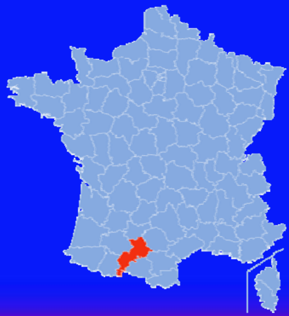
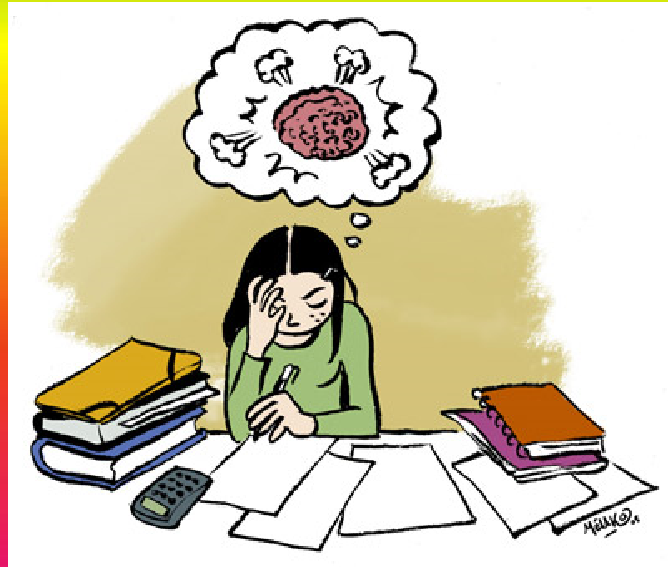


UNIVERSITÉ TOULOUSE III PAUL SABATIER

MATHEMATICAL REASONINGS, SOLVING PROBLEMS OR CONJECTURES, EXPLOITING SURPRISES, ETC. VIA SOME MATHEMATICAL "TAPAS"

Pierre (de) Fermat

Paul Sabatier



and… you

Author: **JB HU** http://www.math.univ-toulouse.fr/~jbhu/

Summary:

With the help of non-standard questions or exercises (called "tapas") at the undergraduate level, we illustrate what mathematics can bring to the general scientific training of students: a rigorous reasoning, solving geometric or extremal problems, explaining surprises, etc. Leitmotivs for these tapas are: ***"It shows or proves something interesting"*****, or** ***"If you solve it, you learn something"*****, or** ***"You will be surprised by the result"*****. These tapas are, for most of them, extracted from two recent books by the author, published at the ends of 2016 and 2017, in English, by the publisher house Springer.**

胡乡伯

法国图卢兹第三大学

"What mathematics usually consists of is problems and solutions"

P. Halmos (1916-2006)

"The best way to learn mathematics is to solve problems"

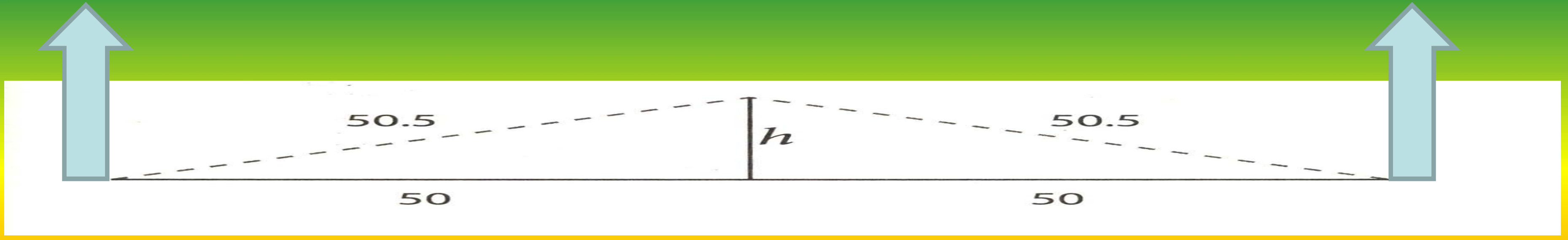
J. Lelong-Ferrand (1918-2014)

"Obvious is the most dangerous word in mathematics"
E.T.BELL (1833-1960)

