BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MODEL

Admission Exam 2019
Written test of MATHEMATICS

PART A

IMPORTANT: Problems in Part A have one or more correct answers.

1. (6 points) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be the sequences defined by

$$a_n = \lim_{x\to 0}\left(1 - x\operatorname{arctg}(nx)\right)^{\frac{1}{x^2}} \quad \text{and} \quad b_n = a_1 + \cdots + a_n,$$

respectively. Then:

[A] $a_n = e^{-n}$ for every $n \geq 1$;
[B] the sequence $(b_n)_{n\geq 1}$ is monotone;
[C] the sequence $(b_n)_{n\geq 1}$ is unbounded;
[D] $\lim\limits_{n\to\infty} b_n \in (0,1)$.

2. (6 points) The value of the limit $\lim\limits_{x\to 0}\dfrac{x - \sin x\cos x}{x^3}$ is

[A] $\dfrac{1}{6}$; [B] it does not exist; [C] $\dfrac{2}{3}$; [D] $-\dfrac{2}{3}$.

3. (6 points) Let $f:(0,\infty)\to\mathbb{R}$ be the function defined by $f(x) = \dfrac{\ln x}{x}$. Then:

[A] the graph of f has exactly two asymptotes;
[B] f has exactly two local extrema;
[C] the graph of f has a unique inflection point;
[D] f is strictly increasing and concave on $(0,e]$.

4. (6 points) The value of the integral $\displaystyle\int_0^{\pi/2} x\cos^2 x\,\mathrm{d}x$ is

[A] $\dfrac{\pi^2-4}{16}$; [B] $\dfrac{1}{16}$; [C] $\dfrac{\pi^2+4}{16}$; [D] $\dfrac{1}{4}$.

5. (6 points) In the triangle ABC consider the vertices $A(1,2)$, $B(5,-1)$ and the center of gravity $G(-1,-1)$. Then:

[A] the distance from the point C to AB is $\frac{54}{5}$;
[B] the point C belongs to the line $d: x+2y+17=0$;
[C] the area of the triangle ABC is $\frac{29}{6}$;
[D] $C(-9,-4)$.

6. (6 points) Let $\vec{u} = (3a+1)\vec{i} + (a+1)\vec{j}$ and $\vec{v} = (a-3)\vec{i} - a\vec{j}$, where $a \in \mathbb{R}$, and $\vec{i}$ and $\vec{j}$ are non-collinear vectors. The value of the parameter $a \in \mathbb{R}$ for which the vectors $\vec{u}$ and $\vec{v}$ are collinear is

A -2; B $-\frac{3}{4}$; C 1; D it does not exist such an a.

7. (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = m^2x^2 - 2(x+1)^2$, where $m \in \mathbb{R}$. The function f is injective if:

A $m = -\sqrt{2}$; B $m = -1$; C $m = 1$; D $m = \sqrt{2}$.

8. (6 puncte) The number of real solutions of the equation $\log_2 |x| + \log_{|x|} 4 = 3$ is

A 1; B 2; C 3; D 4.

9. (6 points) Let $p \in \mathbb{R}$ and the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ p & 1 & -1 \\ p^2 & 1 & 1 \end{pmatrix} \in M_2(\mathbb{R})$. Then:

A $\text{rank}(A) = 3$ for every $p \in \mathbb{R}$;

B $\text{rank}(A) = 2$ for every $p \in \mathbb{R}$;

C there exists a unique number $p \in \mathbb{R}$ for which $\text{rank}(A) = 2$;

D there exist two distinct numbers $p \in \mathbb{R}$ for which $\text{rank}(A) = 2$.

10. (6 points) Let $A = \{a - ia \mid a \in \mathbb{R}\}$ and denote by "+" and "·" the usual operations of addition and multiplication in $\mathbb{C}$. Then:

A A is a subgroup of the group $(\mathbb{C}, +)$;

B A is a subgroup of the group $(\mathbb{C}^*, \cdot)$;

C $(A, +, \cdot)$ is a ring;

D $(A, +, \cdot)$ is a field.

PART B

IMPORTANT: For problems in Part B complete solutions are required.

1. (10 points) Draw the graph of the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x & \text{if } x \le 1 \\ \frac{x^2}{x-1} & \text{if } x > 1. \end{cases}$$

2. (10 points) (a) Solve the equation $\cos 5x + \sin \frac{5x}{2} = 1$ on the set $[0, \pi]$.
(b) Determine the cardinal of the set $A = \left\{\sin \frac{2k\pi}{5} | k \in \mathbb{Z}\right\}$.

3. (10 points) Consider the linear system of equations with real coefficients:

$$\begin{cases} x + y - z = 1 \\ x + ay - z = 1 \\ x + y + az = a \end{cases}.$$

Determine the values $a \in \mathbb{R}$ for which the system is compatible determinate, and in this case solve it.

NOTE: All subjects are compulsory. 10 points are given by default. The work time is 3 hours.

Answers and solutions

PART A

1. $\boxed{\text{A}}$, $\boxed{\text{B}}$, $\boxed{\text{D}}$; **2.** $\boxed{\text{C}}$; **3.** $\boxed{\text{A}}$, $\boxed{\text{C}}$, $\boxed{\text{D}}$; **4.** $\boxed{\text{A}}$; **5.** $\boxed{\text{A}}$, $\boxed{\text{B}}$, $\boxed{\text{D}}$;

6. $\boxed{\text{B}}$, $\boxed{\text{C}}$; **7.** $\boxed{\text{B}}$, $\boxed{\text{C}}$; **8.** $\boxed{\text{D}}$; **9.** $\boxed{\text{D}}$; **10.** $\boxed{\text{A}}$.

