

# Functional modelling with Gaussian Processes

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## Table of Contents





## Modelling





## Motivation

**D. Donoho**

Data, Data, Data!
Challenges and opportunities of the coming **Data Deluge**

Several data types:

- classification problem – needed in decision systems: frequently there are data of very high dimension and several hundred classes to take into account;
- regression/prediction problems.

# Machine learning

Historical background / Motivation:

- Huge amount of data, that should automatically be processed,
- Mathematics provides general solutions, solutions are i.e. not for a given problem,
- Need for "science", that uses mathematics machinery for solving practical problems.



# Definition of Machine Learning

**Machine learning**

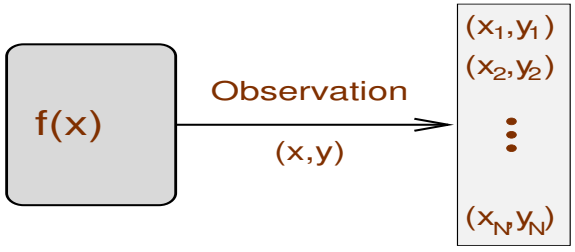
Collection of methods (from statistics, probability theory) to solve problems met in practice.

- noise filtering for
  - non-linear regression and/or
  - non-Gaussian noise
- Classification:
  - binary,
  - multiclass,
  - partially labeled
- Clustering,
- Inversion problems,
- density estimation, novelty detection.

Generally, we need to model the data,



# Modelling Data



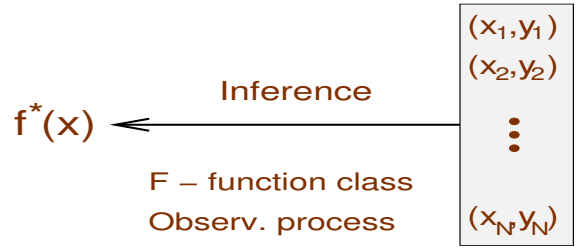

- Real world: there "is" a function $y = f(x)$
- Observation process: a corrupted datum is collected for a sample $x_n$:

$$t_n = y_n + \epsilon \quad \text{additive noise}$$
$$t_n = h(y_n, \epsilon) \quad h \text{ distortion function}$$

- **Problem: find function** $y = f(x)$



# Latent variable models




- **Data set** – collected.
- Assume a function class.
  - polynomial,
  - Fourier expansion,
  - Wavelet;
- Observation process – encodes the noise;
- Find the optimal function from the class.

# Latent variable models II

- We have the **data set** $\mathcal{D} = \{(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_N, y_N)\}$.
- Consider a function class:

$$(1) \quad \mathcal{F} = \left\{ \boldsymbol{w}^T \boldsymbol{x} + b | \boldsymbol{w} \in \mathbb{R}^d,\ b \in \mathbb{R} \right\}$$

$$(2) \quad \mathcal{F} = \left\{ a_0 + \sum_{k=1}^{K} a_k \sin(2\pi k x) + \sum_{k=1}^{K} b_k \cos(2\pi k x) \right.$$
$$\left. | \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^K,\ a_0 \in \mathbb{R} \right\}$$

- Assume an observation process:

$$y_n = f(\boldsymbol{x}_n) + \epsilon \quad \text{with } \epsilon \sim N(0, \sigma^2).$$

Functional Modelling
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Modelling
Machine Learning
Latent variable models
Estimation methods
Summary
Nonparametrics
SVM
$\mathcal{GP}$'s
$\mathcal{GP}$ applications
Ref

# Latent variable models III

1. The **data set:** $\mathcal{D} = \{(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_N, y_N)\}$.
2. Assume a function class:

$$\mathcal{F} = \left\{ f(\boldsymbol{x}, \boldsymbol{\theta}) | \boldsymbol{\theta} \in \mathbb{R}^p \right\}$$

   $\mathcal{F}$ – polynomial, etc.

3. Assume an observation process. Define a loss function:

$$L\left(y_n, f(\boldsymbol{x}_n, \boldsymbol{\theta})\right)$$

   For the Gaussian noise:

$$L(y_n, f(\boldsymbol{x}_n, \boldsymbol{\theta})) = (y_n - f(\boldsymbol{x}_n, \boldsymbol{\theta}))^2.$$

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Modelling
Machine Learning
Latent variable models
Estimation methods
Summary
Nonparametrics
SVM
$\mathcal{GP}$'s
$\mathcal{GP}$ applications
Ref

# Parameter estimation

Estimating parameters:

Finding the **optimal value to $\boldsymbol{\theta}$**:

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta} \in \Omega} L(\mathcal{D}, \boldsymbol{\theta})$$

where

- $\Omega$ is the domain of the parameters.
- $L(\mathcal{D}, \boldsymbol{\theta})$ is a "loss function" for the data set. Example:

$$L(\mathcal{D}, \boldsymbol{\theta}) = \sum_{n=1}^{N} L(y_n, f(\boldsymbol{x}_n, \boldsymbol{\theta}))$$

Functional Modelling
Lehel Csató
Modelling
Machine Learning
Latent variable models
Estimation methods
Summary
Nonparametrics
SVM
$\mathcal{GP}$'s
$\mathcal{GP}$ applications
Ref

# Parameter estimation – M.L.

$L(\mathcal{D}, \boldsymbol{\theta})$ – (log)likelihood function.

Maximum likelihood estimation of the model:

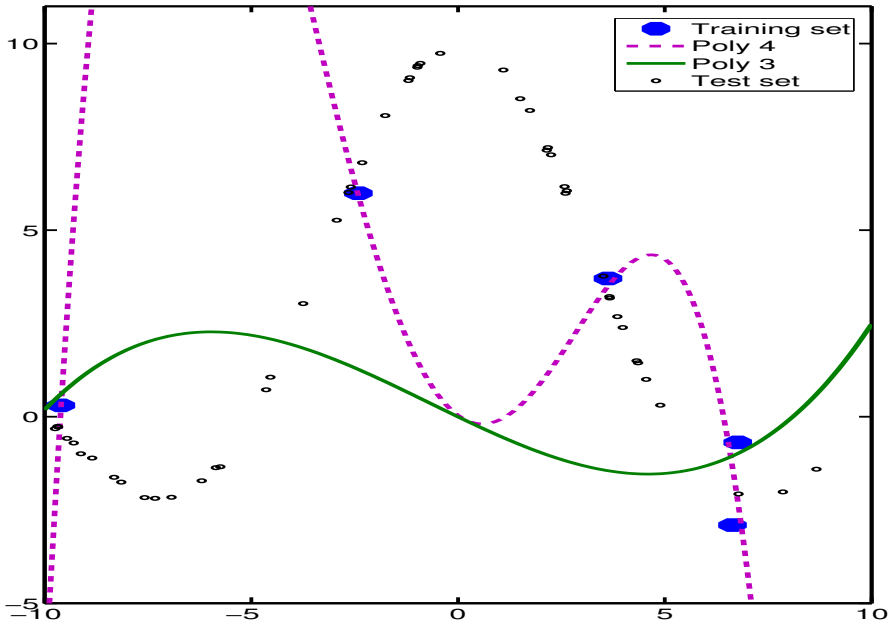
$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta} \in \Omega} L(\mathcal{D}, \boldsymbol{\theta})$$

Example – quadratic regression:

$$L(\mathcal{D}, \boldsymbol{\theta}) = \sum_{n=1}^{N} (y_n - f(\boldsymbol{x}_n, \boldsymbol{\theta}))^2 \quad \text{– factorisation}$$

Drawback: produces perfect fit to the data – **over-fitting**.

# Maximum Likelihood – Over-fitting






# Maximum a–posteriori estimation

M.A.P. – assigning probabilities to the

- Data $\mathcal{D}$: log–likelihood function: the probability of the data drawn using $\boldsymbol{\theta}$

$$P(y_n|\boldsymbol{x}_n, \boldsymbol{\theta}, \boldsymbol{\mathcal{F}}) \propto \exp\left[-L(y_n, f(\boldsymbol{x}_n, \boldsymbol{\theta}))\right]$$

$\propto$ – a normalisation constant missing.

- Parameters $\boldsymbol{\theta}$: probability that $\boldsymbol{\theta}$ could have had a given value

$$p_0(\boldsymbol{\theta}) \propto \exp\left[-\frac{\|\boldsymbol{\theta}\|^2}{2}\right]$$

prior probability.



# M.A.P. estimation II

Combining prior with observation – likelihood – using Bayes' rule:

A–posteriori probability of the parameters:

$$p(\boldsymbol{\theta}|\mathcal{D}, \boldsymbol{\mathcal{F}}) = \frac{P(\mathcal{D}|\boldsymbol{\theta})p_0(\boldsymbol{\theta})}{p(\mathcal{D}|\boldsymbol{\mathcal{F}})}$$

$$p(\mathcal{D}|\boldsymbol{\mathcal{F}}) = \int_\Omega d\boldsymbol{\theta}\ \ P(\mathcal{D}|\boldsymbol{\theta})p_0(\boldsymbol{\theta})$$

$p(\mathcal{D}|\boldsymbol{\mathcal{F}})$ – probability of the data set under a given family of models.



# M.A.P. estimation III

**M.A.P. estimation** – aims at finding $\boldsymbol{\theta}$ with the largest probability:

$$\boldsymbol{\theta}^*_{MAP} = \arg\max_{\boldsymbol{\theta}\in\Omega} p(\boldsymbol{\theta}|\mathcal{D}, \boldsymbol{\mathcal{F}})$$

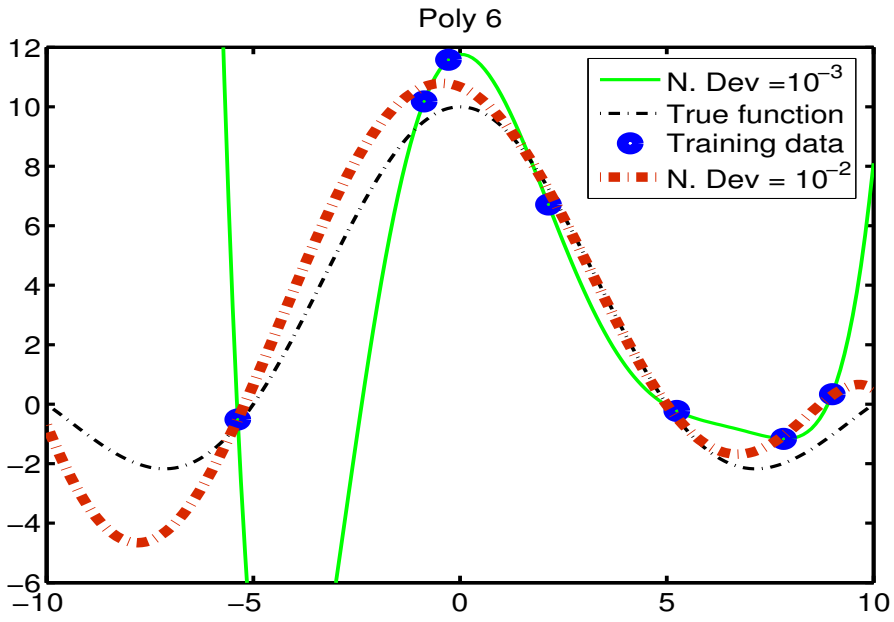
Example:
Using $L(y_n, f(\boldsymbol{x}_n, \boldsymbol{\theta}))$ in defining the likelihood and Gaussian prior:

$$\boldsymbol{\theta}^*_{MAP} = \arg\max_{\boldsymbol{\theta}\in\Omega} K - \sum_n L(y_n, f(\boldsymbol{x}_n, \boldsymbol{\theta})) - \frac{\|\boldsymbol{\theta}\|^2}{2\sigma_o^2}$$

For $\sigma_o^2 \to \infty$ we have maximum likelihood

after a change of sign and max $\to$ min

# M.A.P. – Example






# Parameter estimation – Bayes

We use Bayes' rule:

$$p(\boldsymbol{\theta}|\mathcal{D},\boldsymbol{\mathcal{F}}) = \frac{P(\mathcal{D}|\boldsymbol{\theta})p_0(\boldsymbol{\theta})}{p(\mathcal{D}|\boldsymbol{\mathcal{F}})}$$

$$p(\mathcal{D}|\boldsymbol{\mathcal{F}}) = \int_\Omega d\boldsymbol{\theta}\ \ P(\mathcal{D}|\boldsymbol{\theta})p_0(\boldsymbol{\theta})$$

and try to store the whole distribution of the possible values.

