] &\leq \min_y \left[\max_x E(x, y)\right] \end{aligned}$$

$\{(1) \Rightarrow (3)\}$

$$\max_x \left[\min_y E(x,y)\right] = \min_y \left[\max_x E(x,y)\right]$$

Let $f(x) = \min_y E(x,y)$. From calculus, there exists x^* such that $f(x)$ attains its maximum value at x^*. Hence

$$\min_y E(x^*,y) = \max_x \left[\min_y E(x,y)\right]$$

$\{(3) \Rightarrow (4)\}$ This is direct from the duality theorem of LP. (See Chapter 13 of Dantzig's text.)

■

Question 2.5. Can the LP problem in section (4) of Theorem 2.5 have alternate optimal solutions. If so, how does that affect the choice of (x^*, y^*)?[4]

2.2 Two-Person General-Sum Games

2.2.1 Basic ideas

Two-person general-sum games (sometimes called "bi-matrix games") can be represented by two $(m \times n)$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ where a_{ij} is the "payoff" to Player 1 and b_{ij} is the "payoff" to Player 2. If $A = -B$ then we get a two-person zero-sum game, A.

Note 2.7. These are non-cooperative games with no side payments.

Definition 2.2. *The (pure) strategy* (i^*, j^*) *is a* **Nash equilibrium solution** *to the game* (A, B) *if*

$$\begin{aligned} a_{i^*,j^*} &\geq a_{i,j^*} \qquad \forall\, i \\ b_{i^*,j^*} &\geq b_{i^*,j} \qquad \forall\, j \end{aligned}$$

Note 2.8. If both players are placed on their respective Nash equilibrium strategies (i^*, j^*), then each player cannot unilaterally move away from that strategy and improve his payoff.

[4]Thanks to Esra E. Aleisa for this question.

Question 2.6. Show that if $A = -B$ (zero-sum case), the above definition of a Nash solution corresponds to our previous definition of a saddle point.

Note 2.9. Not every game has a Nash solution using pure strategies.

Note 2.10. A Nash solution need not be the best solution, or even a reasonable solution for a game. It's merely a stable solution against unilateral moves by a single player. For example, consider the game

$$(A, B) = \begin{bmatrix} (4,0) & (4,1) \\ (5,3) & (3,2) \end{bmatrix}$$

This game has two Nash equilibrium strategies, $(4, 1)$ and $(5, 3)$. Note that both players prefer $(5, 3)$ when compared with $(4, 1)$.

Question 2.7. What is the solution to the following simple modification of the above game:[5]

$$(A, B) = \begin{bmatrix} (4,0) & (4,1) \\ (4,2) & (3,2) \end{bmatrix}$$

Example 2.1. (Prisoner's Dilemma) Two suspects in a crime have been picked up by police and placed in separate rooms. If both confess (C), each will be sentenced to 3 years in prison. If only one confesses, he will be set free and the other (who didn't confess (NC)) will be sent to prison for 4 years. If neither confesses, they will both go to prison for 1 year.

This game can be represented in strategic form, as follows:

	C	NC
C	(-3,-3)	(0,-4)
NC	(-4,0)	(-1,-1)

This game has one Nash equilibrium strategy, $(-3, -3)$. When compared with the other solutions, note that it represents one of the worst outcomes for both players.

2.2.2 Properties of Nash strategies

[5] Thanks to Esra E. Aleisa for this question.

Definition 2.3. *The pure strategy pair* (i_1, j_1) **weakly dominates** (i_2, j_2) *if and only if*

$$\begin{aligned} a_{i_1,j_1} &\geq a_{i_2,j_2} \\ b_{i_1,j_1} &\geq b_{i_2,j_2} \end{aligned}$$

and one of the above inequalities is strict.

