BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MODEL

BBU Math-CS Competition 2019
Written test of MATHEMATICS

PART A

IMPORTANT: Problems in Part A have one or more correct answers.

1. (6 points) Let $z_1, z_2 \in \mathbb{C}$ be numbers such that $z_1^2 + z_2^2 = 0$ and $|z_1| = |z_2| = 1$. Then:

A such numbers do not exist;

B there exist $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$ satisfying the given conditions;

C there exist $z_1, z_2 \in \mathbb{R}$ satisfying the given conditions;

D $|z_1 + z_2| = \sqrt{2}$.

2. (6 points) A solution of the inequation $A_{x+2}^3 + C_{x+3}^2 > 5(x+2)$ is

A $x = 0$; B $x = 1$; C $x = 2$; D $x = 3$.

3. (6 points) Consider the polynomials $f = 1 + X + 3X^2 + 5X^3 + \cdots + 2019X^{1010}$, $g = X - 1 \in \mathbb{R}[X]$. The remainder of the division of f by g is

A 1020100; B 2020; C 1020101; D 2039191.

4. (6 points) The number of group isomorphisms from the group $(\mathbb{Z}_3, +)$ to itself is

A 1; B 2; C 3; D 9.

5. (6 points) The function $f\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\} \to \mathbb{R}$, defined by $f(x) = \frac{1}{\sin x} - \frac{1}{x}$,

A does not have a limit at 0, because the one-sided limits at 0 are different;

B is continuous at 0;

C has limit at 0 equal to 0;

D has finite limit at 0.

6. (6 points) Let $a = \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\sin^2 x} \mathrm{d}x$. Then:

A $a = 1$; B $a \in (0,1)$; C $a = \frac{1}{2}$; D $a = \ln 2$.

7. (6 points) Let $a,\ b \in \mathbb{R}$ with $a \neq 0$, and $f\colon \mathbb{R} \to \mathbb{R}$ the function defined by $f(x) = |a+bx|$. Then:

A the function f is not differentiable at 0;

B the function f is differentiable at 0 $\Leftrightarrow a > 0$;

C the function f is differentiable at 0 and $f'(0) \in \{-b, b\}$;

D if $a > 0$, then the function f is differentiable at 0 and $f'(0) = b$.

8. (6 points) Let $f\colon D \to \mathbb{R}$ be the function defined by $f(x) = x + \sqrt{x^2 + 2x}$, where D is the maximal domain of f. The function f

- [A] has a slant asymptote at $+\infty$;
- [B] does not have vertical asymptotes;
- [C] has a horizontal asymptote at $-\infty$;
- [D] has the line $x = 0$ as a vertical asymptote.

9. (6 points) Consider the points $A(0, -1)$ and $B(-2, 1)$. The equation of a line d located at a distance of $5\sqrt{2}$ units from the mediator of the segment $[AB]$, is

[A] $d : y = x - 9$; [B] $d : y = x + 1 + 5\sqrt{2}$; [C] $d : y = x + 1 - 5\sqrt{2}$; [D] $d : y = x + 11$.

10. (6 points) Consider the parallelogram $ABCD$ and the points $M \in AB$ and $N \in AC$ such that $\overrightarrow{AM} = \dfrac{1}{x}\overrightarrow{AB}$ and $\overrightarrow{AN} = \dfrac{1}{y}\overrightarrow{AC}$, where $x, y \in \mathbb{R}^*$. The points D, N, M are collinear if there is the following relationship between the numbers x and y:

[A] $x = 1 - y$; [B] $x = y - 1$; [C] $x = y - \frac{2}{3}$; [D] $x = 2y$.

PART B

IMPORTANT: For problems in Part B complete solutions are required.

1. (10 points) Let $\varepsilon \in \mathbb{C} \setminus \mathbb{R}$ be such that $\varepsilon^3 = 1$ and $A = \begin{pmatrix} 1 & 0 \\ \varepsilon & \varepsilon^2 \end{pmatrix} \in M_2(\mathbb{C})$.

(a) Show that $\varepsilon^2 + \varepsilon + 1 = 0$.
(b) Show that the set $G = \{A, A^2, I_2\}$ is a group with respect to matrix multiplication.
(c) Compute $S = I_2 + A + A^2 + \cdots + A^n$ for every $n \in \mathbb{N}^*$.

2. (10 points) Prove that for every $x \in \mathbb{R}$ the inequality $xe^{-x^2} \leq \frac{1}{\sqrt{2e}}$ holds.

3. (10 points) Let $a \in (0, \frac{\pi}{4})$ be such that $\sin a + \cos a = \frac{\sqrt{7}}{2}$. Compute $\operatorname{tg} \frac{a}{2}$.

NOTE: All subjects are compulsory. 10 points are given by default. The work time is 3.5 hours.