We operate therefore with

$$p_{\text{post}}(\boldsymbol{\theta}|\mathcal{D},\boldsymbol{\mathcal{F}})$$



# Bayes estimation II

When computing $p_{\text{post}}(\boldsymbol{\theta}|\mathcal{D},\boldsymbol{\mathcal{F}})$ we assumed that the posterior can be represented analytically.

This is not the case.

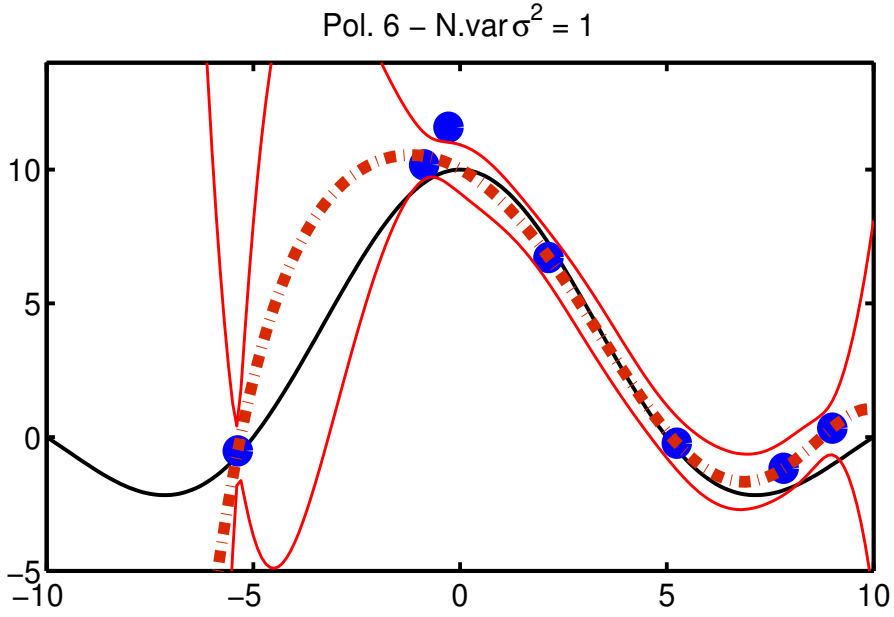
Approximations are needed for the

- posterior distribution
- predictive distribution

In Bayesian modelling an important issue is how we approximate the posterior distribution.

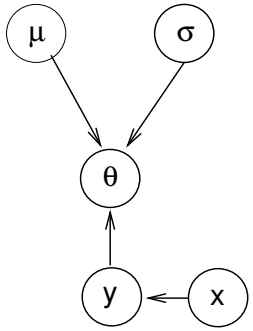


# Bayes – Example



# Graphical Models

- M.L. estimation: no prior
- M.A.P. estimation: no distribution
- Bayes est.




Bayes' models – if approximations used – we have Level II Maximum Likelihood.



# Drawbacks of Parametric Models

Model complexity – is an important "parameter":

- choosing $\mathcal{F}$ is an important decision in modelling the data:
- Example – for polynomial functions:
  - linear – too simple
  - quadratic – "good" for medium-sized data
  - ...
- The model complexity should be changed if we have more data available.



# Non-parametric models





# Non-parametric Models

**Non-parametric models:** use the available data for prediction.

1. Density estimation:
   - Histogram method;
   - Parzen window;
2. Regression/Classification:
   - K-Nearest Neighbour Rule;
   - Support Vector Machine;
   - Gaussian Processes;

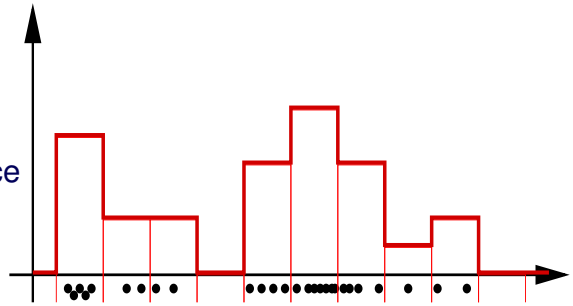
# Density Estimation

**Histogram method:**

"Parameters"
- bin width;
- locations;

Sensitive to the choice of parameters.

Not usable for higher dimensions



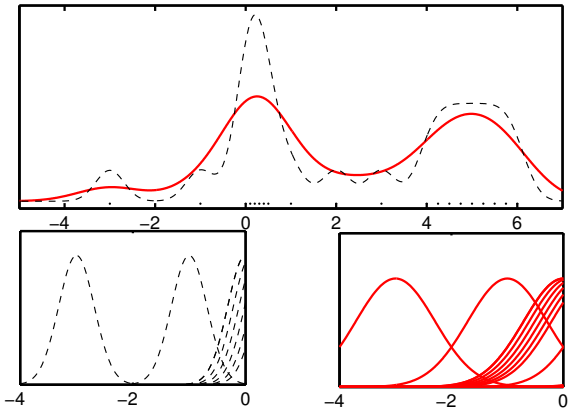


# Density Estimation II

**Parzen window:**

"Parameters"
- function width;
- data locations

Not usable for large data-sets.




$$h(\boldsymbol{x}) = \frac{1}{N}\sum_{n=1}^{N} k(\boldsymbol{x}_n, \boldsymbol{x})$$

**Nonparametric:** summation scales with $N$.

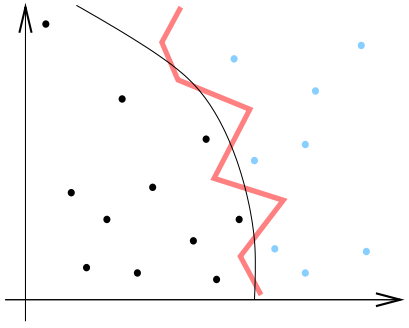


# Non–parametric Classification

**K–Nearest Neighbour:** (Knn)

"Parameters"
- # of neighbours

Slow for large data.



**Nonparametric:** all data taken into account.



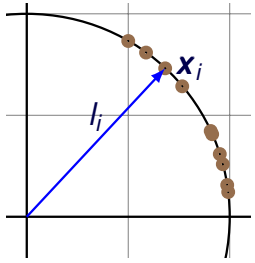
# Knn for high-dimensions I

**We generally:**
- normalise and center the data:

$$\begin{aligned} \boldsymbol{x}_i &\Leftarrow \boldsymbol{x}_i - \overline{\boldsymbol{x}} & \overline{\boldsymbol{x}} &\text{ – mean} \\ \boldsymbol{x}_{ij} &\Leftarrow \boldsymbol{x}_{ij}/\sigma_j & \sigma_j^2 &\text{ is var. for } j\text{-th comp.} \end{aligned}$$

- Each $x_{ij}$ has zero mean and unit variance.
- the length of the random vector $\boldsymbol{x}_i$ is

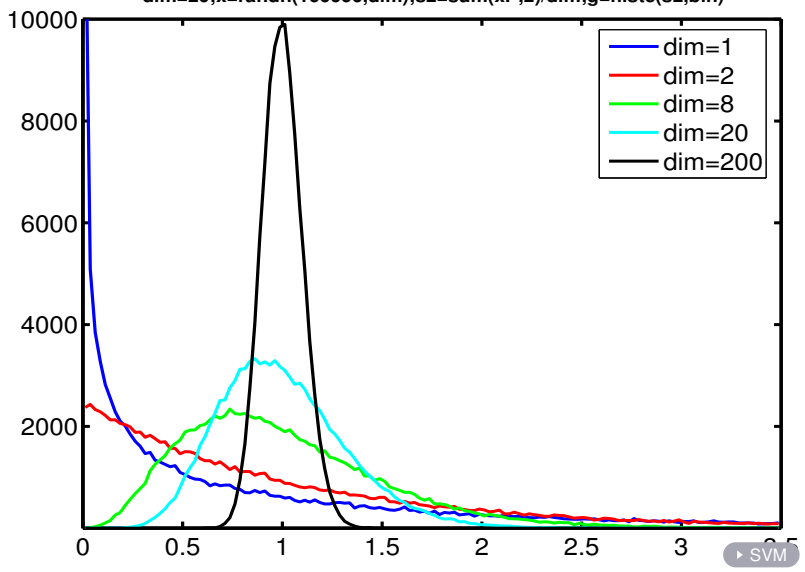
$$l_i^2 = \sum_j x_{ij}^2$$




**cf. central limit theorem**

the larger the dimension, the more concentrated the average length is around the mean.

**Example:** dim=20;x=randn(100000,dim);s2=sum(x.^2,2)/dim;g=histc(s2,bin)




SVM

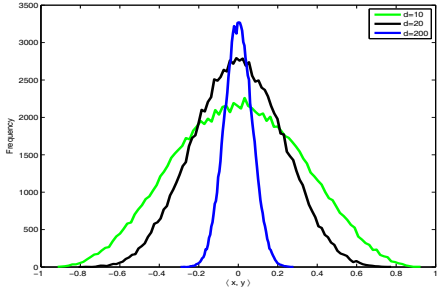




The angle:

$$\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle = \sum_{\ell=1}^{d} x_{i\ell} x_{j\ell}$$

with average value 0.




**Central limit theorem**

The larger the dimension

- the more orthogonal random vectors are;
- the more difficult is to select a representative.











**Motivation:**
Knn

- is too demanding for large datasets;
- works only in low-dimensions;

HD

**Algorithm that works "well" in high-dimensions**

$$f(\boldsymbol{x}) = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \left\{ \sum_n L\left(y_n, f(\boldsymbol{x}_n)\right) + \|\mathbf{P} f(\cdot)\|^2 \right\}$$

**Learning Algorithm**

Design of a method that simultaneously minimises the empirical error (first term) and selects "cleverly" (second term) from a large family $\mathcal{F}$ of available functions.

Details of the elements: ...

Skip HighDim

**Algorithm that works "well" in high-dimensions**

$$f(\boldsymbol{x}) = \operatorname*{argmin}_{f \in \mathcal{F}} \left\{ \sum_n L\left(y_n, f(\boldsymbol{x}_n)\right) + \|\mathbf{P} f(\cdot)\|^2 \right\}$$

**Learning Algorithm**

Design of a method that simultaneously minimises the empirical error (first term) and selects "cleverly" (second term) from a large family $\mathcal{F}$ of available functions.

Within the **SVM framework** we look for candidates

- in a large family of functions; and
- we penalise the complexity of the functions;

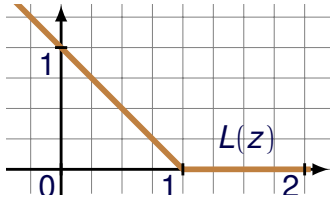
Explain loss function and $\mathcal{F}$.





Hinge Loss:

$$L(z) = \begin{cases} 0 & \text{if } z \geq 1 \\ 1 - z & \text{if } z < 1 \end{cases}$$




For classification with labels $y_n = \pm 1$ the loss function for the data-set $\mathcal{D}$:

$$L(\mathcal{D}) \stackrel{\text{def}}{=} \sum_n L\left(y_n f(\boldsymbol{x}_n)\right)$$

**Minimality**

$L(\mathcal{D})$ is minimal if all elements separated with margin $\geq 1$.





**Loss functions:**

Hinge loss – in classification – penalises data away from boundary;

- assuming class labels $y_i = \pm 1$, we have

$$L\left(y_n, f(\boldsymbol{x}_n)\right) = H_+\left(1 - y_n f(\boldsymbol{x}_n)\right)$$

Logit loss – in binary classification, returns the log-probabilities:

$$L_{\text{logit}}(y_n, f(\boldsymbol{x}_n)) = -\log\left(1 + \exp\left(-y_n f(\boldsymbol{x})\right)\right)$$

- when $y_n f(\boldsymbol{x}_n) \to \infty$, $L_{\text{logit}} \to 0$
- in probit model $(1 + \exp(\cdot))^{-1}$ replaced with $\Phi(\cdot)$.

