Unbounded solutions of third order three-point boundary value problems on a half-line

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Outline

1 Introduction

Outline

1 Introduction

2 Preliminaries

Outline

1. Introduction
2. Preliminaries
3. Main Results

Outline

1. Introduction
2. Preliminaries
3. Main Results
4. Examples

Abstract

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We consider the following third order three-point boundary value problem on a half-line

$$\begin{aligned} &x'''(t) + q(t)f(t, x(t), x'(t), x''(t)) = 0, \quad t \in (0, +\infty), \\ &x'(0) = A, \quad x(\eta) = B, \quad x''(+\infty) = C, \end{aligned}$$

where $\eta \in (0, +\infty)$, but fixed, and $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies Nagumo's condition. We apply Schäuder's fixed point theorem, the upper and lower solution method, and topological degree theory, to establish existence theory for at least one unbounded solution, and at least three unbounded solutions. To demonstrate the usefulness of our results we illustrate two examples.

It is well-known that third order differential equations arise in a wide variety of different areas of applied mathematics and physics, for example, electromagnetic waves or gravity driven flows, a three layers beam, and the deflection of a curved beam having a constant or a varying cross section.

- M. Greguš, *Third Order Linear Differential Equations, Mathematics and its Applications,* Reidel Publishing Co., Dordrecht, 1987.

Numerous physical phenomena such as the free convection problems in boundary layer theory, and the draining or coating fluid flow problems can be reduced to third order differential equations on an infinite interval.

- E.O. Tuck, L.W. Schwartz, *A numerical and asymptotic study of some third-order ordinary differential equations relevant to draining and coating flows*, SIAM Review, 32 (1990) 453-469.

- W.C. Troy, *Solutions of third-order differential equations relevant to draining and coating flows*, SIAM J. Math. Anal., 24 (1993) 155-171.
- F. Bernis, L.A. Peletier, *Two problems from draining flows involving third-order right focal boundary value problems*, SIAM J. Math. Anal., 27 (1996) 515-527.
- R.P. Agarwal, D. O'Regan, *Singular problems on the infinite intervals modeling pheonmena in draining flows*, IMA J. Appl. Math., 66 (2001) 621-635.
- J.S. Guo and J.C. Tsai, *The structure of solutions for a third order differential equation in boundary layer theory*, Japan J. Industrial Appl. Math., 22 (2005) 311-351.

For more details of second order, third order and higher boundary value problems on infinite interval, see, for instance

- H. Lian, J. Zhao, *Existence of unbounded solutions for third-order boundary value problem on infinite intervals,* Discrete Dyn. Nat. Soc., Volume 2012, Article ID 357697, 14 pages.
- P.K. Palamides, R.P. Agarwal, *An existence theorem for a singular third-order boundary value problem on* $[0,+\infty)$, Appl. Math. Lett., 21 (2008) 1254-1259.
- R.P. Agarwal, D. O'Regan, *Nonlinear Boundary Value Problems on the Semi-infinite interval:an upper and lower solution approach,* Mathematatika, 49 (2002) 129-140.
- S. Djebali, Q. Saifi, *Upper and lower solutions for* $\phi-$*Laplacian third-order BVPs on the half line,* CUMO A Mathematical Journal, 16(1), (2014) 105-116.
- B. Yan, D. O'Regan, R.P. Agarwal, *Unbounded solutions for singular boundary value problems on the Semi-infinite interval: Upper and lower solutions multiplicity,* J. Comput. Appl. Math., 197 (2006) 365-386.

- P.W. Eloe, E.R. Kaufmann, C.C. Tisdell, *Multiple solutions of a boundary value problem on an unbounded domain,* Dyn Syst Appl., 15(1) (2006) 53-63.
- Y. Zhao, H. Chen, C. Xu, *Existence of multiple solutions for three-point boundary-value problems on infinite intervals in Banach spaces,* Electron. J. Differ. Equ., 44 (2012) 1-11.
- H. Lian, W. Ge, *Solvability for second-order three-point boundary value problems on half line,* Appl. Math. Lett., 19 (2006) 1000-1006.
- H. Lian, J. Zha, R.P. Agarwal, Upper and lower solution method for nth order BVPs on an infinite interval, *Boundary Value Problems, 2014*(2014), 100, 17 pages.
- M. Pei, S.K.Chang, Existence and Uniqueness of Solutions for Higher-Order Three-Point Boundary Value Problems, *Boundary Value Problems, 2009*(2009), 16 pages.
- H. Lian, P. Wang, W. Ge, *Unbounded Upper and Lower Solutions Method for Strum-Liouville BVP on Infinite Intervals,* Nonlinear Anal., 70 (2009) 2627-2633.

Shi et al. considered the following third order three-point boundary value problem on a half-line

$$\begin{aligned} &x'''(t) = f(t, x(t), x'(t), x''(t)), \quad t \in (0, +\infty), \\ &x(0) = \alpha x(\eta), \ \lim_{t \to +\infty} x^{(i)}(t) = 0, \ i = 1, 2, \end{aligned}$$

where $\alpha \neq 1$ and $\eta \in (0, +\infty)$, and f is a Carath$\grave{e}$odary function. They provided sufficient conditions for the existence and uniqueness of the solution by employing Schäuder's continuation theorem.