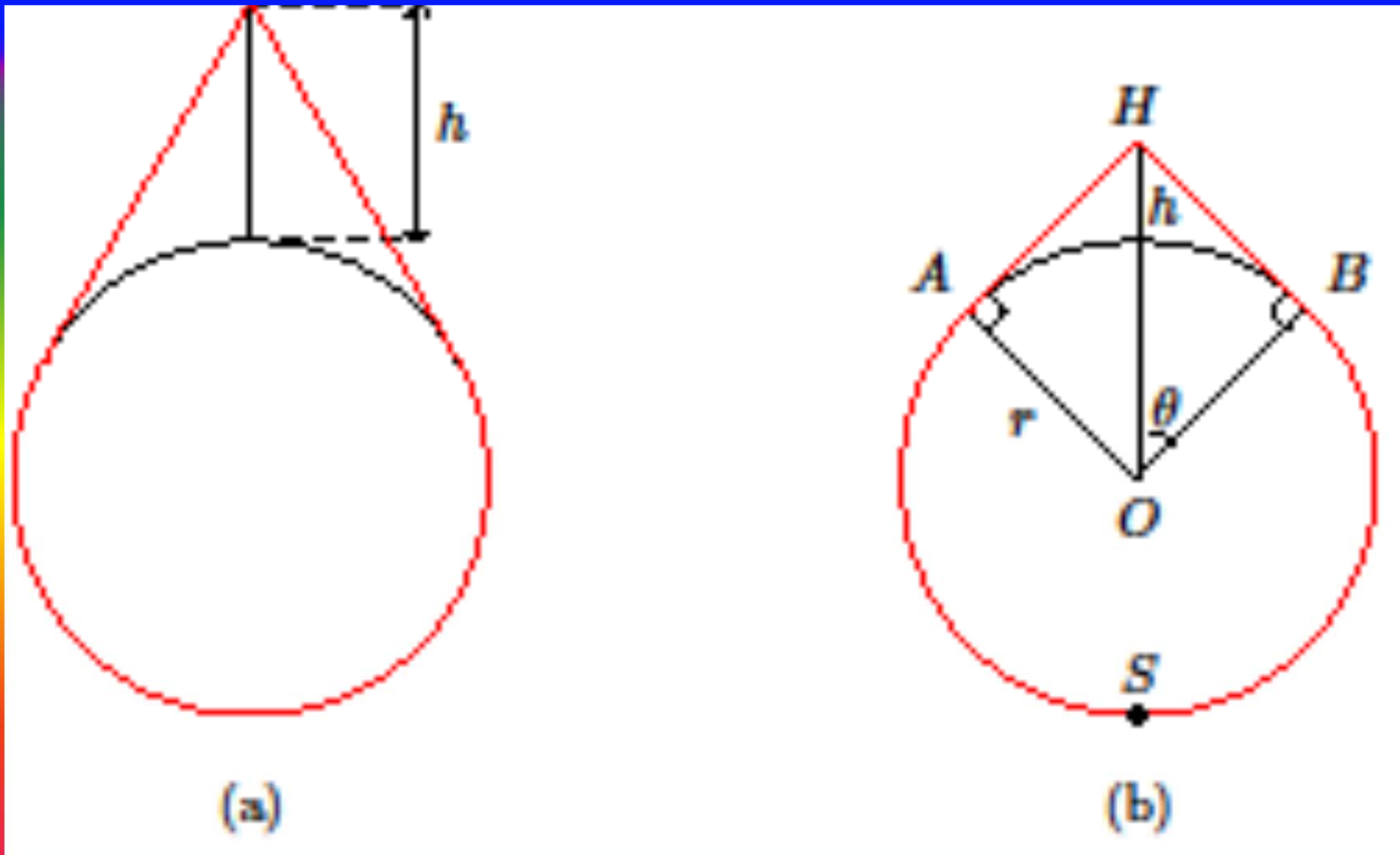
A student in a hurry

A student goes on his bike from his home to the university. He covers the distance with an average velocity of 10 m/h (ten miles per hour). He intends to go back home again on his bike but faster.

What should be the **average velocity on the way back** so that the average velocity on the journey (round-trip) is of 20 m/h?