Quadratic loss – quadratic error:

$$L_{\text{mse}}(y_n, f(\boldsymbol{x}_n)) = (y_n - f(\boldsymbol{x}_n))^2 \equiv (1 - y_n f(\boldsymbol{x}_n))^2$$

- can also be applied to regression problems.

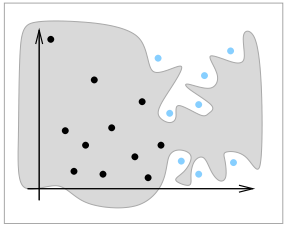




**Which family of functions?** – we have only data $\mathcal{D}$.

**Possibilities:**

- **linear – might be too simple;**
- **complex – might be too complex;**



**In general we**

- want a flexible function class, but
- do not want a too complex solution.

**Solution:**

use a large function class $\mathcal{F}$ **and** define penalties on the complexity of the solution.

- Loss functions help us find the best function
- from a family of functions.

**The family of functions is important**

It should be flexible enough
- for large data–sets
- and to allow for different length–scales.

Solution is of the form ($\mathbf{P}$ penalty):

$$f(\boldsymbol{x}) = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \left\{ \sum_n L\left(y_n, f(\boldsymbol{x}_n)\right) + \|\mathbf{P}f(\cdot)\|^2 \right\}$$

$\|\mathbf{P}f(\cdot)\|^2 = 0 \quad \Rightarrow$ null–space, gives possible solutions.



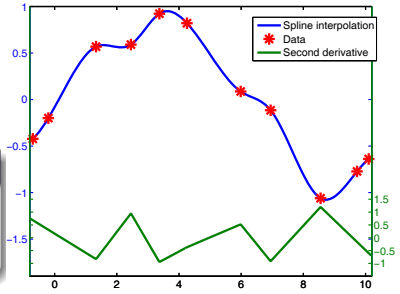


**Example:**

$\mathcal{F}$ – twice diff. functions with cont. second derivatives;

$\mathbf{P} \stackrel{\text{def}}{=} \left(\partial_x^2 f(\cdot)\right)^2$ – the sum of the second derivatives.

**Result:** interpolating splines with knots at the data points.




**Message**

Functional optimisation plausible with reliable results.

Presented in Wahba'90: Splines Models for Observational Data





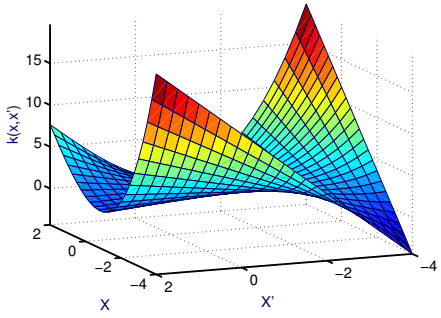
**Result of the optimisation problem:**

$$K_s(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{3} \min\left(\boldsymbol{x}, \boldsymbol{x}'\right)^3 + \frac{1}{2} \left|\boldsymbol{x} - \boldsymbol{x}'\right| \min\left(\boldsymbol{x}, \boldsymbol{x}'\right)^2 + \boldsymbol{x}\boldsymbol{x}' + 1$$

Solution of the argmin:

$$\hat{f}(\boldsymbol{x}) = \sum_n \alpha_n K_s(\boldsymbol{x}, \boldsymbol{x}_n)$$

Linear combination of polynomials.








**Reproducing Kernel Hilbert space:**

Assume $\mathcal{X}$ an index set, $\mathcal{H} = \{f \mid f : \mathcal{X} \to \mathbb{R}\}$ with $\langle \cdot, \cdot \rangle$ the dot product s.t. $\|f\|^2 = \langle f, f \rangle$.

If there exists $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that

1. $k$ has the reproducing property:

$$\forall\, f \in \mathcal{H} \qquad \langle f, k(x, \cdot) \rangle = f(x);$$

2. and $k$ spans $\mathcal{H}$.

**Not a feature map**

RKHS is not equivalent with a feature map.
More than a single feature map for the same $k(\cdot, \cdot)$.

# RKHS II

$k(\cdot,\cdot)$ spans $\mathcal{H}$ $\Rightarrow$ every function $f \in \mathcal{H}$ is a linear combination of $\{k(\boldsymbol{x}_n,\cdot)\}_{n=1}^{N_\mathcal{H}}$:

$$f(\boldsymbol{x}) = \sum_{n=1}^{N_\mathcal{H}} w_i\, k(\boldsymbol{x},\boldsymbol{x}_n) \quad (N_\mathcal{H} = \infty \text{ allowed})$$

**Note:**

- we choose the *support set*:
  - the location of the points and
  - the set size;
- we choose the *weights*.

## Empirical kernel map

The support set $\mathcal{X}$ is the training data set:

$$\mathcal{H}_e = \left\{ \sum\nolimits_n \alpha_n\, k(\boldsymbol{x},\boldsymbol{x}_n) \mid \alpha_n \in \mathbb{R} \right\}$$



# RKHS Mercer

## Mercer kernel

For a positive-definite $k(\cdot,\cdot)$ there exists a set $\{\phi_j\}$ of orthogonal functions and $\{\lambda_j\}$ positive constants s.t.

$$k(\boldsymbol{x},\boldsymbol{x}') = \sum_j \lambda_j \phi_j(\boldsymbol{x})\phi_j(\boldsymbol{x}')$$

**Note:**

- The function $k(\cdot,\cdot)$ defines:
  - the eigen-functions $\{\phi_j\}$
  - the corresponding eigen-values $\{\lambda_j\}$;
- independent of the data-set we are using;
- convergence guarantee: $\sum_j \lambda_j^2 < \infty$.



# Kernels for S.V.M.s Example

**Kernel functions:** Let $\Phi : \mathbb{R} \to \mathbb{R}^3$ be given by:

$$\Phi(\boldsymbol{x}) = \begin{bmatrix} 1 \\ \sqrt{2}\boldsymbol{x} \\ \boldsymbol{x}^2 \end{bmatrix}$$

and compute

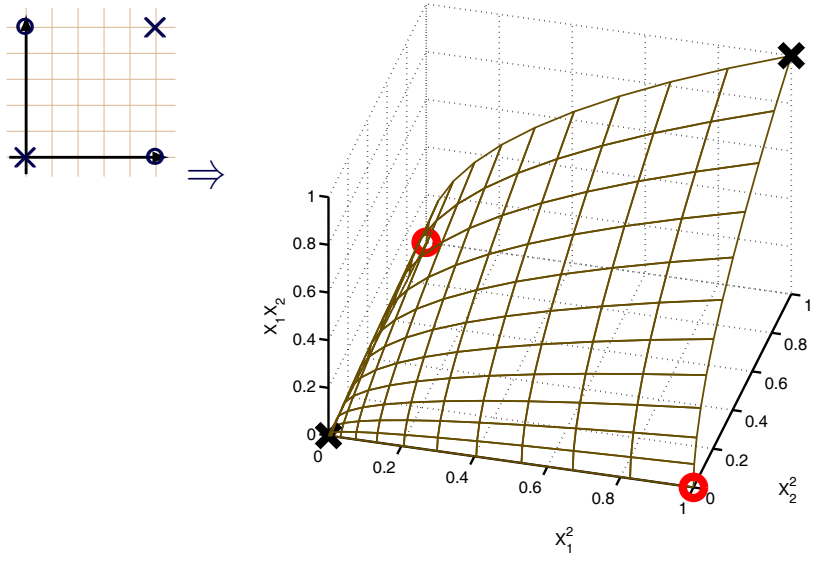
$$\begin{aligned} \Phi(\boldsymbol{x})^T\Phi(\boldsymbol{x}') &= 1\cdot 1 + \sqrt{2}\boldsymbol{x}\cdot\sqrt{2}\boldsymbol{y} + \boldsymbol{x}^2\boldsymbol{y}^2 \\ &= (1+\boldsymbol{x}\boldsymbol{y})^2 \quad \stackrel{\text{def}}{=} \quad K(\boldsymbol{x},\boldsymbol{y}) \end{aligned}$$

## Kernel trick

We can translate linear algorithms into nonlinear ones using a **kernel** function – represented as a scalar product.



# The XOR problem



# Kernels for S.V.M.s II

**Kernels** – two-argument functions that are the generalisation of a matrix to non–countable index sets.

$$K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

Valid kernel functions are positive definite: for any $\boldsymbol{a} = [a_1, a_2, \ldots, a_L]$ and $\mathcal{D} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_L]^T$:

$$\sum_{k=1}^{L} \sum_{l=1}^{L} a_k K(\boldsymbol{x}_k, \boldsymbol{x}_l) a_l \geq 0$$

**Proof idea**: kernel function is $K(\boldsymbol{x}, \boldsymbol{y}) = \Phi(\boldsymbol{x})^T \Phi(\boldsymbol{x}')$
and we have
$\sum_{k,l=1}^{L} a_k \Phi(\boldsymbol{x}_k)^T \Phi(\boldsymbol{x}_l) a_l = (\sum\ a_i \Phi(\boldsymbol{x}_i))^T (\sum\ a_i \Phi(\boldsymbol{x}_i)) = \boldsymbol{s}^T \boldsymbol{s} \geq 0$
where
$\boldsymbol{s} = \sum_{i=1}^{L} a_i \Phi(\boldsymbol{x}_i)$

**What is the dimensionality of $\Phi(\boldsymbol{x})$.**

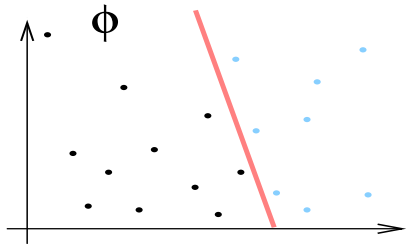
Functional Modelling
Lehel Csató
Modelling
Nonparametrics
SVM
Loss functions
Function class
Kernels
$\mathcal{GP}$'s
$\mathcal{GP}$ applications
Ref

# S.V.M.s Summary

Developed for classification.

Idea – **KERNEL TRICK**:

- use linear algorithms;
- project to $\Phi$;
- solution in $\Phi$;
- back-project;
- NON-linear solution;




Solution to the problem is of the form:

$$f(\boldsymbol{x}) = \Phi_{\boldsymbol{x}}^T \left( \sum_{i \in SV} \alpha_i \Phi_{\boldsymbol{x}_i} \right) \quad = \quad \sum_{i \in SV} \alpha_i K(\boldsymbol{x}, \boldsymbol{x}_i)$$

**Nonparametric:** number of parameters is not fixed.

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Modelling
Nonparametrics
SVM
Loss functions
Function class
Kernels
$\mathcal{GP}$'s
$\mathcal{GP}$ applications
Ref

# Kernel Algorithms

**The projection "trick" is exploited.**

- Based on the success of the S.V.M.-s;
- General recipe:
  - Find/Construct a linear algorithm;
  - Re-express it in $\Phi$ – the space of "features";
  - Write the – non–linear – solution in the space of inputs and use $K(\boldsymbol{x}, \boldsymbol{x}')$.
- Algorithms: Kernel ...
  - ... regression, ... ridge regression;
  - ... Principal/Independent Components;
  - ... Fisher Discriminants;

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Modelling
Nonparametrics
SVM
Loss functions
Function class
Kernels
$\mathcal{GP}$'s
$\mathcal{GP}$ applications
Ref

# Advantage / Price

**Definition:** non-parametric methods $\equiv$ number of parameters/complexity might change with data.

- Advantages:
  - $\uparrow$ Greater complexity;
  - $\uparrow$ Built-in regularisation;
  - $\uparrow$ Performant algorithms.
- Disadvantages:
  - $\downarrow$ Small data sets only;
  - $\downarrow$ No selection of parameters;

# Non-parametric models





# Bayesian Nonparametric Methods

- $\mathcal{GP}$ models
  $\equiv$ functional latent variable models.
  $\equiv$ "simple" random functions.
- Appear in the likelihood:

$$P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) = \prod_i P(y_i|\boldsymbol{x}_i, \boldsymbol{f}) = \prod_i P(y_i|\boldsymbol{x}_i, f_{\boldsymbol{x}_i})$$

Local dependencies only: $\boldsymbol{f} \longrightarrow f_{\boldsymbol{x}}$



# Gaussian processes I

**Gaussian process:** generalisation of a Gaussian.

- $\boldsymbol{f}$ Gaussian random function.

$$\boldsymbol{f} = [f_{\boldsymbol{x}_1}, f_{\boldsymbol{x}_2}, \ldots, f_{\boldsymbol{x}_N}, \ldots]^T, \quad \boldsymbol{x}_n \in \text{domain}.$$

- $\mathcal{GP}$ prior $p_0(\boldsymbol{f})$ characterised with
  - mean function $\langle f_{\boldsymbol{x}} \rangle_0$,
  - covariance kernel $K_0(\boldsymbol{x}, \boldsymbol{x}')$.