- H.Shi, M. Pei, L. Wang, *Solvability of a third- order three-point boundary value problem on half line,* Bull. Malays. Math. Sci. Soc., (2014), DOI 10.1007/s40840-014-0058-0.

Bai and Li considered the third order differential equation on a half-line together with the following boundary conditions

$$x(0) = x'(0) = 0, \ x''(+\infty) = 0.$$

The authors presented sufficient conditions which guarantee the existence of unbounded solutions to the boundary value problem by using the upper and lower solution method.

- C. Bai, C. Li, *Unbounded upper and lower solution method for third-order boundary-value problems on the half-line,* Electron. J. Differ. Equ., 119 (2009) 1-12.

Third order three point boundary value problem

In this lecture, we develop an existence theory of unbounded solutions for third order ordinary differential equations together with boundary conditions on a half-line

$$\begin{aligned} &x'''(t) + q(t)f(t, x(t), x'(t), x''(t)) = 0, \quad t \in (0, +\infty), \\ &x'(0) = A, \quad x(\eta) = B, \quad \lim_{t\to+\infty} x''(t) = x''(+\infty) = C, \end{aligned} \tag{1}$$

where

- $\eta \in (0, +\infty)$, but fixed,
- $q : (0, +\infty) \to (0, +\infty)$ is continuous,
- $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is continuous,
- $A, B \in \mathbb{R}$, $C \geq 0$.

By using the upper and lower solution method, we present easily verifiable sufficient conditions for the existence of solutions of (1).

Preliminaries

Hereafter I want to talk about some necessary definitions and preparatory results which will be needed to prove the the existence of solutions of (1). We begin with constructing Green's function for the linear boundary value problem

$$\begin{aligned} &x'''(t) + v(t) = 0, \quad t \in (0, +\infty), \\ &x'(0) = A, \quad x(\eta) = B, \quad x''(+\infty) = C. \end{aligned} \tag{2}$$

Lemma 1

Let $v \in \mathcal{C}[0, +\infty)$ *and* $\int_0^\infty v(t)dt < +\infty$. *Then* $x \in \mathcal{C}^2[0, +\infty) \cap \mathcal{C}^3(0, +\infty)$ *is a solution of the problem (2) if and only if*

$$x(t) = \left(B - A\eta - \frac{C\eta^2}{2}\right) + At + \frac{C}{2}t^2 + \int_0^\infty G(t,s)v(s)ds$$

where

$$G(t,s) = \begin{cases} s(t-\eta), & s \le \min\{\eta, t\}; \\ \frac{t^2}{2} + \frac{s^2}{2} - s\eta, & t \le s \le \eta; \\ st - \frac{s^2}{2} - \frac{\eta^2}{2}, & \eta \le s \le t; \\ \frac{1}{2}(t^2 - \eta^2), & \max\{\eta, t\} \le s. \end{cases} \tag{3}$$

Lemma 2

- *If* $\eta \leq t$, *then* $G(t,s) \geq 0$, *for* $(t,s) \in [\eta, +\infty) \times [0, +\infty)$,
- *If* $t \leq \eta$, *then* $G(t,s) \leq 0$, *for* $(t,s) \in [0, \eta) \times [0, +\infty)$.

Banach Space

Let

$$X = \left\{ x \in \mathcal{C}^2[0,+\infty) : \lim_{t\to+\infty} \frac{x(t)}{1+t^2},\ \lim_{t\to+\infty} \frac{x'(t)}{1+t} \text{ and } \lim_{t\to+\infty} x''(t) \text{ exist} \right\}$$

with the norm $\|x\| = \max\{\|x\|_1, \|x\|_2, \|x\|_\infty\}$ where

$$\|x\|_1 = \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1+t^2},\quad \|x\|_2 = \sup_{t\in[0,+\infty)} \frac{|x'(t)|}{1+t},\quad \|x\|_\infty = \sup_{t\in[0,+\infty)} |x''(t)|.$$

Then by the standard arguments, it follows that $(X, \|.\|)$ is a Banach space.

Lower and Upper Solutions

Lower and Upper Solutions

A function $\alpha \in X \cap \mathcal{C}^3(0, +\infty)$ is called a lower solution of (1) if

$$\alpha'''(t) + q(t)f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq 0, \quad t \in (0, +\infty), \tag{4}$$
$$\alpha'(0) \leq A, \quad \alpha(\eta) = B, \quad \alpha''(+\infty) \leq C. \tag{5}$$

Similarly, a function $\beta \in X \cap \mathcal{C}^3(0, +\infty)$ is called an upper solution of (1) if

$$\beta'''(t) + q(t)f(t, \beta(t), \beta'(t), \beta''(t)) \leq 0, \quad t \in (0, +\infty), \tag{6}$$
$$\beta'(0) \geq A, \quad \beta(\eta) = B, \quad \beta''(+\infty) \geq C. \tag{7}$$

Also, we say $\alpha(\beta)$ is a strict lower solution (strict upper solution) for problem (1) if in the above inequalities are strict.

Remark

If

$$\alpha'(t) \leq \beta'(t) \text{ for all } t \in (0, +\infty), \tag{8}$$

then on integrating (8) and using the continuity of $\alpha(t)$ and $\beta(t)$, and the fact that $\alpha(\eta) = B = \beta(\eta)$, it follows that

$$\beta(t) \leq \alpha(t) \text{ for all } t \in [0, \eta)$$

and

$$\alpha(t) \leq \beta(t) \text{ for all } t \in [\eta, +\infty)$$

.