Surprizing Pythagoras…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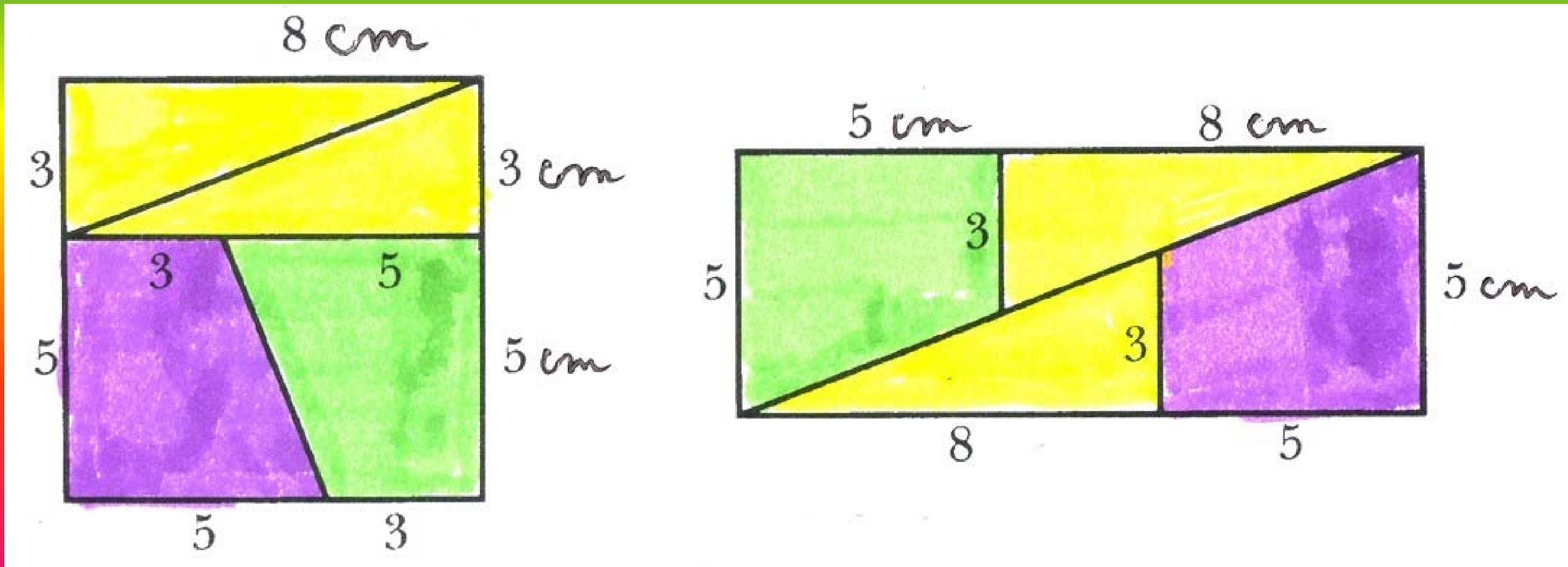
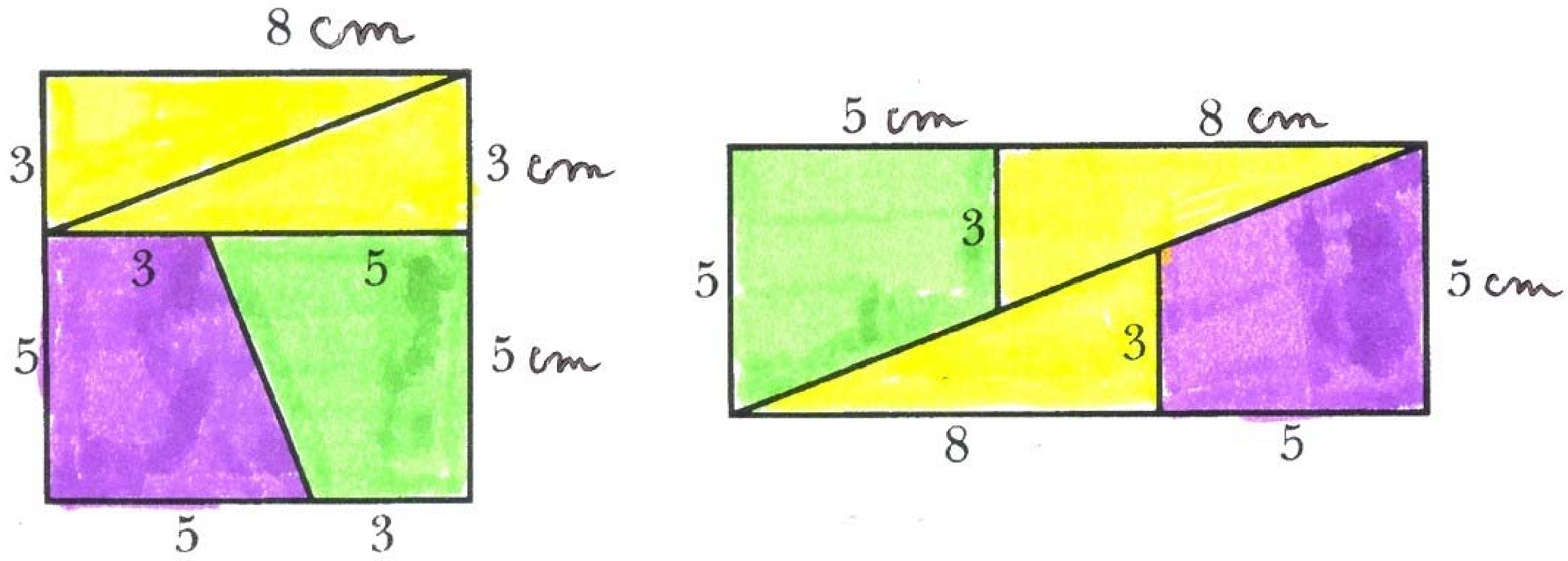
(a)
(b)