Property - for any sample set $\mathcal{D}$, a joint Gaussian r.v.:

$$\boldsymbol{f}_{\mathcal{D}} = [f_{\boldsymbol{x}_1}, \ldots, f_{\boldsymbol{x}_N}] \sim \mathcal{N}(\boldsymbol{f}_{\mathcal{D}} | \langle f_{\mathcal{D}} \rangle_0, \boldsymbol{K}_0)$$

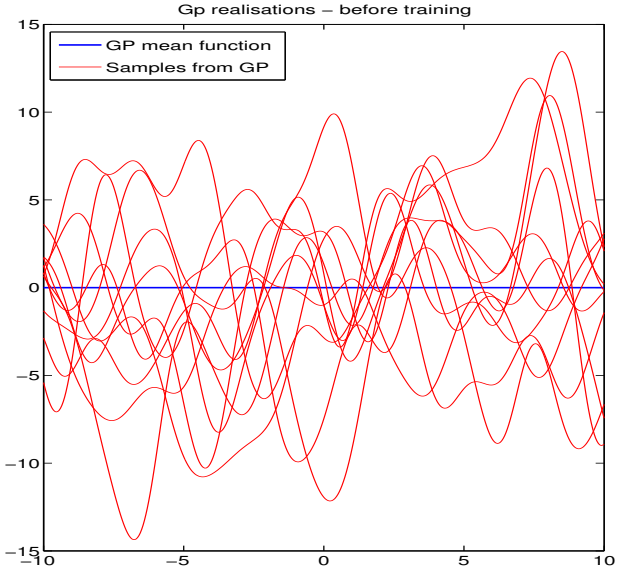


# Gaussian processes II

**Gaussian process:** random function

"Parameters"
- mean function
- covariance kernel

DEMO



# $\mathcal{GP}$ parameters

**Gaussian process parameters**

- mean function – usually is 0.
- **parameter:** the class of the kernel function
- parameters hidden into the kernel function.
  Example:

$$K(\boldsymbol{x}, \boldsymbol{x}'|\boldsymbol{\theta}) = \exp\left[\theta_0 - \frac{1}{2}\sum_{i=1}^{d}\theta_i\left(x_i - x_i'\right)^2\right]$$

$\boldsymbol{\theta} = [\theta_0, \theta_1, \ldots, \theta_d]^T$ – parameter vector.



# Kernel Functions

**Kernel functions:**

- Generate the covariance matrix.
- Need to be positive definite functions/matrices

$$\forall \mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T\}$$
$$\boldsymbol{K}_{\mathcal{X}} = \begin{bmatrix} K(\boldsymbol{x}_1, \boldsymbol{x}_1) & \ldots & K(\boldsymbol{x}_1, \boldsymbol{x}_T) \\ \vdots & \ddots & \vdots \\ K(\boldsymbol{x}_T, \boldsymbol{x}_1) & \ldots & K(\boldsymbol{x}_T, \boldsymbol{x}_T) \end{bmatrix}$$

must be a positive definite matrix.

- A construction of kernels as covariances:



# Kernel constructions I

Let $\{x_k = k\Delta_N\}_{k=1}^N$ be the **location** of Gaussian independent r.v.-s

$$v_k \sim \mathcal{N}(0, \sigma_0^2\Delta_N)$$

where $\Delta_N = 1/N$. Let $t$ denote the the index where $t = k_t\Delta_N$.
If $b_t \stackrel{\text{def}}{=} \sum_{i=1}^{t} v_i$, then $\langle b_t \rangle = 0$ and the covariance:

$$\langle b_s b_t \rangle = \left\langle \sum_{i_s=1}^{s}\sum_{i_t=1}^{t} v_{i_s} v_{i_t} \right\rangle = \sum_{i_s=1}^{\min(s,t)} \langle v_{i_s}^2 \rangle$$
$$= \sum_{i_s=1}^{\min(s,t)} \sigma_0^2\Delta_N = \min(s,t)\sigma_0^2$$

in the following we take $\sigma_0^2 = 1$.

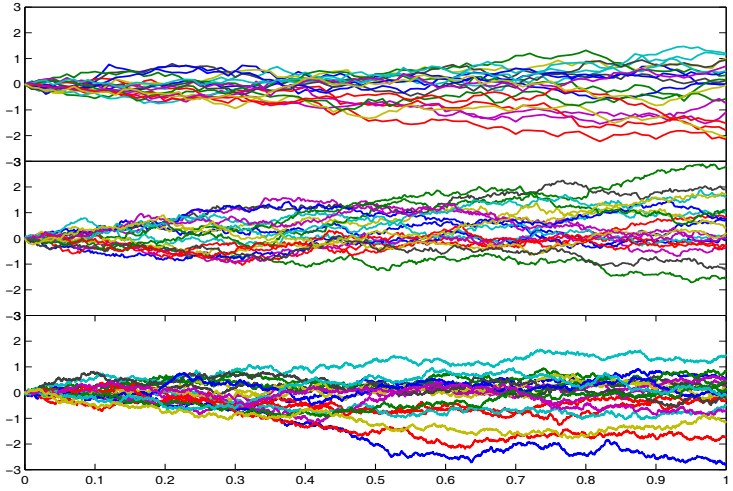
**Brownian motion**

A stochastic process with covariance kernel $K(s,t) = \min(s,t)$ is a Brownian motion.
Its derivative – $v_t$ – is the Wiener process.



# Kernel constructions II

Images of the Brownian motion at different resolutions (different $N$-s).



# Kernel constructions III

Let us integrate (sum) the Brownian motion.

Define

$$s_t \stackrel{\text{def}}{=} \sum_{i_t=1}^{t} b_{i_t}$$

leading to $\langle s_t \rangle = 0$ and

$$K(s,t) = \langle s_s s_t \rangle = \left\langle \sum_{i_s=1}^{s} \sum_{i_t=1}^{t} s_{i_s} s_{i_t} \right\rangle$$
$$= \sum_{i_s=1}^{s} \sum_{i_t=1}^{t} \langle s_{i_s} s_{i_t} \rangle = \int_0^s dz_s \int_0^t dz_t \, \min(z_s, z_t)$$

For the integration we assume

$$s < t \quad \Longrightarrow \quad z_s < t \quad \Longrightarrow \quad [0,t] = [0,z_s] \cup [z_s,t] \,.$$



# Kernel constructions IV

Using $[0,t] = [0,z_s] \cup [z_s,t]$

$$K(s,t) = \int_0^s dz_s \left( \int_0^{z_s} dz_t \, z_t + \int_{z_s}^t dz_t \, z_s \right) = \int_0^s dz_s \left( \frac{z_s^2}{2} + z_s(t - z_s) \right)$$
$$= \int_0^s dz_s \left( z_s t - \frac{z_s^2}{2} \right)$$
$$= \frac{s^2 t}{2} - \frac{s^3}{6} \quad \text{assuming } s < t$$

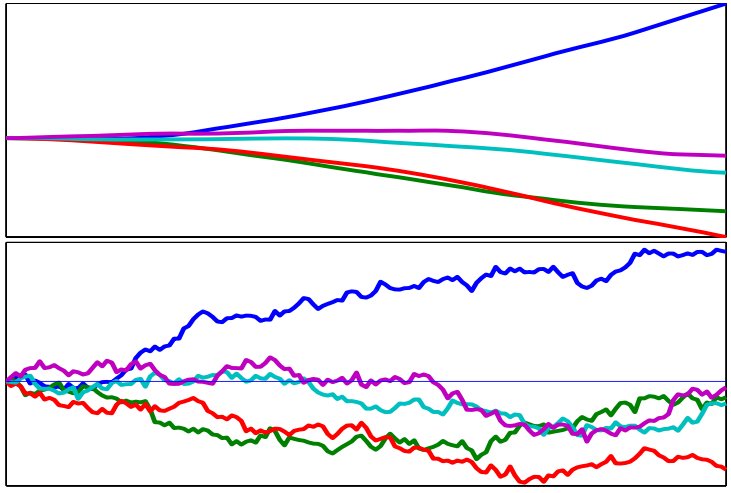
After symmetrization (writing the $s > t$ case and unifying)

$$K(s,t) = \frac{st \min(s,t)}{2} - \frac{\min(s,t)^3}{6}$$
$$= \frac{1}{2} \min(s,t)^2 \, |s-t| + \frac{1}{3} \min(s,t)^3$$



# Kernel constructions V

Samples from the integrated Brownian motion
(Brownian motion on the bottom).





# Kernel constructions VI

- The integrated Brownian motion: $s(0) = s'(0) = 0$.
- For generality we add a *constant* and a *linear* term:

$$s_2(x) = w_0 + w_1 x + s(x)$$

where $w_0$, $w_1$ are i.i.d. Gaussian r.v-s.

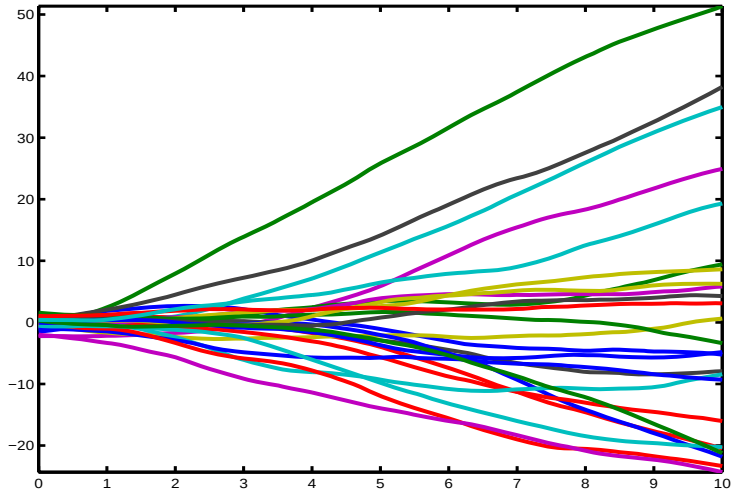
- Means that the kernel is:

$$K_2(s,t) = \langle s_2(s) s_2(t) \rangle = 1 + st + K_s(s,t)$$

$$K_2(s,t) = 1 + st + \frac{1}{2} \min(s,t)^2 |s-t| + \frac{1}{3} \min(s,t)^3$$

# Kernel constructions VII

## Random splines:






# Kernels from conditioning VIII

- Consider the Brownian motion: $K_b(s,t) = \min(s,t)$.
- We generate random functions with $x_1 = 0$.
  Called Brownian bridge
- To calculate the covariance, we have to condition the r.v.-s $x_s$ and $x_t$ on $x_1 = 0$.