Nagumo's condition

Let $\alpha, \beta \in X \cap \mathcal{C}^3(0, +\infty)$ be a pair of lower and upper solutions of (1) satisfying

$$\alpha'(t) \leq \beta'(t), \ t \in [0, +\infty)$$

A continuous function $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is said to satisfy Nagumo's condition with respect to the pair of functions α, β, if there exist a nonnegative function $\phi \in \mathcal{C}[0, +\infty)$ and a positive function $h \in \mathcal{C}[0, +\infty)$ such that

$$|f(t, y, z, w)| \leq \phi(t)h(|w|) \tag{9}$$

for all $(t, y, z, w) \in [0, \eta) \times [\beta(t), \alpha(t)] \times [\alpha'(t), \beta'(t)] \times \mathbb{R}$ and $(t, y, z, w) \in [\eta, +\infty) \times [\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times \mathbb{R}$ and

$$\int_0^\infty \frac{s}{h(s)} ds = +\infty. \tag{10}$$

Theorem 1

Assume that α, β are lower and upper solutions of (1) satisfying

$$\alpha'(t) \leq \beta'(t), \ t \in [0, +\infty),$$

and suppose that $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is continuous satisfying Nagumo's condition with respect to the pair of functions α, β. Further, assume that

$$f(t, \alpha(t), z, w) \leq f(t, y, z, w) \leq f(t, \beta(t), z, w) \tag{11}$$

for

$$(t, y, z, w) \in [0, \eta) \times [\beta(t), \alpha(t)] \times [\alpha'(t), \beta'(t)] \times \mathbb{R}$$

and

$$(t, y, z, w) \in [\eta, +\infty) \times [\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times \mathbb{R}.$$

Theorem 1

If

$$\int_0^{\infty} \max\{s,1\} q(s) ds < +\infty, \ \int_0^{\infty} \max\{s,1\} \phi(s) q(s) ds < +\infty \qquad (12)$$

and there exists a constant $\gamma > 1$ such that

$$m = \sup_{t \in [0,+\infty)} (1+t)^{\gamma} q(t) \phi(t) < +\infty \qquad (13)$$

where $\phi(t)$ is the function in Nagumo's condition of f, then (1) has at least one solution $x \in X \cap \mathcal{C}^3(0,+\infty)$ satisfying

$$\beta(t) \le x(t) \le \alpha(t), \quad t \in [0,\eta), \quad \alpha(t) \le x(t) \le \beta(t), \quad t \in [\eta,+\infty),$$

$$\alpha'(t) \le x'(t) \le \beta'(t), \quad |x''(t)| < N \quad t \in [0,+\infty);$$

here, N is a constant depending on α, β, C and h.

Proof.

We can choose an r such that

$$r \geq \max\left\{ \sup_{t\in[0,+\infty)} |\alpha''(t)|, \sup_{t\in[0,+\infty)} |\beta''(t)|, C \right\}$$

and an $N > r$ such that

$$\begin{aligned} \int_r^N \frac{s}{h(s)} ds &> m\left(\sup_{t\in[0,+\infty)} \frac{\beta'(t)}{(1+t)^\gamma} - \inf_{t\in[0,+\infty)} \frac{\alpha'(t)}{(1+t)^\gamma} \right. \\ &+ \left. \frac{\gamma}{\gamma-1} \max\{\|\beta\|_2, \|\alpha\|_2\} \right). \end{aligned}$$

Proof.

We define the following auxiliary functions

$$f_1(t,y,z,w) = \begin{cases} t \in [0,\eta), & \begin{cases} f(t,\beta,z,w), & y < \beta(t); \\ f(t,y,z,w), & \beta(t) \le y \le \alpha(t); \\ f(t,\alpha,z,w), & y > \alpha(t), \end{cases} \\ t \in [\eta,+\infty), & \begin{cases} f(t,\beta,z,w), & y > \beta(t); \\ f(t,y,z,w), & \alpha(t) \le y \le \beta(t); \\ f(t,\alpha,z,w), & y < \alpha(t), \end{cases} \end{cases}$$

$$f^*(t,y,z,w) = \begin{cases} f_1(t,y,\beta',w^*) + \frac{\beta'(t)-z}{1+|\beta'(t)-z|}, & z > \beta'(t); \\ f_1(t,y,z,w^*), & \alpha'(t) \le z \le \beta'(t); \\ f_1(t,y,\alpha',w^*) + \frac{\alpha'(t)-z}{1+|\alpha'(t)-z|}, & z < \alpha'(t), \end{cases} \quad (14)$$

where

$$w^* = \begin{cases} N, & w > N; \\ w, & -N \le w \le N; \\ -N, & w < -N. \end{cases}$$

Proof.

Now we consider the modified problem

$$\begin{aligned} & x'''(t) + q(t) f^*(t, x(t), x'(t), x''(t)) = 0, \quad t \in (0, +\infty), \\ & x'(0) = A, \quad x(\eta) = B, \quad x''(+\infty) = C. \end{aligned} \tag{15}$$

As an application of Schäuder's fixed point theorem first we will prove that (15) has at least one solution x satisfying

$$\alpha'(t) \leq x'(t) \leq \beta'(t), \quad t \in [0, +\infty)$$

$$\beta(t) \leq x(t) \leq \alpha(t), \quad t \in [0, \eta), \quad \alpha(t) \leq x(t) \leq \beta(t), \quad t \in [\eta, +\infty).$$

Proof.