Definition 2.4. *The pure strategy pair* (i_1, j_1) **strongly dominates** (i_2, j_2) *if and only if*

$$\begin{aligned} a_{i_1,j_1} &> a_{i_2,j_2} \\ b_{i_1,j_1} &> b_{i_2,j_2} \end{aligned}$$

Definition 2.5. (Weiss [8]) *The pure strategy pair* (i, j) *is* **inadmissible** *if there exists some strategy pair* $(i\prime, j\prime)$ *that weakly dominates* (i, j).

Definition 2.6. (Weiss [8]) *The pure strategy pair* (i, j) *is* **admissible** *if it is not* **inadmissible**.

Example 2.2. Consider again the game

$$(A, B) = \begin{bmatrix} (4, 0) & (4, 1) \\ (5, 3) & (3, 2) \end{bmatrix}$$

With Nash equilibrium strategies, $(4, 1)$ and $(5, 3)$. Only $(5, 3)$ is admissible.

Note 2.11. If there exists multiple admissible Nash equilibria, then side-payments (with collusion) may yield a “better” solution for all players.

Definition 2.7. *Two bi-matrix games* $(A.B)$ *and* (C, D) *are* **strategically equivalent** *if there exists* $\alpha_1 > 0$, $\alpha_2 > 0$ *and scalars* β_1, β_2 *such that*

$$\begin{aligned} a_{ij} &= \alpha_1 c_{ij} + \beta_1 \qquad \forall\, i, j \\ b_{ij} &= \alpha_2 d_{ij} + \beta_2 \qquad \forall\, i, j \end{aligned}$$

Theorem 2.6. *If bi-matrix games* $(A.B)$ *and* (C, D) *are strategically equivalent and* (i^*, j^*) *is a Nash strategy for* (A, B), *then* (i^*, j^*) *is also a Nash strategy for* (C, D).

Note 2.12. This was used to modify the original matrices for the Prisoners’ Dilemma problem in Example 2.1.

2.2.3 Nash equilibria using mixed strategies

Sometimes the bi-matrix game (A, B) does not have a Nash strategy using pure strategies. As before, we can use mixed strategies for such games.

Definition 2.8. *The (mixed) strategy (x^*, y^*) is a* **Nash equilibrium solution** *to the game (A, B) if*

$$\begin{array}{rcll} x^* A y^{*\mathrm{T}} & \geq & x A y^{*\mathrm{T}} & \forall\, x \in \Xi^m \\ x^* B y^{*\mathrm{T}} & \geq & x^* B y^{\mathrm{T}} & \forall\, y \in \Xi^n \end{array}$$

where Ξ^r is the r-dimensional probability simplex.

Question 2.8. Consider the game

$$(A, B) = \begin{bmatrix} (-2, -4) & (0, -3) \\ (-3, 0) & (1, -1) \end{bmatrix}$$

Can we find mixed strategies (x^*, y^*) that provide a Nash solution as defined above?

Theorem 2.7. *Every bi-matrix game has at least one Nash equilibrium solution in mixed strategies.*

Proof: (This is the sketch provided by the text for Proposition 33.1; see Chapter 3 for a complete proofs for $N \geq 2$ players.)

Consider the sets Ξ^n and Ξ^m consisting of the mixed strategies for Player 1 and Player 2, respectively. Note that $\Xi^n \times \Xi^m$ is non-empty, convex and compact. Since the expected payoff functions xAy^{T} and xBy^{T} are linear in (x, y), the result follows using Brouwer's fixed point theorem, ■

2.2.4 Finding Nash mixed strategies

Consider again the game

$$(A, B) = \begin{bmatrix} (-2, -4) & (0, -3) \\ (-3, 0) & (1, -1) \end{bmatrix}$$

For Player 1

$$\begin{array}{rcl} xAy^{\mathrm{T}} & = & -2x_1y_1 - 3(1 - x_1)y_1 + (1 - x_1)(1 - y_1) \\ & = & 2x_1y_1 - x_1 - 4y_1 + 1 \end{array}$$

For Player 2

$$xBy^{\mathrm{T}} = -2x_1y_1 - 2x_1 + y_1 - 1$$

In order for (x^*, y^*) to be a Nash equilibrium, we must have for all $0 \leq x_1 \leq 1$