We identify the kernel from the conditioned **joint** Gaussian distribution

$$p(x_s, x_t | x_1 = 0) \propto \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_s \\ x_t \\ 0 \end{bmatrix}^T \begin{bmatrix} K_b(s,s) & K_b(s,t) & s \\ K_b(t,s) & K_b(t,t) & t \\ s & t & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_s \\ x_t \\ 0 \end{bmatrix} \right\}$$

$$\propto \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_s \\ x_t \\ 0 \end{bmatrix}^T \begin{bmatrix} \left( \begin{bmatrix} K_b(s,s) & K_b(s,t) \\ K_b(t,s) & K_b(t,t) \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}^T \right)^{-1} & E \\ E^T & F \end{bmatrix} \begin{bmatrix} x_s \\ x_t \\ 0 \end{bmatrix} \right\}$$

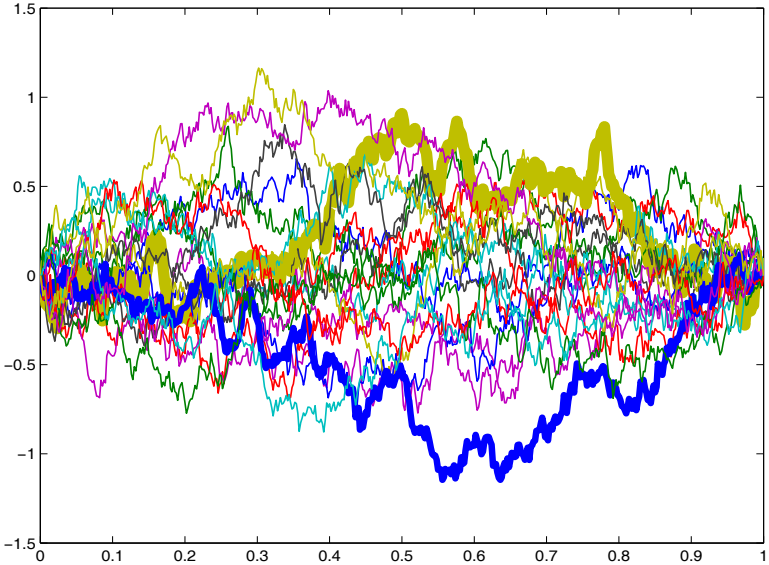
where we used the matrix inversion lemma

$$\Rightarrow \qquad K_0(s,t) = K_b(s,t) - st$$



# Kernels from conditioning IX

## Samples from a Brownian bridge:






# Kernels from conditioning X

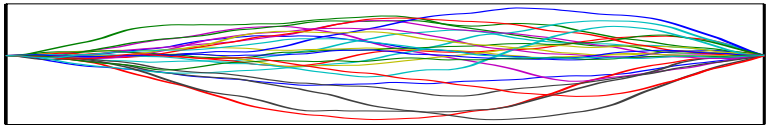
**Exercise:**

- Find the mean and kernel functions corresponding to the Brownian bridge where $x_1 = 1$.
- Consider the spline kernel

$$K_s(s,t) = \frac{1}{2} \min(s,t)^2 |s-t| + \frac{1}{3} \min(s,t)^3$$

Similarly to the Brownian bridge, find the kernel function for the splines conditioned on $x_1 = 0$.

**Samples from the second family look like this:**



# Generating random samples XI

```
clear all;
N = 100;  T=25;  D = 0.99;
t = linspace(0,D,N+1);
t = t(2:end);

% put covariance function here
k_bb= inline('min(s,t)-s*t','s','t');

Ks = zeros(N,N);
for ii=1:N;
    for jj=1:N;
        Ks(ii,jj) = k_sb(t(ii),t(jj));
    end;
end;

kks = chol(Ks);
yr = randn(T,N);
ys = zeros(N+2,T);
ys(2:N+1,:) = (yr*kks)';

t0=[0,t,1];
figure(1); cla; box on; hold on;
plot(t0,ys);
```



# Gaussian Process Inference I

- $\mathcal{GP}$ inference: application of Bayes' rule.

$$p_{\text{post}}(\boldsymbol{f}, \boldsymbol{f}_{\mathcal{D}}) \propto P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})\; p_0(\boldsymbol{f}_{\mathcal{D}}, \boldsymbol{f})$$

- For any collection of indexes $\mathcal{X}$ the posterior:

$$p_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}) = \frac{1}{Z_{\mathcal{D}}} \int d\boldsymbol{f}_{\mathcal{D}}\; P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})\; p_0(\boldsymbol{f}_{\mathcal{D}}, \boldsymbol{f}_{\mathcal{X}})$$

where

$$Z_{\mathcal{D}} = \int d\boldsymbol{f}_{\mathcal{D}}\; P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})\; p_0(\boldsymbol{f}_{\mathcal{D}})$$

Data probability – conditioned on the model.

▸ A.R.D.

*obs: NO specific $P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})$*



# Gaussian Process Inference II

**Problems with the representation**

$$p_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}) \propto \int d\boldsymbol{f}_{\mathcal{D}}\; P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})\; p_0(\boldsymbol{f}_{\mathcal{D}}, \boldsymbol{f}_{\mathcal{X}})$$

Integral evaluation necessary for posterior distribution.

- Representation – How to represent the posterior?
  - Finite representation of the posterior process;
  - Non–Gaussian posterior processes: approximations to them



# Gaussian Process Parametrisation I

**Property of Gaussian averages:**

$$\langle f_{\boldsymbol{x}} \rangle_{\text{post}} \;=\; \langle f_{\boldsymbol{x}} \rangle_0 + \sum_i K_0(\boldsymbol{x}, \boldsymbol{x}_i)\alpha(i)$$

Where coefficients:

$$\alpha(i) = \frac{\partial}{\partial \langle f_i \rangle_0} \ln \left\langle P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) \right\rangle_0$$

Provide **parametrisation** (see Kimeldorf-Wahba).

## Gaussian Process Parametrisation II

**For the posterior kernel:**

$$K_{\text{post}}(\boldsymbol{x}, \boldsymbol{x}') = K_0(\boldsymbol{x}, \boldsymbol{x}') + \sum_{i,j} K_0(\boldsymbol{x}, \boldsymbol{x}_i) C(ij) K_0(\boldsymbol{x}_j, \boldsymbol{x}')$$

Where coefficients:

$$C(ij) = \frac{\partial^2}{\partial \langle f_i \rangle_0 \, \partial \langle f_j \rangle_0} \ln \left\langle P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) \right\rangle_0$$

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Modelling
Nonparametrics
SVM
$\mathcal{GP}$'s
Kernels II
Inf./Pred.
Regression
Approximations
A.R.D
Sparsity
$\mathcal{GP}$ applications
Ref

## $\mathcal{GP}$s in Feature space I

$K_0(\boldsymbol{x}, \boldsymbol{x}')$ defines a *feature space* $\mathcal{F}$ :

$$\phi_{\boldsymbol{x}}, \phi_{\boldsymbol{x}'} \in \mathcal{F} \quad \text{and} \quad K_0(\boldsymbol{x}, \boldsymbol{x}') = \phi_{\boldsymbol{x}}^T \phi_{\boldsymbol{x}'}$$

Using $\mathcal{F}$ and the scalar product:

$$\langle f_{\boldsymbol{x}} \rangle_{\text{post}} = \phi_{\boldsymbol{x}}^T \sum_{i=1}^{N} \alpha(i) \phi_i = \phi_{\boldsymbol{x}}^T \boldsymbol{\mu}_{\text{post}}$$

$$K_{\text{post}}(\boldsymbol{x}, \boldsymbol{x}') = \phi_{\boldsymbol{x}}^T \left( \boldsymbol{I}_{\mathcal{F}} + \sum_{i,j=1}^{N} \phi_i C(ij) \phi_j^T \right) \phi_{\boldsymbol{x}'} = \phi_{\boldsymbol{x}}^T \boldsymbol{\Sigma}_{\text{post}} \phi_{\boldsymbol{x}'}$$

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Modelling
Nonparametrics
SVM
$\mathcal{GP}$'s
Kernels II
Inf./Pred.
Regression
Approximations
A.R.D
Sparsity
$\mathcal{GP}$ applications
Ref

## $\mathcal{GP}$s in Feature space II

$K_0(\boldsymbol{x}, \boldsymbol{x}')$ defines a *feature space* $\mathcal{F}$ :

$$\phi_{\boldsymbol{x}}, \phi_{\boldsymbol{x}'} \in \mathcal{F} \quad \text{and} \quad K_0(\boldsymbol{x}, \boldsymbol{x}') = \phi_{\boldsymbol{x}}^T \phi_{\boldsymbol{x}'}$$

$$\begin{aligned} \langle f_{\boldsymbol{x}} \rangle_{\text{post}} &\Longleftrightarrow \quad \boldsymbol{\mu}_{\text{post}} \\ K_{\text{post}}(\boldsymbol{x}, \boldsymbol{x}') &\Longleftrightarrow \quad \boldsymbol{\Sigma}_{\text{post}} \end{aligned}$$

**$\mathcal{GP}$ inference:**

Estimating a Gaussian distribution in $\mathcal{F}$

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Nonparametrics
SVM
$\mathcal{GP}$'s
Kernels II
Inf./Pred.
Regression
Approximations
A.R.D
Sparsity
$\mathcal{GP}$ applications
Ref

## Prediction with Gaussian processes

Given: $\boldsymbol{x}^*$ - for which we require answer $y^*$.

$$\begin{aligned} p(y^*|\boldsymbol{x}^*, \mathcal{D}) &= \int df^* \int d\boldsymbol{f}_{\mathcal{D}} \; p(y^*, \boldsymbol{f}_{\mathcal{D}}, f^*|\boldsymbol{x}^*, \mathcal{D}) \\ &= \int df^* \; P(y^*|\boldsymbol{x}^*, f^*) \int d\boldsymbol{f}_{\mathcal{D}} \; p_{\text{post}}(\boldsymbol{f}_{\mathcal{D}}, f^*|\mathcal{D}) \\ &= \int df^* \; P(y^*|\boldsymbol{x}^*, f^*) p_{\text{post}}(f^*|\mathcal{D}) \end{aligned}$$

where $f^* = f_{\boldsymbol{x}^*}$ – random variable associated to $\boldsymbol{x}^*$.

**We use posterior process:**

- irrespective of the likelihood;
- if not Gaussian, we approximate.

# Regression with Gaussian noise

- Gaussian noise:

$$P(\boldsymbol{y}_{\mathcal{D}}|\boldsymbol{f}_{\mathcal{D}}) \propto \exp\left[-\frac{1}{2\sigma_o^2}\|\boldsymbol{y}_{\mathcal{D}} - \boldsymbol{f}_{\mathcal{D}}\|^2\right]$$

- Gaussian latent variables:

$$P(\boldsymbol{f}_{\mathcal{D}}|K(\cdot,\cdot|\boldsymbol{\theta})) \propto \exp\left[-\frac{1}{2}(\boldsymbol{f}_{\mathcal{D}} - \boldsymbol{\mu}_{\mathcal{D}})^T \boldsymbol{K}_{\mathcal{D}}{}^{-1}(\boldsymbol{f}_{\mathcal{D}} - \boldsymbol{\mu}_{\mathcal{D}})\right]$$

- Combining: product quadratic $\Longrightarrow$ Gaussian



# Posterior distribution – Gaussian noise

**Gaussian distribution –** reading off the coefficients:

$$\boldsymbol{\mu}_{\text{post}} = \left(\boldsymbol{K}_{\mathcal{D}}{}^{-1} + \frac{1}{\sigma_o^2}\boldsymbol{I}_{\mathcal{N}}\right)^{-1}\left[\boldsymbol{K}_{\mathcal{D}}{}^{-1}\mu_D + \frac{1}{\sigma_o^2}\boldsymbol{y}_{\mathcal{D}}\right]$$
$$\boldsymbol{\Sigma}_{\text{post}} = \left(\boldsymbol{K}_{\mathcal{D}}{}^{-1} + \frac{1}{\sigma_o^2}\boldsymbol{I}_{\mathcal{N}}\right)^{-1}$$

The **joint** distribution of all r.v.-s is a Gaussian:

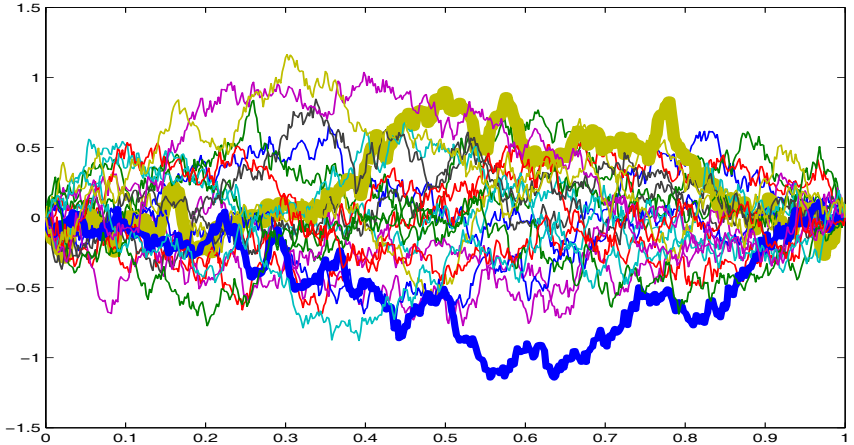
$$\boldsymbol{f}_{\mathcal{D}} \sim \mathcal{N}(\boldsymbol{\mu}_{\text{post}}, \boldsymbol{\Sigma}_{\text{post}})$$

The distribution of the r.v.-s at **training locations**.