For this we need to prove the following steps:

1. The modified problem (15) has a solution x. To show this, for $x \in X$, we define an operator as follows

$$\begin{aligned}(Tx)(t) = & \left(B - A\eta - \frac{C\eta^2}{2}\right) + At + \frac{C}{2}t^2 \\ & + \int_0^\infty G(t,s)q(s)f^*(s,x(s),x'(s),x''(s))ds \end{aligned} \tag{16}$$

for $t \in [0, +\infty)$.
We want to show that the operator T is completely continuous. For this, we need the following generalized Arzela-Ascoli lemma.

- R.P. Agarwal and D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, 2001.

Arzela-Ascoli Lemma

$M \subset X$ is relatively compact if the following conditions hold:

1. all functions belonging to M are uniformly bounded,
2. all functions belonging to M are equi-continuous on any compact sub-interval of $[0, +\infty)$,
3. all functions from M are equi-convergent at infinity, that is, for any $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that for all $t \geq T$ and any $u \in M$,

$$\left| \frac{u(t)}{1+t^2} - \lim_{t\to+\infty} \frac{u(t)}{1+t^2} \right| < \epsilon, \ \left| \frac{u'(t)}{1+t} - \lim_{t\to+\infty} \frac{u'(t)}{1+t} \right| < \epsilon,$$
$$\text{and } |u''(t) - \lim_{t\to+\infty} u''(t)| < \epsilon.$$

Proof.

We split the proof in the following parts:
(a) $T : X \to X$ is well defined.
(b) $T : X \to X$ is continuous.
(c) $T : X \to X$ is compact. (TM is uniformly bounded and TM is equi-continuous.)
(d) TM is equi-convergent at infinity.

Hence all conditions of Arzela-Ascoli Lemma are fulfilled, $T : X \to X$ is completely continuous.

(e) The Schäuder fixed point theorem now guarantees that the operator T has at least one fixed point which is a solution of BVP (15).

Proof.

2. Every solution x of the problem (15) satisfies

$$\alpha'(t) \le x'(t) \le \beta'(t), \quad t \in [0, +\infty) \tag{17}$$

$$\beta(t) \le x(t) \le \alpha(t), \quad t \in [0, \eta), \quad \alpha(t) \le x(t) \le \beta(t), \quad t \in [\eta, +\infty). \tag{18}$$

For this, we shall show that $x'(t) \le \beta'(t)$ for all $t \in [0, +\infty)$. If this is not true then there exists a $t_0 \in [0, +\infty)$ such that

$$x'(t_0) - \beta'(t_0) = \sup_{t \in [0, +\infty)} (x'(t) - \beta'(t)) > 0.$$

Now in view of $\lim\limits_{t \to +\infty} (x''(t) - \beta''(t)) \le 0$, there are three cases.

Proof.

Case I. If $t_0 = 0$, then
$x'(0) - \beta'(0) = \lim_{t \to 0^+} x'(t) - \beta'(t) = \sup_{t \in [0,+\infty)} (x'(t) - \beta'(t)) > 0$. From the boundary condition (7), we have the contradiction $x'(0) - \beta'(0) \leq 0$.

Proof.

Case II. If $t_0 \in (0, +\infty)$ then, we have $x''(t_0) = \beta''(t_0)$ and $x'''(t_0) \leq \beta'''(t_0)$. But then from (14), (15) and $N > \sup_{t\in[0,+\infty)} |\beta''(t)|$, we find

$$\begin{aligned} x'''(t_0) &= -q(t_0) f^*(t_0, x(t_0), x'(t_0), x''(t_0)) \\ &= -q(t_0)\left[f_1(t_0, x(t_0), \beta'(t_0), \beta''(t_0)) + \frac{\beta'(t_0) - x'(t_0)}{1 + |\beta'(t_0) - x'(t_0)|}\right]. \end{aligned}$$

Now using the condition (11) and the definition of f_1 respectively, we obtain for $t_0 \in [0, \eta)$
if $x(t_0) \geq \beta(t_0)$,

$$x'''(t_0) = -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) + q(t_0)\frac{x'(t_0) - \beta'(t_0)}{1 + |\beta'(t_0) - x'(t_0)|},$$

and if $x(t_0) < \beta(t_0)$,

$$x'''(t_0) \geq -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) + q(t_0)\frac{x'(t_0) - \beta'(t_0)}{1 + |\beta'(t_0) - x'(t_0)|}, \ .$$

Proof.

and using the definition of f_1 and the condition (11) respectively, we obtain for $t_0 \in [\eta, +\infty)$,
if $x(t_0) > \beta(t_0)$,

$$x'''(t_0) = -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) + q(t_0) \frac{x'(t_0) - \beta'(t_0)}{1 + |\beta'(t_0) - x'(t_0)|},$$

and if $x(t_0) \le \beta(t_0)$,

$$x'''(t_0) \ge -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) + q(t_0) \frac{x'(t_0) - \beta'(t_0)}{1 + |\beta'(t_0) - x'(t_0)|}.$$

Therefore, it follows that

$$\begin{aligned} x'''(t_0) &\ge -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) + q(t_0) \frac{x'(t_0) - \beta'(t_0)}{1 + |\beta'(t_0) - x'(t_0)|} \\ &> -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) \ge \beta'''(t_0), \end{aligned}$$

which is a contradiction.