(4) $$x^*Ay^{*\mathrm{T}} \geq xAy^{*\mathrm{T}} \qquad \forall\, x \in \Xi^m$$

(5) $$x^*By^{*\mathrm{T}} \geq x^*By^{\mathrm{T}} \qquad \forall\, y \in \Xi^n$$

For Player 1 this means that we want (x^*, y^*) so that for all x_1

$$\begin{aligned} 2x_1^*y_1^* - x_1^* - 4y_1^* + 1 &\geq 2x_1y_1^* - x_1 - 4y_1^* + 1 \\ 2x_1^*y_1^* - x_1^* &\geq 2x_1y_1^* - x_1 \end{aligned}$$

Let's try $y_1^* = \frac{1}{2}$. We get

$$\begin{aligned} 2x_1^*\left(\frac{1}{2}\right) - x_1^* &\geq 2x_1\left(\frac{1}{2}\right) - x_1 \\ 0 &\geq 0 \end{aligned}$$

Therefore, if $y^* = (\frac{1}{2}, \frac{1}{2})$ then any x^* can be chosen and condition (4) will be satisfied.

Note that only condition (4) and Player 1's matrix A was used to get Player 2's strategy y^*.

For Player 2 the same thing happens if we use $x_1^* = \frac{1}{2}$ and condition (5). That is, for all $0 \leq y_1 \leq 1$

$$\begin{aligned} -2x_1^*y_1^* - 2x_1^* + y_1^* - 1 &\geq -2x_1y_1^* - 2x_1 + y_1^* - 1 \\ -2x_1^*y_1^* + y_1^* &\geq -2x_1y_1^* + y_1 \\ -2\left(\frac{1}{2}\right)y_1^* + y_1^* &\geq -2\left(\frac{1}{2}\right)y_1^* + y_1 \\ 0 &\geq 0 \end{aligned}$$

How can we get the values of (x^*, y^*) that will work? One suggested approach from (Başar and Olsder [2]) uses the following:

Theorem 2.8. *Any mixed Nash equilibrium solution* (x^*, y^*) *in the interior of* $\Xi^m \times \Xi^n$ *must satisfy*

$$\sum_{j=1}^{n} y_j^*(a_{ij} - a_{1j}) = 0 \qquad \forall\, i \neq 1 \tag{6}$$

$$\sum_{i=1}^{m} x_i^*(b_{ij} - b_{i1}) = 0 \qquad \forall\, j \neq 1 \tag{7}$$

Proof: Recall that

$$\begin{aligned} E(x,y) = xAy^{\mathrm{T}} &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j a_{ij} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} x_i y_j a_{ij} \end{aligned}$$

Since $x_1 = 1 - \sum_{i=2}^{m} x_i$, we have

$$\begin{aligned} xAy^{\mathrm{T}} &= \sum_{j=1}^{n}\left[\sum_{i=2}^{m} x_i y_j a_{ij} + \left(1 - \sum_{i=2}^{m} x_i\right) y_j a_{1j}\right] \\ &= \sum_{j=1}^{n}\left[y_j a_{1j} + y_j \sum_{i=2}^{m} x_i (a_{ij} - a_{1j})\right] \\ &= \sum_{j=1}^{n}\left[y_j a_{1j} + \sum_{i=2}^{m} x_i \sum_{j=1}^{n} y_j (a_{ij} - a_{1j})\right] \end{aligned}$$

If (x^*, y^*) is an interior maximum (or minimum) then

$$\frac{\partial}{\partial x_i} xAy^{\mathrm{T}} = \sum_{j=1}^{n} y_j (a_{ij} - a_{1j}) = 0 \qquad \text{for } i = 2, \ldots, m$$

Which provide the Equations 6.