# Posterior distribution Brownian bridge

- **Assume** that there was an observation for the Brownian motion $k(\boldsymbol{x},\boldsymbol{x}') = \min(\boldsymbol{x},\boldsymbol{x}')$;
- at 1 the value of the process is 0.
- It generates a new process: $k_B(\boldsymbol{x},\boldsymbol{x}') = \min(\boldsymbol{x},\boldsymbol{x}') - \boldsymbol{x}\boldsymbol{x}'$






# Predictive distributions – Gaussian noise

**For test location $\boldsymbol{x}^*$:**

$$\mu^* = \boldsymbol{\alpha}_{\mathcal{D}}{}^T\boldsymbol{k}_* \qquad \boldsymbol{\alpha}_{\mathcal{D}} = \boldsymbol{C}_{\mathcal{D}} * \boldsymbol{y}_{\mathcal{D}}$$
$$\sigma^* = K(\boldsymbol{x}^*,\boldsymbol{x}^*) - \boldsymbol{k}_*^T\boldsymbol{C}_{\mathcal{D}}\boldsymbol{k}_* \quad \text{with} \quad \boldsymbol{C}_{\mathcal{D}} = \left(\boldsymbol{K}_{\mathcal{D}} + \sigma_o^2\boldsymbol{I}_N\right)^{-1}$$

where $\boldsymbol{k}_* = [K(\boldsymbol{x}^*,\boldsymbol{x}_1),\ldots,K(\boldsymbol{x}^*,\boldsymbol{x}_N)]^T$

Posterior mean and covariance functions:

$$\mu(\boldsymbol{x}) = \boldsymbol{\alpha}_{\mathcal{D}}{}^T\boldsymbol{k}_{\boldsymbol{x}}$$
$$K_{\text{post}}(\boldsymbol{x},\boldsymbol{x}') = K(\boldsymbol{x},\boldsymbol{x}') - \boldsymbol{k}_{\boldsymbol{x}}^T\boldsymbol{C}_{\mathcal{D}}\boldsymbol{k}_{\boldsymbol{x}'}$$

where $\boldsymbol{k}_{\boldsymbol{x}} = [K(\boldsymbol{x},\boldsymbol{x}_1),\ldots,K(\boldsymbol{x},\boldsymbol{x}_N)]^T$.

# Non-computable Posteriors

- **If likelihood non–Gaussian** $\Rightarrow$ posterior does not have analytical form.
  (No "summarising" statistics)

- Methods to obtain posterior:
  - Sampling;
  - Analytic approximations;

# Sampling from the posterior I

**Sampling**

$$p_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}) = \frac{1}{Z_{\mathcal{D}}} \int d\boldsymbol{f}_{\mathcal{D}}\; P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) p_0(\boldsymbol{f}_{\mathcal{D}}, \boldsymbol{f}_{\mathcal{X}})$$

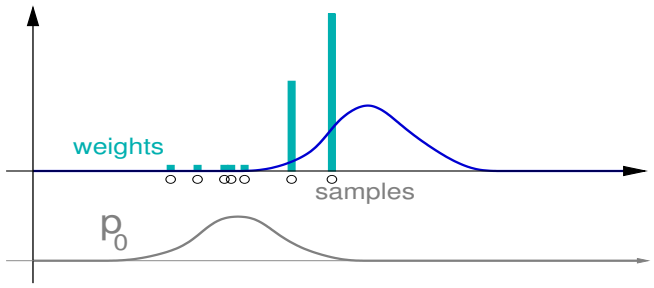
**In practise:** – joint sampling from $p_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}, \boldsymbol{f}_{\mathcal{D}})$,
– keeping only $\boldsymbol{f}_{\mathcal{X}}$.

**Implementation:** sampling from $p_0(\boldsymbol{f}_{\mathcal{D}}, \boldsymbol{f}_{\mathcal{X}})$ + weighting:

$$p_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}) \approx \frac{1}{C_T} \sum_{t=1}^{T} P(\boldsymbol{y}_N|\boldsymbol{f}_{\mathcal{D}}^{(i)})\; \delta(\boldsymbol{f}_{\mathcal{X}} - \boldsymbol{f}_{\mathcal{X}}^{(i)})$$

with $C_T = \sum_{t=1}^{T} P(\boldsymbol{y}_N|\boldsymbol{f}_{\mathcal{D}}^{(i)})$

# Sampling from the posterior II




Sampling methods:

- powerful *i.e.* allow flexibility in modelling
- Hard to assess convergence
- Sampling algorithms suited for different models.
- Can be incredibly slow (tempering, MCMC)

# Laplace Approximation I

**Log–Posterior:**

$$\begin{aligned}\log p_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}, \boldsymbol{f}_{\mathcal{D}}) &= K + \log P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) + \log p_0(\boldsymbol{f}_{\mathcal{D}}, \boldsymbol{f}_{\mathcal{X}}) \\ &= \underbrace{\log P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) + \log p_0(\boldsymbol{f}_{\mathcal{D}})}_{g_{\mathcal{D}}(\boldsymbol{f}_{\mathcal{D}})} + \underbrace{\log p_0(\boldsymbol{f}_{\mathcal{X}}|\boldsymbol{f}_{\mathcal{D}})}_{g_{\mathcal{X}}(\boldsymbol{f}_{\mathcal{X}})}\end{aligned}$$

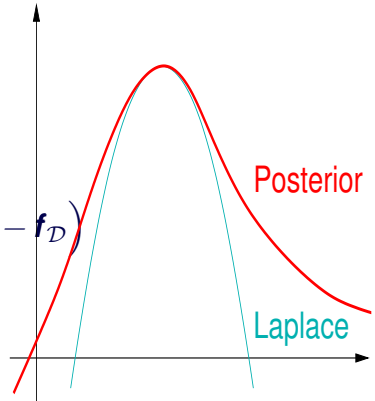
Finding maximum of $\boldsymbol{f}_{\mathcal{X}}$ and $\boldsymbol{f}_{\mathcal{D}}$:

$$\hat{\boldsymbol{f}}_{\mathcal{X}} = \boldsymbol{P}_{\mathcal{X}\mathcal{D}}\, \boldsymbol{f}_{\mathcal{D}}$$
$$\hat{\boldsymbol{f}}_{\mathcal{D}} = \arg\max g_{\mathcal{D}}(\boldsymbol{f}_{\mathcal{D}})$$

Taylor expansion around $\hat{\boldsymbol{f}}_{\mathcal{D}}$:

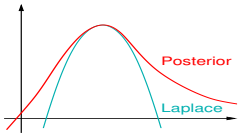
$$g_{\mathcal{D}}(\boldsymbol{f}_{\mathcal{D}}) \approx \left(\hat{\boldsymbol{f}}_{\mathcal{D}} - \boldsymbol{f}_{\mathcal{D}}\right)^T \left[H_g(\hat{\boldsymbol{f}}_{\mathcal{D}})\right] \left(\hat{\boldsymbol{f}}_{\mathcal{D}} - \boldsymbol{f}_{\mathcal{D}}\right) + \boldsymbol{0}\left(\hat{\boldsymbol{f}}_{\mathcal{D}} - \boldsymbol{f}_{\mathcal{D}}\right) + g_{\mathcal{D}}(\hat{\boldsymbol{f}}_{\mathcal{D}})$$

$\Rightarrow$ **Gaussian**



# Laplace approximation II

**Gaussian approximation:**




$$\hat{p}_{\text{post}}(\boldsymbol{f}_{\mathcal{X}}, \boldsymbol{f}_{\mathcal{D}}) \propto p_0(\boldsymbol{f}_{\mathcal{X}}, \boldsymbol{f}_{\mathcal{D}}) \underbrace{\frac{\mathcal{N}_L\left(\boldsymbol{f}_{\mathcal{D}} \middle| \hat{\boldsymbol{f}}_{\mathcal{D}}, \left[H_g(\hat{\boldsymbol{f}}_{\mathcal{D}})\right]^{-1}\right)}{p_0(\boldsymbol{f}_{\mathcal{D}})}}_{\hat{P}(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})}$$

Defines an approximation to the **likelihood**:

$$\hat{P}(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}}) \propto \frac{\mathcal{N}_L\left(\boldsymbol{f}_{\mathcal{D}} \middle| \hat{\boldsymbol{f}}_{\mathcal{D}}, \left[H_g(\hat{\boldsymbol{f}}_{\mathcal{D}})\right]^{-1}\right)}{p_0(\boldsymbol{f}_{\mathcal{D}})}$$



# Laplace approximation Summary

**The Laplace approximation:**

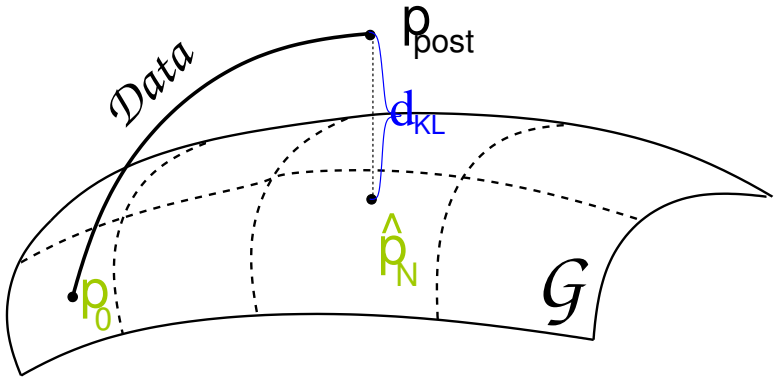
- (+) generates an approximation to the *likelihood*;
- (−) applicable only for differentiable likelihood functions;
- (+) defines an approximation to the *whole* process;
- (−) the Hessian has to be positive definite and **"smaller" than the prior**



# Analytic approximations – I

**Aim:** approximate the posterior distribution – or the posterior process.

$\mathcal{GP}$ **prior** $\longrightarrow$ $\mathcal{GP}$ **approximation to posterior. projection** – closest $\mathcal{GP}$ .






# Analytic approximations – II

**Choice of projection:** Kullback-Leibler divergence

$$\text{KL}(\mathcal{GP}_{\text{post}} \| \mathcal{GP}) = \int d\mathcal{GP}_{\text{post}}(\boldsymbol{f}) \log \frac{d\mathcal{GP}_{\text{post}}(\boldsymbol{f})}{d\mathcal{GP}(\boldsymbol{f})}$$

$$\mathcal{GP}^* = \arg\min_{\mathcal{GP}} \text{KL}(\mathcal{GP}_{\text{post}} \| \mathcal{GP})$$

The minimiser:

$$\begin{aligned} \langle f_x \rangle_{\mathcal{GP}^*} &\stackrel{\text{def}}{=} \langle f_x \rangle_{\text{post}} \\ K_{\mathcal{GP}^*}(\boldsymbol{x}, \boldsymbol{x}') &\stackrel{\text{def}}{=} K_{\text{post}}(\boldsymbol{x}, \boldsymbol{x}') \end{aligned}$$

**Implies** that the KL-approximation the $\mathcal{GP} \Leftrightarrow (\boldsymbol{\alpha}_{\mathcal{D}}, \boldsymbol{C}_{\mathcal{D}})$.