Creating area…



8 . (5 + 3) = 64….. **(8 + 5) . 5 = 65 !!!**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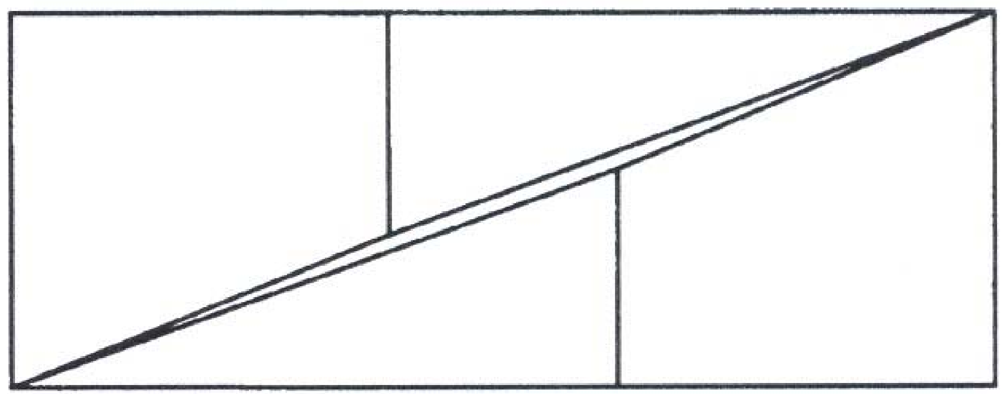


Extremal problems…

2. ★ *Triangles of largest area inscribed in a square*

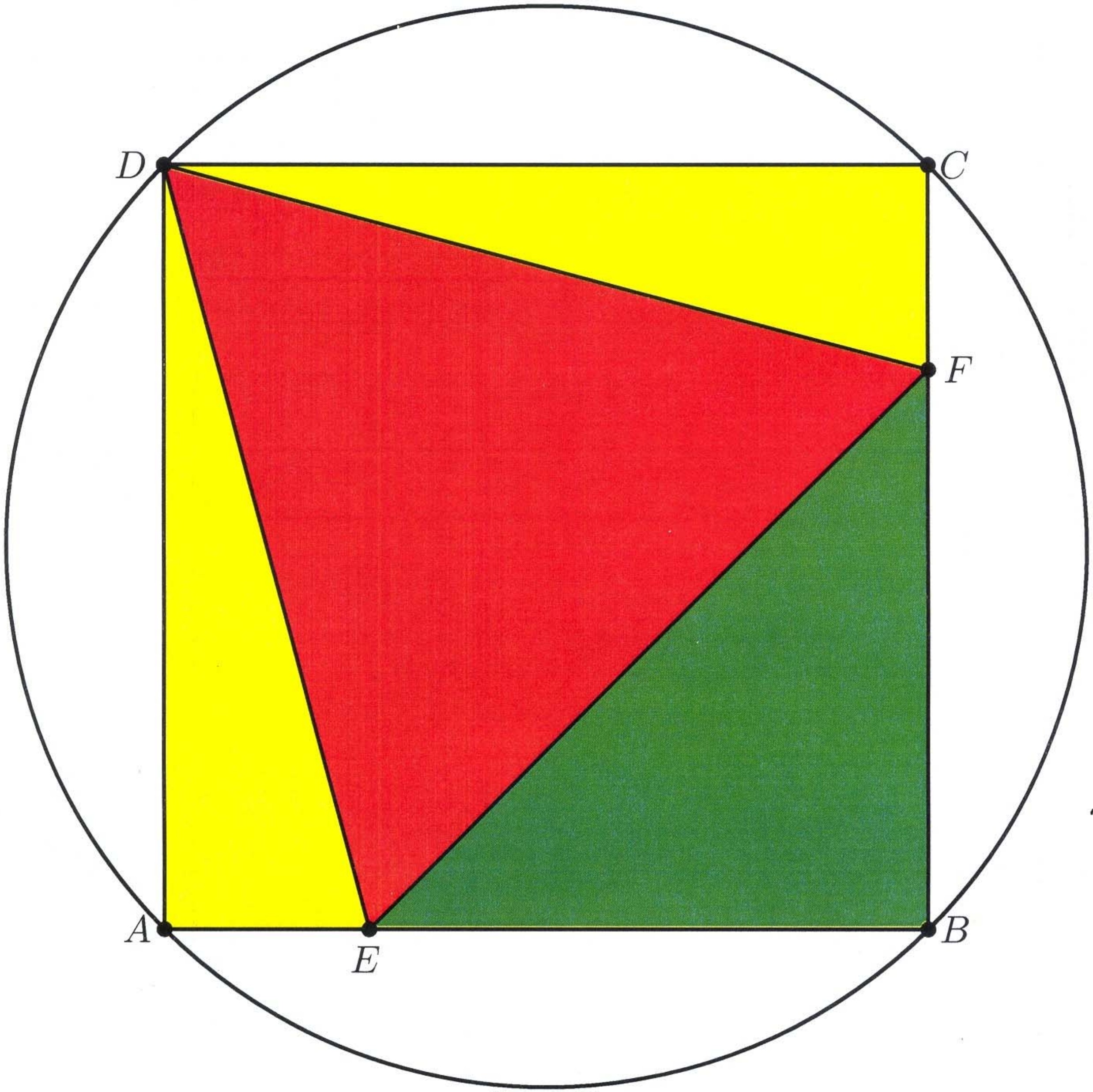
Let DEF be a triangle contained in a square $ABCD$.