The derivation of Equations 7 is similar. ■

Note 2.13. In the proof we have the equation

$$xAy^{\mathrm{T}} = \sum_{j=1}^{n} y_j a_{1j} + \sum_{i=2}^{m} x_i \left[\sum_{j=1}^{n} y_j (a_{ij} - a_{1j})\right]$$

Any Nash solution (x^*, y^*) in the interior of $\Xi^m \times \Xi^n$ has

$$\sum_{j=1}^{n} y_j^*(a_{ij} - a_{1j}) = 0 \qquad \forall\, i \neq 1$$

So this choice of y^* produces

$$xAy^{\mathrm{T}} = \sum_{j=1}^{n} y_j a_{1j} + \sum_{i=2}^{m} x_i\, [0]$$

making this expression independent of x.

Note 2.14. Equations 6 and 7 only provide necessary (not sufficient) conditions, and only characterize solutions on the interior of the probability simplex (i.e., where every component of x and y are strictly positive).

For our example, these equations produce

$$\begin{aligned} y_1^*(a_{21} - a_{11}) + y_2^*(a_{22} - a_{12}) &= 0 \\ x_1^*(b_{12} - b_{11}) + x_2^*(b_{22} - b_{21}) &= 0 \end{aligned}$$

Since $x_2^* = 1 - x_1^*$ and $y_2^* = 1 - y_1^*$, we get

$$\begin{aligned} y_1^*(-3 - (-2)) + (1 - y_1^*)(1 - 0) &= 0 \\ -y_1^* + (1 - y_1^*) &= 0 \\ y_1^* &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} x_1^*(-3 - (-4)) + (1 - x_1^*)(-1 - 0) &= 0 \\ x_1^* - (1 - x_1^*) &= 0 \\ x_1^* &= \frac{1}{2} \end{aligned}$$

But, in addition, one must check that $x_1^* = \frac{1}{2}$ and $y_1^* = \frac{1}{2}$ are actually Nash solutions.

2.2.5 The Lemke-Howson algorithm

Lemke and Howson [6] developed a quadratic programming technique for finding mixed Nash strategies for two-person general sum games (A, B) in strategic form. Their method is based on the following fact, provided in their paper:

Let e_k denote a column vector of k ones, and let x and y be row vectors of dimension m and n, respectively. Let p and q denote scalars. We will also assume that A and B are matrices, each with m rows and n columns.

A mixed strategy is defined by a pair (x, y) such that

$$xe_m = ye_n = 1, \quad \text{and} \quad x \geq 0, y \geq 0 \tag{8}$$

with expected payoffs

$$xAy^{\mathrm{T}} \quad \text{and} \quad xBy^{\mathrm{T}}. \tag{9}$$

A Nash equilibrium solution is a pair $(\bar{x}, \bar{y})$ satisfying (8) such that for all (x, y) satisfying (8),

$$xA\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}} \quad \text{and} \quad \bar{x}By^{\mathrm{T}} \leq \bar{x}B\bar{y}^{\mathrm{T}}. \tag{10}$$

But this implies that

$$A\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}}e_m \quad \text{and} \quad \bar{x}B \leq \bar{x}B\bar{y}^{\mathrm{T}}e_n^{\mathrm{T}}. \tag{11}$$

Conversely, suppose (11) holds for $(\bar{x}, \bar{y})$ satisfying (8). Now choose an arbitrary (x, y) satisfying (8). Multiply the first expression in (11) on the left by x and second expression in (11) on the right by y^{T} to get (10). Hence, (8) and (11) are, together, equivalent to (8) and (10).