# Computing KL-distances

Between $\mathcal{GP}$s with the same prior:
$\mathcal{GP}_1 = \mathcal{N}(\mu_1, \Sigma_1)$ and $\mathcal{GP}_2 = \mathcal{N}(\mu_2, \Sigma_2)$, the KL-distance is:

$$2\mathrm{KL}(\mathcal{GP}_1 \| \mathcal{GP}_2) = (\mu_2 - \mu_1)^T \Sigma_2^{-1} (\mu_2 - \mu_1) + \mathrm{tr}\left(\Sigma_1 \Sigma_2^{-1} - \boldsymbol{I}_{\mathcal{F}}\right) - \ln \left|\Sigma_1 \Sigma_2^{-1}\right|$$

With parameters ($\boldsymbol{Q}_{\mathcal{BV}} = \boldsymbol{K}_{\mathcal{BV}}^{-1}$):

$$(\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1) (\boldsymbol{C}_2 + \boldsymbol{Q}_{\mathcal{BV}})^{-1} (\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1)$$
$$\mathrm{tr}\left[(\boldsymbol{C}_1 - \boldsymbol{C}_2)(\boldsymbol{C}_2 + \boldsymbol{Q}_{\mathcal{BV}})^{-1}\right] - \ln \left|(\boldsymbol{C}_1 + \boldsymbol{Q}_{\mathcal{BV}}) (\boldsymbol{C}_2 + \boldsymbol{Q}_{\mathcal{BV}})^{-1}\right|$$

Assumptions: the kernel matrix on the $\mathcal{BV}$ set is non-singular $|\boldsymbol{K}_{\mathcal{BV}}| \neq 0$.

# Bayesian Online Learning

Learning: propagating the mean and the kernel:

$$\langle f_{\boldsymbol{x}} \rangle_{t+1} = \langle f_{\boldsymbol{x}} \rangle_t + q\, K_t(\boldsymbol{x}, \boldsymbol{x}_{t+1})$$
$$K_{t+1}(\boldsymbol{x}, \boldsymbol{x}') = K_t(\boldsymbol{x}, \boldsymbol{x}') + r\, K_t(\boldsymbol{x}, \boldsymbol{x}_{t+1}) K_t(\boldsymbol{x}_{t+1}, \boldsymbol{x}')$$

$q$, $r$ functions of the **single** likelihood:

$$q = q^{(t+1)} = \frac{\partial}{\partial \langle f_{t+1} \rangle_t} \ln \langle P(y_{t+1} | f_{t+1}) \rangle_t$$

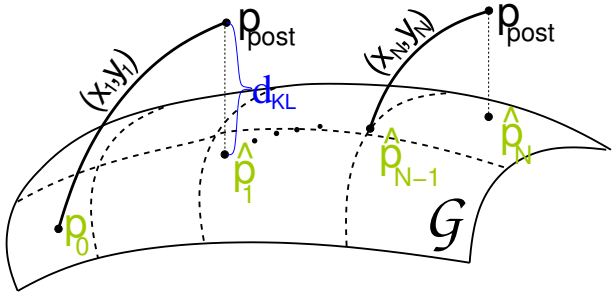
where $\langle \cdot \rangle_t$ average w.r.to $f_{t+1} \sim \mathcal{N}_t(\langle f_{t+1} \rangle, \sigma^2_{t+1})$.

# Analytic Approximations – III

**Bayesian Online Learning (recursive)**

- Instead of $|\mathcal{D}| = N$ uses $\mathcal{D} = (\boldsymbol{x}_{t+1}, y_{t+1})$ and;
- For prior process $\langle f_{\boldsymbol{x}} \rangle_t$, $K_t(\boldsymbol{x}, \boldsymbol{x}')$.

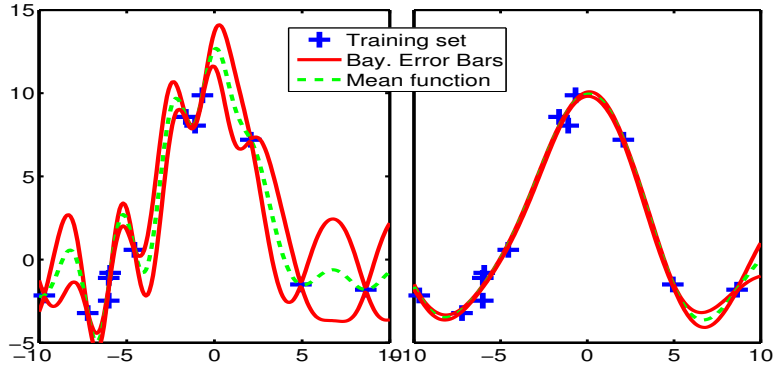
$\mathrm{KL}(\mathcal{GP}^{t+1}_{\mathrm{post}} \| \mathcal{GP}^*)$ smaller approximation




# Optimising hyper-parameters I

$\mathcal{GP}$ **kernel** parameters $\Leftrightarrow$ model choice.
Exemplu:

RBF kernel: $K(\boldsymbol{x}, \boldsymbol{x}') = A \exp\left[-\sum (x - x')^2 \beta\right]$




$\beta = 0.1$ $\beta = 1.$

# Optimising hyper-parameters II

Model evidence:

$$Z_{\mathcal{D}}(\boldsymbol{\theta}) = P(\mathcal{D}|\boldsymbol{\theta}) = \int d\boldsymbol{f}_{\mathcal{D}}\; P(\mathcal{D}|\boldsymbol{f}_{\mathcal{D}})\; p_0(\boldsymbol{f}_{\mathcal{D}}|\theta)$$

**Maximum Likelihood II** inference

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}\in\Omega} P(\boldsymbol{\theta}|\mathcal{D}) = \arg\min_{\boldsymbol{\theta}\in\Omega} \frac{P(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta}|\mathcal{M})}{p(\mathcal{D}|\mathcal{M})}$$

if $p(\boldsymbol{\theta}|\mathcal{M})$ "flat"

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}\in\Omega} Z_{\mathcal{D}}(\boldsymbol{\theta})$$

Evidence maximisation.

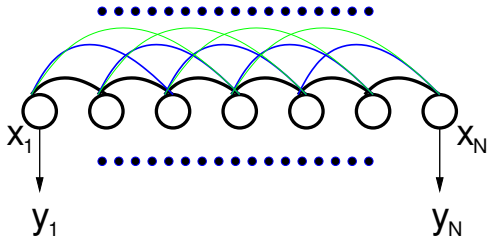
Gradient/conj.grad. methods are used.

MacKay Evidence



# Sparse representations – Motivation

**Gaussian Processes** are fully connected graphical models.




$\Longrightarrow$ Computing estimates is difficult. E.g for the posterior mean:

$$\langle f_{\boldsymbol{x}}\rangle_{\text{post}} = \boldsymbol{y}^T \left(\boldsymbol{K}_N + \sigma_o^2 \boldsymbol{I}_N\right)^{-1} \boldsymbol{k}_{\boldsymbol{x}}$$

inversion requires $\mathcal{O}(N^3)$ time.

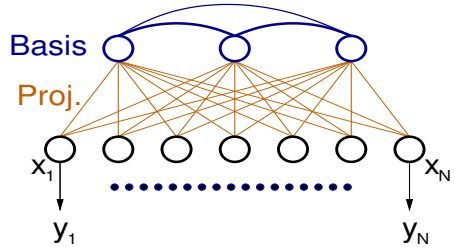
Regr



# Sparse representations – a solution

Condition all training locations on a set of basis locations.

$$\boldsymbol{f}_{\mathcal{BV}} \sim \mathcal{N}\left(\mu_{\mathcal{BV}}, \Sigma_{\mathcal{BV}}\right)$$




The pseudo-latents $\boldsymbol{f}_{\mathcal{X}}$ are conditioned on $\boldsymbol{f}_{\mathcal{BV}}$:

$$\boldsymbol{f}_{\mathcal{X}}|\boldsymbol{f}_{\mathcal{BV}} \sim \mathcal{N}\left(\boldsymbol{P}\,\mu_{\mathcal{BV}}, \boldsymbol{P}\,\Sigma_{\mathcal{BV}}\,\boldsymbol{P}^T\right)$$

where $\boldsymbol{P}$ is the projection matrix:

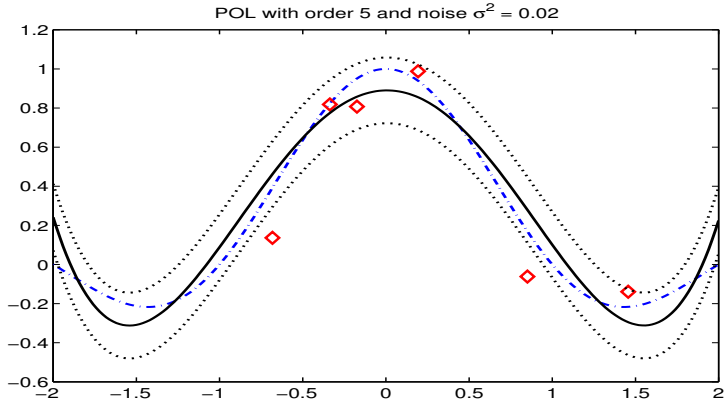
$$\boldsymbol{P} = \boldsymbol{P}_{\boldsymbol{X},\mathcal{BV}} = \boldsymbol{K}_{\mathcal{X},\mathcal{BV}}\,\boldsymbol{K}_{\mathcal{BV}}^{-1}$$



# Gaussian Regression

Artificial data: $y = \sin(x)/x$ and polynomial kernel $K_0(x, x') = (1 + \boldsymbol{x}^T\boldsymbol{x}')^k$.

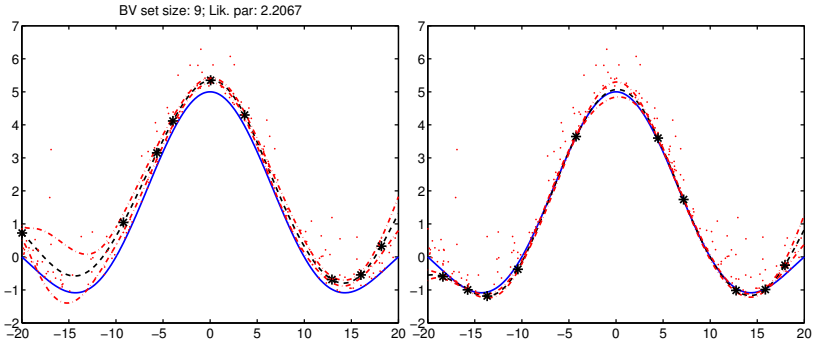
Number of training points: 1000 with added Gaussian noise $\sigma^2 = 0.02$



# Robust one-sided regression

Exponential, one-sided, additive noise.

$$P(y|f_{\boldsymbol{x}}) = \begin{cases} \lambda \exp\left[-\lambda(y - f_{\boldsymbol{x}})\right] & \text{if } y > f_{\boldsymbol{x}}. \\ 0 & \text{otherwise.} \end{cases}$$






# Classification

For each location $\boldsymbol{x}$ we have $\pm 1$. The likelihood function for this model is:

$$P\left(y|f(\boldsymbol{x})\right) = \mathrm{Erf}\left(\frac{y\ f_{\boldsymbol{x}}}{\sigma_0}\right)$$

Erf the incomplete Gaussian ($\sim$ sigmoid):

$$\mathrm{Erf}(x) = \int_{-\infty}^{x} dt\ \exp(-t^2/2)/\sqrt{2\pi}$$

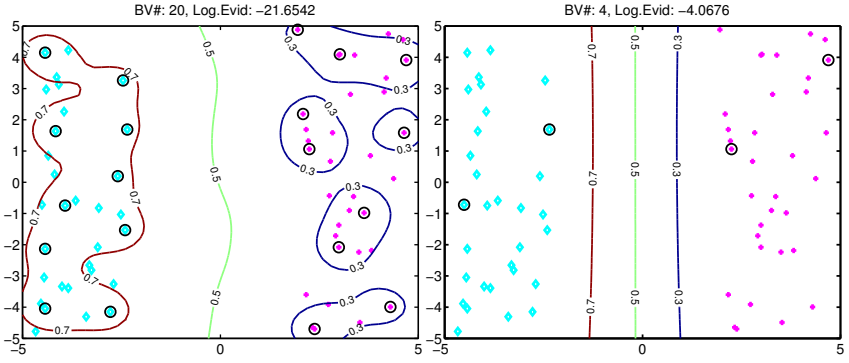
- Posterior is not Gaussian.
- For single data, mean-var computable $\Rightarrow$ iterative methods can be used.