1°) No constraint on the shape of the triangle.

What is the **largest possible area** for DEF?

2°) The triangle is constrained to be equilateral.

Find an **equilateral triangle of largest area** DEF contained in the square $ABCD$? What is then the maximal area?


D
C
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10. ★★★ *Prime numbers which are sums of two squares*

2017... is a prime number... But it is more than that, it is the sum of two squares of integers: $44^2 + 9^2$. There are not so many numbers like that... Indeed, between 2000 and 2050, only two numbers are prime and sums of two squares: 2017 as we said above, and $2029 = 2^2 + 45^2$. **How to characterize such integers?**

Theorem by FERMAT and EULER: **a prime number (greater than 3) is the sum of two squares of integers if and only if it is of the form $4k + 1$, with k positive integer.**

Combinatorial proof. The main (and only ingredients).

Let S be a finite set and let $f : S \to S$.

- If f is an **involution** which has **only one fixed point**, then the number of elements in S is **odd**.

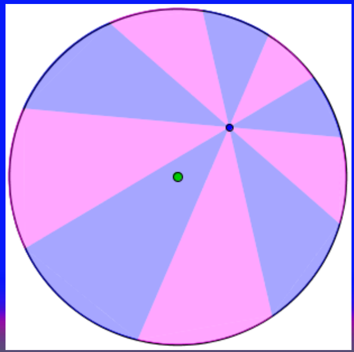
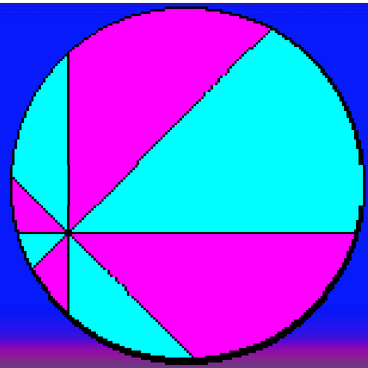
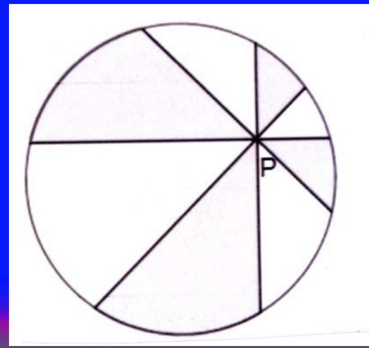
- If the number of elements in S is **odd**, any **involution** f on S posseses **at least one fixed point**.

Slices in a spherical bread

Before going to the university, a student takes breakfast with slices in a spherical loaf of bread. This spherical loaf, of diameter $2r$, is sliced into n pieces of **equal thickness** $h = 2r/n$.

1°) Which slice, with cuts at a and $a+h$, has the *most bread*?

2°) Which slice, with cuts at a and $a+h$, has the *most crust*?