Proof.

Case III. If $t_0 = +\infty$ then
$$x'(+\infty) - \beta'(+\infty) = \lim_{t\to+\infty} x'(t) - \beta'(t) = \sup_{t\in[0,+\infty)} (x'(t) - \beta'(t)) > 0.$$
Obviously, from the boundary condition (7), we obtain the contradiction $x'''(+\infty) - \beta'''(+\infty) \leq 0$. Consequently, $x'(t) \leq \beta'(t)$ for $t \in [0, +\infty)$. The proof of $\alpha'(t) \leq x'(t)$ for $t \in [0, +\infty)$ is similar. Hence (17) holds, and consequently (18) follows.

Proof.

3. The solution x satisfies

$$|x''(t)| < N \text{ for all } t \in [0, +\infty).$$

Suppose there exists a $t_0 \in [0, +\infty)$ such that $|x''(t_0)| \geq N$. Since $\lim_{t \to +\infty} x''(t) = C < N$, there exists a $T > 0$ such that

$$|x''(t)| < N \text{ for } t \geq T.$$

Let $t_1 = \inf\{t \leq T : |x''(s)| < N, \forall s \in [t, +\infty)\}$. Then $|x''(t_1)| = N$ and $|x''(t)| < N$ for all $t > t_1$, and there exists a $t_2 < t_1$ such that $|x''(t)| \geq N$ for $t \in [t_2, t_1]$. We need to consider two cases $x''(t_1) = N$ and $x''(t) \geq N$ for $t \in [t_2, t_1]$ or $x''(t_1) = -N$ and $x''(t) \leq -N$ for $t \in [t_2, t_1]$. We assume that $x''(t_1) = N$ and $x''(t) \geq N$ for $t \in [t_2, t_1]$, then we have

Proof.

$$
\begin{aligned}
\int_r^N \frac{s}{h(s)}ds \leq \int_C^N \frac{s}{h(s)}ds = - \int_{t_1}^{\infty} \frac{x''(s)}{h(x''(s))}x'''(s)ds \\
= & \quad - \int_{t_1}^{\infty} \frac{-q(s)f(s,x(s),x'(s),x''(s))x''(s)}{h(x''(s))}ds \\
\leq & \quad \int_{t_1}^{\infty} q(s)\phi(s)x''(s)ds \leq m \int_{t_1}^{\infty} \frac{x''(s)}{(1+s)^{\gamma}}ds \\
= & \quad m \left(\int_{t_1}^{\infty} \left(\frac{x'(s)}{(1+s)^{\gamma}} \right)' ds - \int_{t_1}^{\infty} x'(s) \left(\frac{1}{(1+s)^{\gamma}} \right)' ds \right) \\
\leq & \quad m \left(\sup_{t\in[0,+\infty)} \frac{\beta'(t)}{(1+t)^{\gamma}} - \inf_{t\in[0,+\infty)} \frac{\alpha'(t)}{(1+t)^{\gamma}} \frac{\gamma}{\gamma-1} \max\{\|\beta\|_2, \|\alpha\|_2\} \right) \\
< & \quad \int_r^N \frac{s}{h(s)}ds
\end{aligned}
$$

which is a contradiction.

Proof.

Consequently, from (17),(18) and auxiliary function f^*, f_1 we have

$$\begin{aligned} x'''(t) &= -q(t)f^*(t, x(t), x'(t), x''(t)) \\ &= -q(t)f_1(t, x(t), x'(t), x''(t)) \\ &= -q(t)f(t, x(t), x'(t), x''(t)), \end{aligned}$$

and hence, x is a solution of (1). □

Theorem 2

Assume that there exist strict lower and upper solutions $\alpha_2,\ \beta_1$, and lower and upper solutions $\alpha_1,\ \beta_2$ of BVP (1), satisfying

$$\alpha_1'(t) \le \alpha_2'(t) \le \beta_2'(t),\ \ \alpha_1'(t) \le \beta_1'(t) \le \beta_2'(t),\ \alpha_2'(t) \nleq \beta_1'(t), \qquad (19)$$

for all $t \in [0,+\infty)$. Suppose further that $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is continuous satisfying Nagumo's condition with respect to the pair of functions α_1, β_2, and

$$f(t,\alpha_1(t),z,w) \le f(t,y,z,w) \le f(t,\beta_2(t),z,w)$$

for $(t,y,z,w) \in [0,\eta) \times [\beta_2(t),\alpha_1(t)] \times [\alpha_1'(t),\beta_2'(t)] \times \mathbb{R}$, and
$(t,y,z,w) \in [\eta,+\infty) \times [\alpha_1(t),\beta_2(t)] \times [\alpha_1'(t),\beta_2'(t)] \times \mathbb{R}$.
If (12) and (13) hold then (1) has at least three solutions $x_1, x_2, x_3 \in X \cap \mathcal{C}^3(0,+\infty)$ satisfying

$$\beta_i(t) \le x_i(t) \le \alpha_i(t),\ t \in [0,\eta),\ \alpha_i(t) \le x_i(t) \le \beta_i(t),\ t \in [\eta,+\infty), i = 1,2$$

$$\alpha_1'(t) \le x_1'(t) \le \beta_1'(t),\ \alpha_2'(t) \le x_2'(t) \le \beta_2'(t),\ t \in [0,+\infty),$$

$$\beta_2(t) \le x_3(t) \le \alpha_1(t),\ t \in [0,\eta),\ \alpha_1(t) \le x_3(t) \le \beta_2(t),\ t \in [\eta,+\infty)$$

Proof.