Answers and solutions

PART A

1. $\boxed{\text{B}}$, $\boxed{\text{D}}$; **2.** $\boxed{\text{A}}$, $\boxed{\text{B}}$; **3.** $\boxed{\text{C}}$; **4.** $\boxed{\text{B}}$; **5.** $\boxed{\text{C}}$, $\boxed{\text{D}}$;

6. $\boxed{\text{B}}$, $\boxed{\text{D}}$; **7.** $\boxed{\text{C}}$, $\boxed{\text{D}}$; **8.** $\boxed{\text{A}}$, $\boxed{\text{B}}$, $\boxed{\text{C}}$; **9.** $\boxed{\text{A}}$, $\boxed{\text{D}}$; **10.** $\boxed{\text{B}}$.

PART B

1. (a) The condition $\varepsilon^3 = 1$ implies $0 = \varepsilon^3 - 1 = (\varepsilon - 1)(\varepsilon^2 + \varepsilon + 1)$. Using the fact that $\varepsilon \notin \mathbb{R}$, we obtain the required equality.

(b) Given $\varepsilon^3 = 1$ and $\varepsilon^2 + \varepsilon + 1 = 0$, we can easily compute

$$A^2 = \begin{pmatrix} 1 & 0 \\ \varepsilon + 1 & \varepsilon \end{pmatrix} \quad \text{and} \quad A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

hence $A^{-1} = A^2$ and $(G, \cdot)$ is a group.

(c) If $n = 3k$, then $S = k(A^2 + A + I_2) + I_2 = \begin{pmatrix} 3k+1 & 0 \\ k + 2k\varepsilon & 1 \end{pmatrix}$.

If $n = 3k+1$, then $S = k(A^2 + A + I_2) + A + I_2 = \begin{pmatrix} 3k+2 & 0 \\ k + (2k+1)\varepsilon & \varepsilon^2 + 1 \end{pmatrix}$.

If $n = 3k+2$, then $S = (k+1)(A^2 + A + I_2) = \begin{pmatrix} 3k+3 & 0 \\ k + 1 + (2k+2)\varepsilon & 0 \end{pmatrix}$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = xe^{-x^2}$. We have $f'(x) = (1 - 2x^2)e^{-x^2}$, for every $x \in \mathbb{R}$. It follows that the equation $f'(x) = 0$ has the solutions $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$, while

$$f'(x) > 0 \Leftrightarrow x \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and}$$

$$f'(x) < 0 \Leftrightarrow x \in \left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, \infty\right).$$

Hence the function f is strictly decreasing on the interval $\left(-\infty, -\frac{\sqrt{2}}{2}\right]$, strictly increasing on the interval $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ and strictly decreasing on the interval $\left[\frac{\sqrt{2}}{2}, \infty\right)$. As

$$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0,$$

we finally deduce that the point $\frac{\sqrt{2}}{2}$ is a global maximum point of the function f. Therefore $f(x) \le f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{\sqrt{2e}}$, for every $x \in \mathbb{R}$.

3. Using the formulas

$$\sin a = \frac{2\text{tg}\frac{a}{2}}{1 + \text{tg}^2\frac{a}{2}}, \quad \cos a = \frac{1 - \text{tg}^2\frac{a}{2}}{1 + \text{tg}^2\frac{a}{2}},$$

and the notation $x = \text{tg}\frac{a}{2}$ respectively, we obtain the equation

$$-(2 + \sqrt{7})x^2 + 4x + 2 - \sqrt{7} = 0,$$

whence $x_1 = \frac{3}{2+\sqrt{7}} = -2 + \sqrt{7}$ and $x_2 = \frac{1}{2+\sqrt{7}} = \frac{-2+\sqrt{7}}{3}$.

Since $\frac{a}{2} \in \left(0, \frac{\pi}{8}\right)$ and the function tg is increasing, the obtained solutions must be less than $\operatorname{tg}\frac{\pi}{8}$. For instance, this value can be computed by using the formula $\operatorname{tg} a = \dfrac{2\operatorname{tg}\frac{a}{2}}{1 - \operatorname{tg}^2\frac{a}{2}}$. Thus $1 = \operatorname{tg}\frac{\pi}{4} = \dfrac{2\operatorname{tg}\frac{\pi}{8}}{1 - \operatorname{tg}^2\frac{\pi}{8}}$. Using the fact that $\operatorname{tg}\frac{\pi}{8} > 0$, we obtain $\operatorname{tg}\frac{\pi}{8} = \sqrt{2} - 1$. After verification we can observe that $x_1 > \sqrt{2} - 1$ and $x_2 < \sqrt{2} - 1$, hence the only solution is $\operatorname{tg}\frac{a}{2} = x_2 = \frac{-2+\sqrt{7}}{3}$.