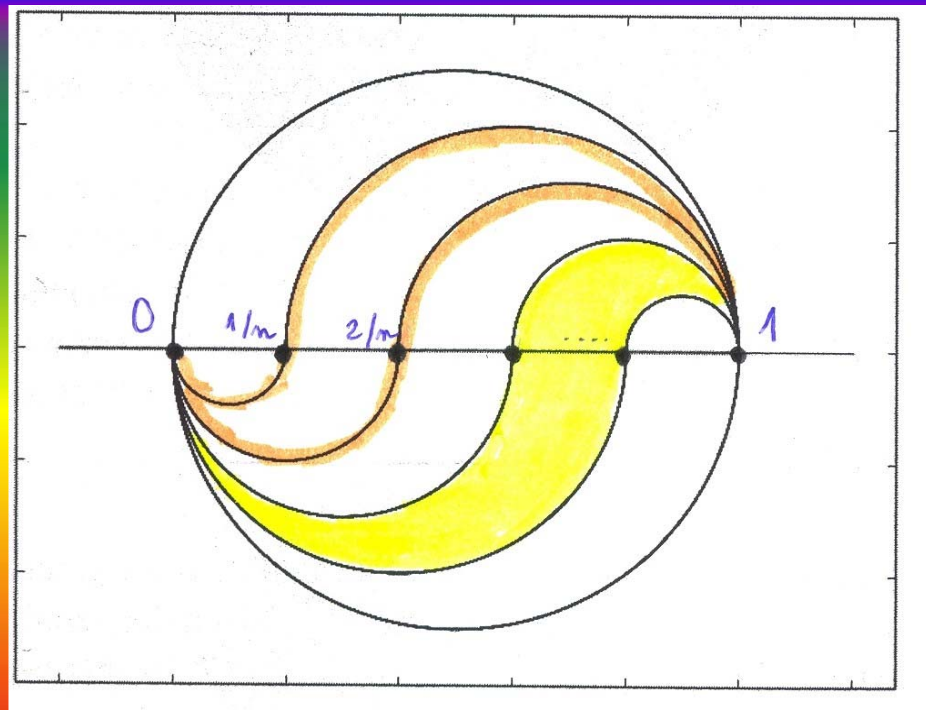
1. ★★ *Fair division of a pizza into 8 parts*

Let us consider a circular pizza that we want to divide into 8 parts, 4 for yourself, 4 for your friend, both getting the same amount of pizza.

Suppose that the center for cuttings is a point P in the pizza, not necessarily the center: with four adjacent cuts rotated by an angle of $\pi/4 = 45^{\circ}$, we get 8 different pieces, similar to sectors of a circle but they are not really ones; however we call them "sectors" for simplicity reasons. You take every second "sector" and your friend the remaining ones. So, each has got 4 pieces of pizza, of different shapes of course.

Prove that **the pizza has been equally divided**, that is to say: the total area of the 4 pieces of pizza is the same for you and your friend!

AIRBUS



In mathematics, the art of proposing a question must be held of higher value than solving it"
G.CANTOR (1845-1918)

"Numerically equal" or "mathematically equal"?
Consider the following "large" integers:

$$S = (3987)^{12} \; ; \; T = (4365)^{12}$$
$$U = (4472)^{12} .$$

1°) Compute, using the PYTHON programming language for example, $S+T$ and U.
2°) Calculate the relative difference $\varepsilon_r = \frac{(S+T)-U}{S+T+U}$.
Hints. 1°) $S+T$ and U are integers with 44 digits; emphasize the first 10 ones.
2°) ε_r is very small, suggesting that $S+T$ and U are "numerically equal".
Answers. 1°) $S+T =$ **6397665634**...........$\times \mathbf{10^{34}}$
$U =$ **6397665634**............ $\times \mathbf{10^{34}}$
2°) $\boldsymbol{\varepsilon_r} \simeq 9.47 \times 10^{-12} < \mathbf{10^{-11}}$.

Let $f : I \to \mathbb{R}$ be assumed differentiable on the open interval I. We suppose the following: for all $a < b$ in I, there is only one $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad (1)$$

In other words, the intermediate c appearing in the expression of the mean value theorem is **unique**.

What can be said about f?

Comment. If one strengthens the assumption above by requiring that the unique c for which (1) holds true is always $(a + b)/2$, then f is necessarily quadratic, $f(x) = ax^2 = bx + c$, with $a \neq 0$.

The square of a derivative function is not necessarily a derivative function

Let f and g be two functions defined on $\mathbb{R}$ as following:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 \text{ if } x = 0; \end{cases}$$

$$g(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

1°) Show that both f and g are derivative functions, that is to say: there are differentiable functions F and G such that $F' = f$ and $G' = g$.

2°) Show that either f^2 or g^2 **is not** a derivative function.