We define an auxiliary function f_1^* same as f^* in Theorem 19 except α and β replaced by α_1 and β_2, respectively. We consider the modified problem

$$\begin{array}{l} x'''(t) + q(t) f_1^*(t, x(t), x'(t), x''(t)) = 0, \quad t \in (0, +\infty), \\ x'(0) = A, \quad x(\eta) = B, \quad x''(+\infty) = C. \end{array} \tag{20}$$

We want to show that (20) has at least three solutions. For this we define an operator by

$$\begin{aligned} (T_1 x)(t) &= \left(B - A\eta - \frac{C\eta^2}{2} \right) + At + \frac{C}{2} t^2 \\ &\quad + \int_0^\infty G(t,s) q(s) f_1^*(s, x(s), x'(s), x''(s)) ds. \end{aligned}$$

As for T in Theorem 1 we can show that $T_1 : X \to X$ is completely continuous.

Proof.

Now by using the degree theory, we will show that T_1 has at least three fixed points which are solutions of (20). For $x \in X$, as in Theorem 19, we have

$$\|T_1x\|_1 \leq \left|B - A\eta - \frac{C\eta^2}{2}\right| + |A| + \frac{C}{2} + \max\left\{\frac{1}{2}, \eta\right\} \int_0^\infty sq(s)(H_3\phi(s) + 1)ds := k_1,$$

$$\|T_1x\|_2 \leq |A| + C + \int_0^\infty sq(s)(H_3\phi(s) + 1)ds := k_2,$$

$$\|(T_1x)\|_\infty \leq C + \int_0^\infty q(s)(H_3\phi(s) + 1)ds := k_3,$$

where $H_3 = \sup_{0 \leq t \leq \|x\|_\infty} h(t) < +\infty$. Let $\Omega = \{x \in X : \|x\| < K\}$ where $K > \max\{k_1, k_2, k_3\}$.

Proof.

Then we have $\|T_1 x\| < K$, which implies that $T_1\overline{\Omega} \subset \Omega$. Thus, $\deg(I - T_1, \Omega, 0) = 1$. Next, we set

$$\Omega_{\alpha_2} = \{x \in \Omega : x'(t) > \alpha_2'(t),\ t \in [0, +\infty)\},$$

$$\Omega^{\beta_1} = \{x \in \Omega : x'(t) < \beta_1'(t),\ t \in [0, +\infty)\}.$$

Since $\alpha_1'(t) \le \alpha_2'(t) \le \beta_2'(t),\ \ \alpha_1'(t) \le \beta_1'(t) \le \beta_2'(t),\ \alpha_2'(t) \not\le \beta_1'(t)$ and $\alpha_2'(t) \not\le \beta_1'(t),\ t \in [0, +\infty)$, we find $\Omega_{\alpha_2} \neq \emptyset \neq \Omega^{\beta_1}$ and $\overline{\Omega}_{\alpha_2} \cap \overline{\Omega^{\beta_1}} = \emptyset$ whereas the set $\Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}} \neq \emptyset$. Hence in view of the strict upper and lower solutions β_1 and α_2, T_2 has no solution in $\partial\Omega_{\alpha_2} \cup \partial\Omega^{\beta_1}$.

Proof.

The additivity of degree implies that

$$\begin{aligned}\deg(I-T_1,\Omega,0) &= \deg(I-T_1,\Omega\setminus\overline{\Omega_{\alpha_2}\cup\Omega^{\beta_1}},0)\\ &+ \deg(I-T_1,\Omega_{\alpha_2},0)+\deg(I-T_1,\Omega^{\beta_1},0).\end{aligned} \quad (21)$$

Now we shall show that $\deg(I-T_2,\Omega_{\alpha_1},0)=1$. For this, we define a completely continuous operator $T_2:\overline{\Omega}\to\overline{\Omega}$ by

$$(T_2x)(t)=\left(B-A\eta-\frac{C\eta^2}{2}\right)+At+\frac{C}{2}t^2+\int_0^\infty G(t,s)q(s)f_2^*(s,.,.,.)ds,$$

where the function f_2^* is the same as f_1^* except α_1 is replaced with α_2. Again as in Theorem 19 it is easy to show that T_3 has a fixed point x satisfying $\alpha_2'(t)\le x'(t)\le\beta_2'(t)$, $t\in[0,+\infty)$. Since the lower solution α_2 is strict, $x'(t)\neq\alpha_2'(t)$, $t\in[0,+\infty)$. Therefore, $x\in\Omega_{\alpha_2}$. Hence, it follows that

$$\deg(I-T_2,\Omega\setminus\overline{\Omega}_{\alpha_2},0)=0.$$

Proof.