PART B

1. The graph of the restriction of f to $(-\infty, 1]$ is a closed semiline, situated on the first bisector.

In what follows we study the variation of the restriction of f to the interval $(1, \infty)$.

$\diamond \lim\limits_{x \searrow 1} f(x) = +\infty \quad \Rightarrow \quad$ the line $x = 1$ is a vertical asymptote for G_f;

$\diamond \lim\limits_{x\to\infty} f(x) = +\infty \quad \Rightarrow \quad G_f$ does not have a horizontal asymptote to $+\infty$;

$\diamond \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = 1$ and $\lim\limits_{x\to\infty} \big(f(x) - x\big) = 1 \quad \Rightarrow \quad$ the line $y = x + 1$ is a slant asymptote to $+\infty$ for G_f;

$\diamond f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ and $f''(x) = \dfrac{2}{(x-1)^3}$ for every $x \in (1, \infty)$.

The table with the variation of f is presented below.

x	1			2			$+\infty$
$f'(x)$	|	-	-	0	$+$	$+$	$+$
$f''(x)$	|	$+$	$+$	$+$	$+$	$+$	$+$
$f(x)$	|	$\searrow$	$\searrow$	4	$\nearrow$	$\nearrow$	$\nearrow$

The graph of f is presented in Figure 1.

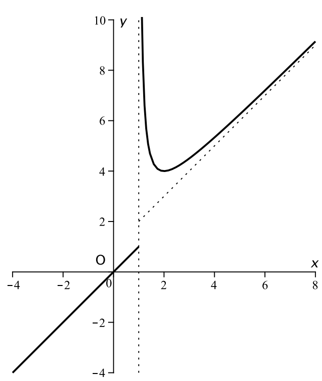

Figure 1: The graph of the function f.

2. (a) We have $\cos 5x + \sin \frac{5x}{2} = 1 \Leftrightarrow 1 - 2\sin^2 \frac{5x}{2} + \sin \frac{5x}{2} = 1 \Leftrightarrow \sin \frac{5x}{2} \left(-2\sin \frac{5x}{2} + 1\right) = 0$.
Hence $\sin \frac{5x}{2} = 0$, that is,

$$x \in \left\{ \frac{2k\pi}{5} \Big| k \in \mathbb{Z} \right\} \cap [0, \pi] = \left\{ 0, \frac{2\pi}{5}, \frac{4\pi}{5} \right\}$$

or $\sin \frac{5x}{2} = \frac{1}{2}$, that is, $\frac{5x}{2} \in \left\{ \frac{\pi}{6} + 2k\pi | k \in \mathbb{Z} \right\} \bigcup \left\{ \frac{5\pi}{6} + 2k\pi | k \in \mathbb{Z} \right\}$, and so

$$x \in \left(\left\{ \frac{\pi}{15} + \frac{4k\pi}{5} \Big| k \in \mathbb{Z} \right\} \bigcup \left\{ \frac{\pi}{3} + \frac{4k\pi}{5} \Big| k \in \mathbb{Z} \right\} \right) \cap [0, \pi] = \left\{ \frac{\pi}{15}, \frac{13\pi}{15}, \frac{\pi}{3} \right\}.$$

Thus the solution of the problem is the set $\left\{0, \frac{\pi}{15}, \frac{\pi}{3}, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{13\pi}{15}\right\}$.
(b) Every $k \in \mathbb{Z}$ may be written as $5l, 5l+1, 5l+2, 5l+3$ or $5l+4$, where $l \in \mathbb{Z}$. Thus

$$\left\{\frac{2k\pi}{5}\middle| k \in \mathbb{Z}\right\} = \left\{2\pi l, 2\pi l + \frac{2\pi}{5}, 2\pi l + \frac{4\pi}{5}, 2\pi l + \frac{6\pi}{5}, 2\pi l + \frac{8\pi}{5}\middle| l \in \mathbb{Z}\right\}.$$

We know that $2\pi l$ ($l \in \mathbb{Z}$) is a period of the function $\sin : \mathbb{R} \to [-1, 1]$, hence

$$A = \left\{\sin 0,\ \sin\frac{2\pi}{5},\ \sin\frac{4\pi}{5},\ \sin\frac{6\pi}{5},\ \sin\frac{8\pi}{5}\right\}.$$

We still need to check if there are equal elements in this set. In order to do that we bring them in the first quadrant and we obtain

$$A = \left\{\sin 0, \sin\frac{2\pi}{5}, \sin\frac{4\pi}{5} = -\sin\frac{\pi}{5}, \sin\frac{6\pi}{5} = \sin\frac{\pi}{5}, \sin\frac{8\pi}{5} = -\sin\frac{2\pi}{5}\right\}.$$

Hence all elements are distinct, and so the cardinal of the set is 5.

3. The determinant of the system is $\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & a & -1 \\ 1 & 1 & a \end{vmatrix} = (a-1)(a+1)$.
The system is compatible determinate $\Leftrightarrow \Delta \neq 0 \Leftrightarrow a \notin \{-1, 1\}$. In this case we have:

$$\Delta_1 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & a & -1 \\ a & 1 & a \end{vmatrix} = 2a(a-1), \quad \Delta_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & a & a \end{vmatrix} = 0, \quad \Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = (a-1)^2.$$

We obtain the unique solution: $x = \frac{\Delta_1}{\Delta} = \frac{2a}{a+1}, \quad y = \frac{\Delta_2}{\Delta} = 0, \quad z = \frac{\Delta_3}{\Delta} = \frac{a-1}{a+1}$.