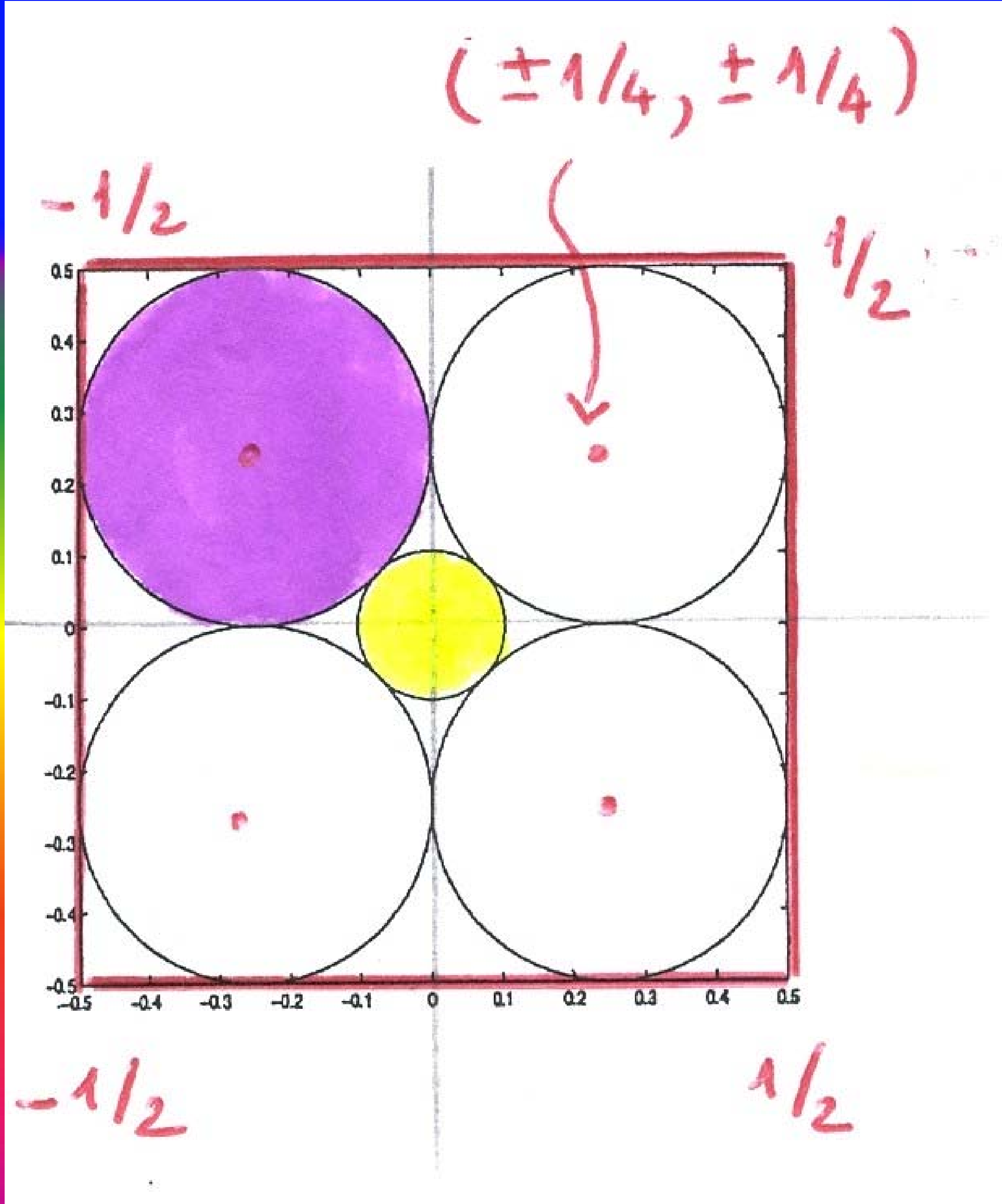
3. ★ *When the n-dimensional jack gets out of its box*

In the space $\mathbb{R}^n$, consider the unit cube centered at the origin, of equation $\max(|x_1|, |x_2|, ..., |x_n|) \leqslant 1/2$, hence of volume (*i.e.*, LEBESGUE measure) 1. In this cube, we place 2^n marbles (small balls) centered at the points $(\pm\frac{1}{4}, ..., \pm\frac{1}{4})$ and of radius $\frac{1}{4}$. At the center of the box, there is still some place for a small one, call it a jack (or a n-dimensional cochonnet) C_n centered at the origin and tangent to the 2^n small balls.

(a) What is the **radius** r_n of C_n? Check that **for** n **large enough, a part of the jack gets out of its box!**

(b) What is the **volume** λ_n (*i.e.*, LEBESGUE measure) of C_n? Using STIRLING's formula, provide an equivalent of λ_n and show that $\lambda_n \to +\infty$ when $n \to +\infty$.