# Toy Classification

RBF kernel: $K(\boldsymbol{x}, \boldsymbol{x}') = \exp\left[-b - \sum_{i=1}^{d}(x_i - x_i')^2 \beta_i\right]$

behaviour of the ARD parameters $\beta_i$



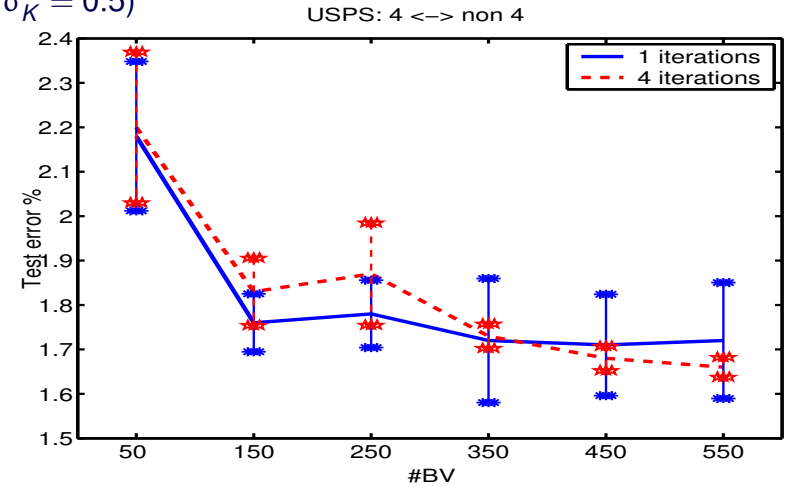

demogp_class_gui



# Classification

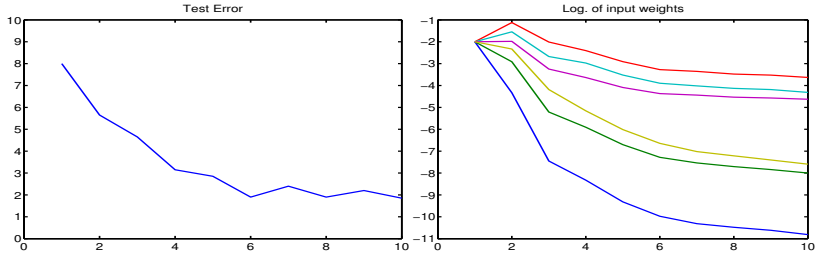
**USPS data-set**

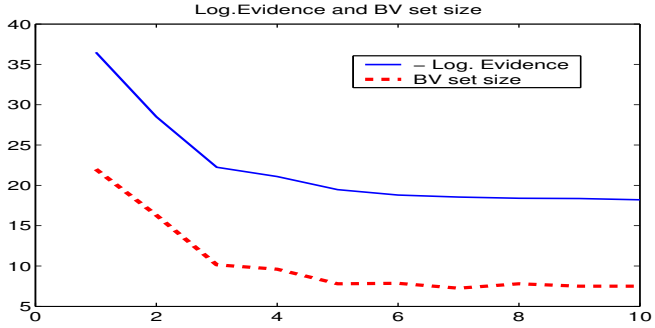
Handwritten image data-set of gray-scale images with 7291 training and 2007 test patterns. (RBF kernel with $\sigma_K^2 = 0.5$)



# Crab data-set

## Using RBF kernels






# Multiclass Classification

**Problem setup:**

- For each location $\boldsymbol{x}$ we have $y \in \{1, \ldots, K\}$.
- Transforming it into $y \in \{0, 1\}^K$ Coding:

$$y = [0, \ldots, 0, 1, 0, \ldots]^T \quad \text{on the } k\text{-th position}$$

- $K$ independent GP's are used. Indep. is a-priori.
- The likelihood function is:

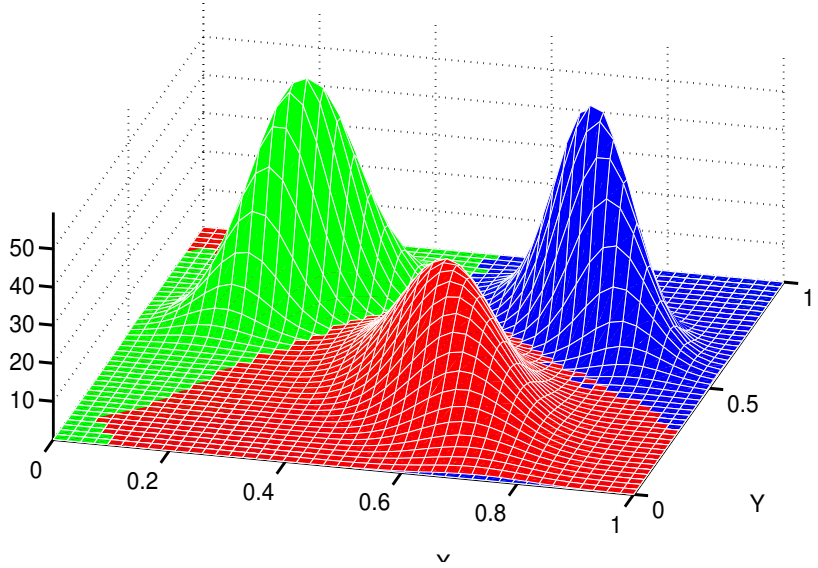
$$P(y|f(\boldsymbol{x})) = \frac{y^T \boldsymbol{s}}{\mathbf{1}^T \boldsymbol{s}} \text{ where } \boldsymbol{s} = \exp\left([f_1(\boldsymbol{x}), \ldots, f_K(\boldsymbol{x})]^T\right).$$

- The posterior processes are not independent.



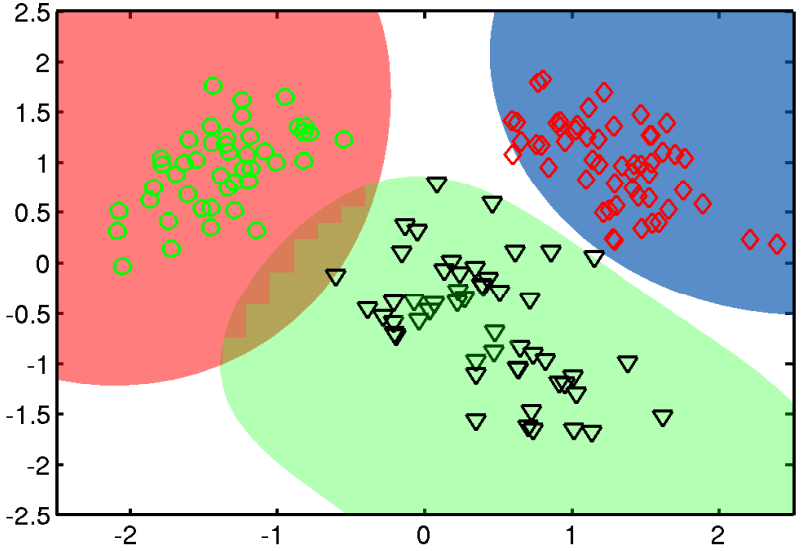
# Multiclass Classification

**Two-dimensional demo:** class-conditional distributions






# Multiclass Classification



# Modelling Inverse Problems

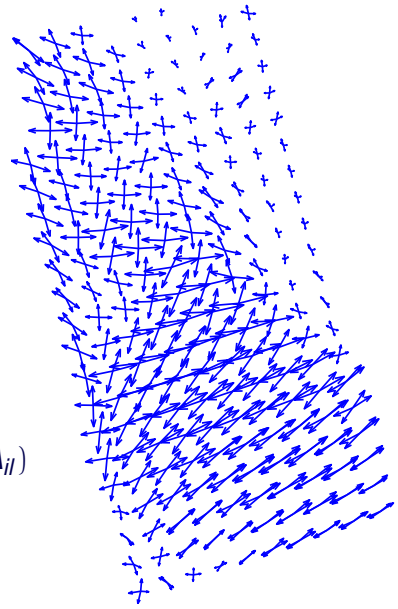
**Likelihood:**
local observations of global wind-fields $(u_i, v_i)$.

Probabilistic framework preferred due to lack of direct observations.

Uncertainty captured in Mixture density networks:

$$P(u_i, v_i|obs) = \sum_{k=1}^{4} \beta_{il} \mathcal{N}(u_i, v_i|\mu_{il}, A_{il})$$

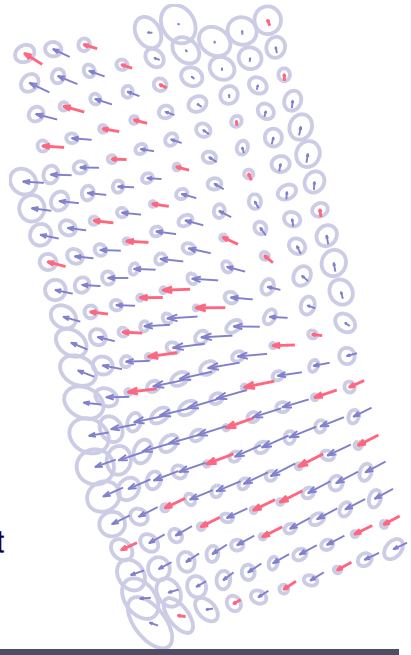
$\beta_{il}, \mu_{il}, A_{il}$ local parameters.





# Modelling Inverse Problems

- Few Basis Vectors retained
- Approximation preserves information about local uncertainty
- The inference process is fast





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▸ Bib 2



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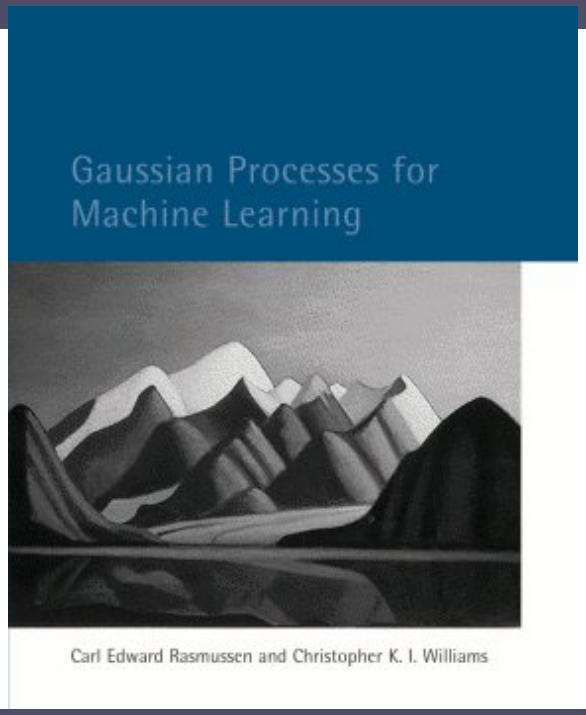


# NIPS'05 I

## NIPS'05 GP-related articles:

- J. Murillo-Fuentes, F. Perez-Cruz: Gaussian Processes for Multiuser Detection in CDMA Receivers
- Y. Shen, A. Ng, M. Seeger: Fast Gaussian Process Regression Using KD-Trees
- A. Shon, K. Grochow, A. Hertzmann, R. Rao: Gaussian Process CCA for Image Synthesis and Robotic Imitation
- R. Der, D. Lee: Beyond Gaussian Processes: On the Distributions of Infinite Networks
- D. Fleet, J. Wang, A. Hertzmann: Gaussian Process Dynamical Models

# NIPS'05 II

## NIPS'05 GP-related articles:

- Y. Engel, P. Szabo, D. Volkinshtein: Learning With Gaussian Process Temporal Difference Methods
- M. Kuss, C. Rasmussen: Assessing Approximations for Gaussian Process Classification
- E. Snelson, Z. Ghahramani: Sparse Parametric Gaussian Processes
- S. Kakade, M. Seeger, D. Foster: Worst-Case Bounds for Gaussian Process Models
- S. Keerthi, W. Chu: A Matching Pursuit Approach to Sparse Gaussian Process Regression
- E. Meeds, S. Osindero: An Alternative Infinite Mixture Of Gaussian Process Experts