Further, we can show that $T_2\overline{\Omega} \subset \Omega$. Then, we have

$$\deg(I - T_2, \Omega, 0) = 1. \tag{22}$$

Since $f_2^* = f$ in the region Ω_{α_2}, we find

$$\begin{aligned}
\deg(I - T_1, \Omega_{\alpha_2}, 0) &= \deg(I - T_2, \Omega_{\alpha_2}, 0) \\
&= \deg(I - T_2, \Omega_{\alpha_2}, 0) + \deg(I - T_2, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) \\
&= \deg(I - T_2, \Omega, 0) = 1.
\end{aligned}$$

Similar to the proof of (23), we also have

$$\deg(I - T_1, \Omega^{\beta_1}, 0) = 1. \tag{23}$$

Thus from (21), (22) and (23), we obtain

$$\deg(I - T_1, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}, 0) = -1.$$

Therefore, T_1 has at least three fixed points $x_1 \in \Omega^{\beta_1}$, $x_2 \in \Omega_{\alpha_2}$, $x_3 \in \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}$ which are solutions of the problem (1). □

Example-1

Consider the third order three-point boundary value problem

$$x'''(t) + \frac{1}{1+t^9}(x(t)-1)^2(2t - x'(t))^2(1+\sin(x''(t))^2) = 0, \quad (24)$$
$$x'(0) = 0, \quad x(1) = 1, \quad x''(+\infty) = 1.$$

for $t \in (0, +\infty)$.

Comparing (24) with (1), we have

$$q(t) = \frac{1}{1+t^9}, \; f(t, y, z, w) = (y-1)^2(2t-z)^2(1+\sin w^2)$$

and $\eta = 1, \; A = 0, \; B = 1, \; C = 1$. It is easy to check that

$$\alpha(t) = 2 - t^2, \; \beta(t) = t^2$$

are lower and upper solutions of (24).

Further, $\alpha, \beta \in X,\ \beta(t) \le \alpha(t),\ t \in [0,1),\ \alpha(t) \le \beta(t),\ t \in [1,+\infty)$ and $\alpha'(t) \le \beta'(t),\ t \in [0,+\infty)$.
Clearly, f is continuous
f is decreasing on $[0,1) \times [t^2, 2-t^2] \times [-2t, 2t] \times \mathbb{R}$
f is increasing on $[1,+\infty) \times [2-t^2, t^2] \times [-2t, 2t] \times \mathbb{R}$ in y.
Thus f satisfies condition (11).
Moreover, f satisfies Nagumo's condition with respect to $\alpha(t) = 2 - t^2$ and $\beta(t) = t^2$; that is, when $0 \le t < 1$, $t^2 \le y \le 2 - t^2$, $-2t \le z \le 2t$, $w \in \mathbb{R}$, and $1 \le t < +\infty$, $2 - t^2 \le y \le t^2$, $-2t \le z \le 2t$, $w \in \mathbb{R}$,

$$|f(t,y,z,w)| \le \phi(t)h(|w|),$$

where $\phi(t) = t^2(t^2-1)^2,\ h(w) = 16(1+w^2)$ and

$$\int_0^\infty \frac{s}{h(s)}ds = \int_0^\infty \frac{s}{16(1+s^2)}ds = +\infty.$$

Also,

$$\int_0^{\infty} \max\{s,1\} q(s) ds = \int_0^1 \frac{1}{1+s^9} ds + \int_1^{\infty} \frac{s}{1+s^9} ds < +\infty,$$

$$\int_0^{\infty} \max\{s,1\} \phi(s) q(s) ds = \int_0^1 \frac{s^2(s^2-1)^2}{1+s^9} ds + \int_1^{\infty} \frac{s^3(s^2-1)^2}{1+s^9} ds < +\infty;$$

i.e., (12) is satisfied. Therefore, from Theorem 1, the boundary problem (24) has at least one solution x such that

$$\beta(t) = t^2 \leq x(t) \leq \alpha(t) = 2 - t^2, \ t \in [0,1),$$

$$\alpha(t) = 2 - t^2 \leq x(t) \leq \beta(t) = t^2, \ t \in [1,+\infty),$$

and

$$\alpha'(t) = -2t \leq x'(t) \leq \beta'(t) = 2t, \ |x''(t)| < N \text{ for all } t \in [0,+\infty),$$

$$\beta(t) = t^2 \le x(t) \le \alpha(t) = 2 - t^2, \; t \in [0, 1),$$
$$\alpha(t) = 2 - t^2 \le x(t) \le \beta(t) = t^2, \; t \in [1, +\infty),$$

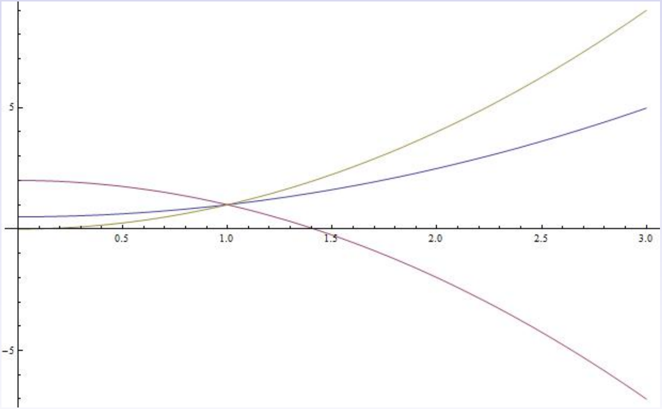

Figure : The solution $x(t)$ of the BVP (24) and $\alpha(t) = 2 - t^2, \beta(t) = t^2$

$$\alpha'(t) = -2t \le x'(t) \le \beta'(t) = 2t \quad \text{for all } t \in [0, +\infty),$$