This serves as the foundation for the proof of the following theorem:

Theorem 2.9. *Any mixed strategy (x^*, y^*) for bi-matrix game (A, B) is a Nash equilibrium solution if and only if x^*, y^*, p^* and q^* solve problem (LH):*

$$\begin{array}{rlcll}
\textit{(LH):} & \max_{x,y,p,q}\{xAy^{\mathrm{T}} + xBy^{\mathrm{T}} - p - q\} & & & \\
\textit{st:} & Ay^{\mathrm{T}} & \leq & pe_m & \\
& B^{\mathrm{T}}x^{\mathrm{T}} & \leq & qe_n & \\
& x_i & \geq & 0 & \forall\, i \\
& y_j & \geq & 0 & \forall\, j \\
& \sum_{i=1}^m x_i & = & 1 & \\
& \sum_{j=1}^n y_j & = & 1 &
\end{array}$$

Proof: $(\Rightarrow)$

Every feasible solution (x, y, p, q) to problem (LH) must satisfy the constraints

$$\begin{array}{rcl}
Ay^{\mathrm{T}} & \leq & pe_m \\
xB & \leq & qe_n^{\mathrm{T}}.
\end{array}$$

Multiply both sides of the first constraint on the left by x and multiply the second constraint on the right by y^{T}. As a result, we see that a feasible (x, y, p, q) must satisfy

$$\begin{array}{rcl} xAy^{\mathrm{T}} & \leq & p \\ xBy^{\mathrm{T}} & \leq & q. \end{array}$$

Hence, for any feasible (x, y, p, q). the objective function must satisfy

$$xAy^{\mathrm{T}} + xBy^{\mathrm{T}} - p - q \leq 0.$$

Suppose (x^*, y^*) is any Nash solution for (A, B). Let

$$\begin{array}{rcl} p^* & = & x^* A y^{*\mathrm{T}} \\ q^* & = & x^* B y^{*\mathrm{T}}. \end{array}$$

Because of (10) and (11), this implies

$$\begin{array}{rcccl} Ay^{*\mathrm{T}} & \leq & x^* A y^{*\mathrm{T}} e_m & = & p^* e_m \\ x^* B & \leq & x^* B y^{*\mathrm{T}} e_n^{\mathrm{T}} & = & q^* e_n^{\mathrm{T}}. \end{array}$$

So this choice of (x^*, y^*, p^*, q^*) is feasible, and results in the objective function equal to zero. Hence it's an optimal solution to problem (LH)

$(\Leftarrow)$

Suppose $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ solves problem (LH). From Theorem 2.7, there is at least one Nash solution (x^*, y^*). Using the above argument, (x^*, y^*) must be an optimal solution to (LH) with an objective function value of zero. Since $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is an optimal solution to (LH), we must then have

$$\bar{x}A\bar{y}^{\mathrm{T}} + \bar{x}B\bar{y}^{\mathrm{T}} - \bar{p} - \bar{q} = 0 \tag{12}$$

with $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ satisfying the constraints

$$A\bar{y}^{\mathrm{T}} \leq \bar{p}e_m \tag{13}$$

$$\bar{x}B \leq \bar{q}e_n^{\mathrm{T}}. \tag{14}$$

Now multiply (13) on the left by $\bar{x}$ and multiply (14) on the right by $\bar{y}^{\mathrm{T}}$ to get

$$\bar{x}A\bar{y}^{\mathrm{T}} \leq \bar{p} \tag{15}$$

$$\bar{x}B\bar{y}^{\mathrm{T}} \leq \bar{q}. \tag{16}$$

Then (12), (15), and (16) together imply

$$\begin{aligned} \bar{x}A\bar{y}^{\mathrm{T}} &= \bar{p} \\ \bar{x}B\bar{y}^{\mathrm{T}} &= \bar{q}. \end{aligned}$$

So (13), and (14) can now be rewritten as

$$A\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}}e_m \tag{17}$$

$$\bar{x}B \leq \bar{x}B\bar{y}^{\mathrm{T}}e_n. \tag{18}$$

Choose an arbitrary $(x, y) \in \Xi^m \times \Xi^n$ and, this time, multiply (17) on the left by x and multiply (18) on the right by y^{T} to get

$$xA\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}} \tag{19}$$

$$\bar{x}By^{\mathrm{T}} \leq \bar{x}B\bar{y}^{\mathrm{T}} \tag{20}$$

for all $(x, y) \in \Xi^m \times \Xi^n$. Hence $(\bar{x}, \bar{y})$ is a Nash equilibrium solution. ■

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