(c) What is the **volume** μ_n of the space occupied by the 2^n balls? What is the behavior of μ_n when when $n \to +\infty$?



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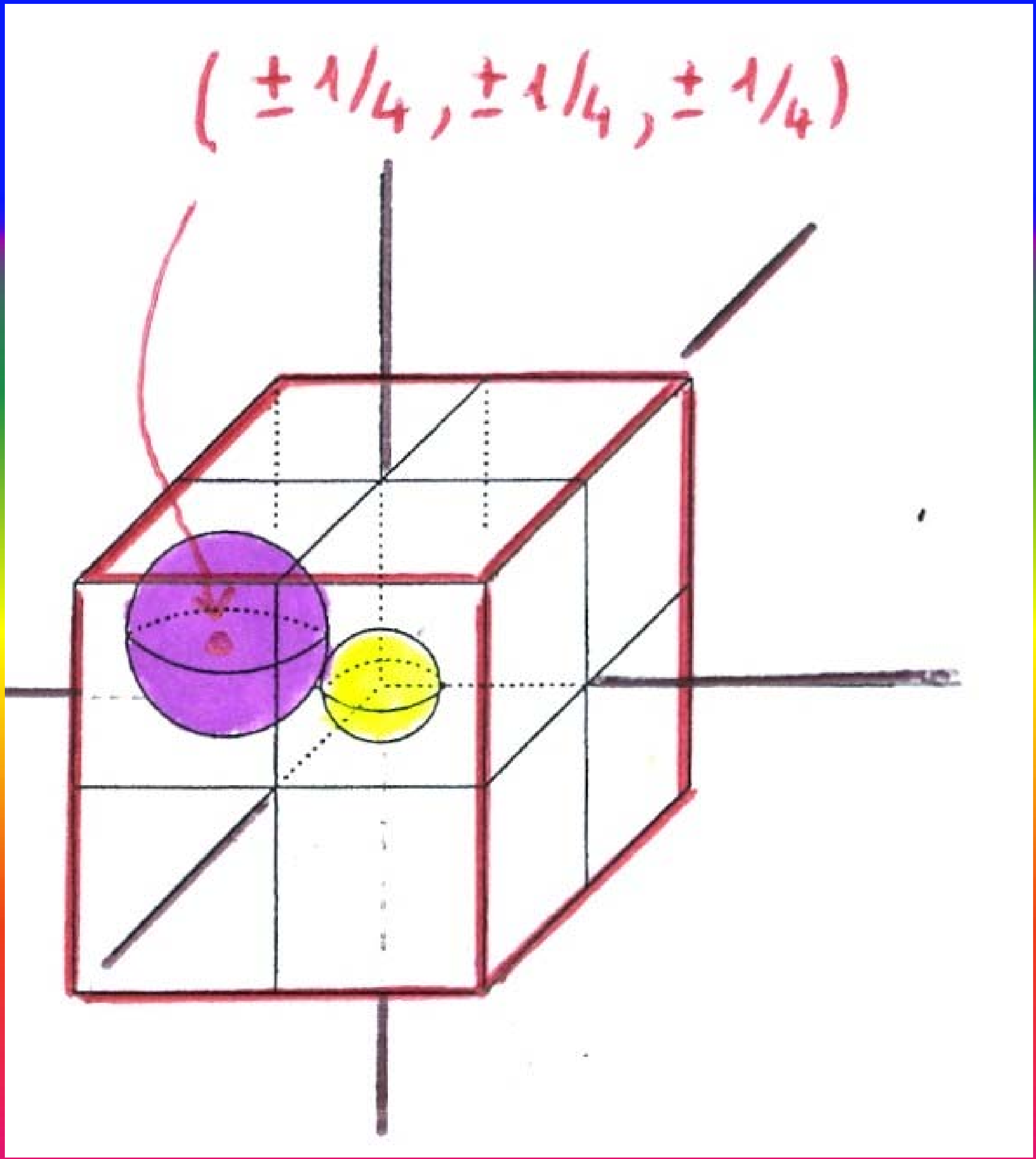
(± 1/4, ± 1/4, ± 1/4)

J.-B. HIRIART-URRUTY

- Volume 1 *(333* items*) Mathematical Tapas* (166 pages) (for Undergraduates) Springer, SUMS series (*published in October 2016*).

- Volume 2 *(222 + 8* items*) Mathematical Tapas* (230 pages) (for Undergraduate to Graduate level) Springer *(published in December 2017).*

- Oui, le mathématiques peuvent surprendre. *Au fil des maths, Association des Professeurs de mathématiques (mars 2019).*