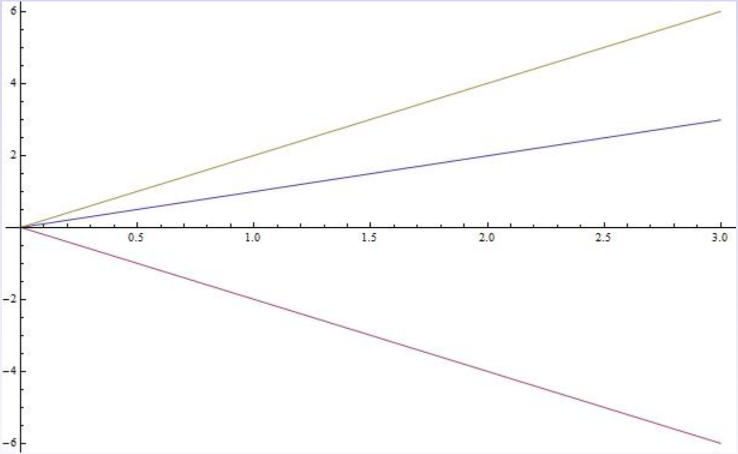

Figure : The solution derivative $x'(t)$ of the BVP (24) and $\alpha'(t) = -2t, \beta'(t) = 2t$

$$|x''(t)| < N \text{ for all } t \in [0, +\infty),$$

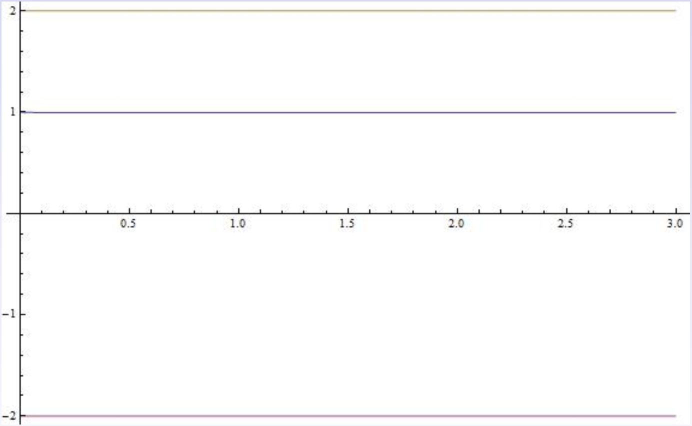

Figure : The solution second derivative of the BVP (24) satisfies $|x''(t)| < N$

Example-2

Consider the third order three-point boundary value problem

$$\begin{aligned} &x'''(t) + e^{-t}(6 + 2t + x'(t))(2 + 3t - x'(t))(\tfrac{3}{2} - x''(t)) = 0, \\ &x'(0) = -\tfrac{19}{4}, \quad x(2) = 0, \quad x''(+\infty) = 1, \end{aligned} \tag{25}$$

$t \in (0, +\infty)$.

Here $\eta = 2,\ A = \frac{-19}{4},\ B = 0,\ C = 1,$

$$q(t) = e^{-t},\ f(t, y, z, w) = (6 + 2t + z)(2 + 3t - z)(\frac{3}{2} - w).$$

It is easy to check that

$$\alpha_1(t) = -t^2 - 6t + 16, \quad \beta_2(t) = \frac{3}{2}t^2 + t - 8$$

are lower and upper solutions of (25) and

$$\alpha_2(t) = \frac{14}{3}(t+1)^{\frac{3}{2}} - 12t + 24 - 14\sqrt{3}, \quad \beta_1(t) = t^2 - \frac{9}{2}t + 5$$

are strict lower and upper solutions of (25). Further, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in X$ and satisfy (19).

Clearly, f is continuous. Moreover, with respect to $\alpha_1(t) = -t^2 - 6t + 16$ and $\beta_2(t) = \frac{3}{2}t^2 + t - 8$; that is, when $0 \le t < +\infty$, $-6 - 2t = \alpha_1'(t) \le z \le \beta_2'(t) = 3t + 1$ and $w \in \mathbb{R}$, f satisfies

$$|f(t, y, z, w)| \le \phi(t)h(|w|),$$

where $\phi(t) = 30t^2 + 92t + 97$, $h(w) = \frac{3}{2} + w$, and hence

$$\int_0^\infty \frac{s}{h(s)}ds = \int_0^\infty \frac{s}{\frac{3}{2} + s}ds = +\infty.$$

$$\int_0^\infty \max\{s, 1\}q(s)ds = \int_0^1 e^{-s}ds + \int_1^\infty se^{-s}ds < +\infty,$$

$$\begin{aligned}\int_0^\infty \max\{s, 1\}\phi(s)q(s)ds &= \int_0^1 (30s^2 + 92s + 97)e^{-s}ds \\ &\quad + \int_1^\infty s(30s^2 + 92s + 97)e^{-s}ds < +\infty;\end{aligned}$$

that is, (12) is also satisfied. Thus, from Theorem 2, confirms that the boundary problem (25) has at least three solutions.

The material presented in this talk is based on the following paper:

- Ravi P. Agarwal, Erbil Çetin, *Unbounded solutions of third order three-point boundary value problems on a half-line,* Advances in Nonlinear Analysis, DOI: 10.1515/anona-2015-0043, May 2015.