BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

MODEL

MATE-INFO UBB COMPETITION and ADMISSIONS EXAM, 2018
Written test in MATHEMATICS

PART A

IMPORTANT: For each problem in Part A at least one answer is correct, and several answers may be correct.

1. (5 points) Consider the points $A(2m^2, m+3)$, $B(-1,1)$ and $C(3,5)$ in the xOy plane. The value $m \in \mathbb{R}$ for which the points A, B and C are collinear is:

$\boxed{\text{A}}$ -1; $\boxed{\text{B}}$ 1; $\boxed{\text{C}}$ $-\frac{1}{2}$; $\boxed{\text{D}}$ $\frac{1}{2}$; $\boxed{\text{E}}$ 0.

2. (5 points) A solution in the interval $[0, 2\pi]$ of equation $\sqrt{3}\sin x + \cos x = 0$ is:

$\boxed{\text{A}}$ $\frac{5\pi}{4}$; $\boxed{\text{B}}$ $\frac{5\pi}{6}$; $\boxed{\text{C}}$ $\frac{\pi}{3}$; $\boxed{\text{D}}$ $\frac{11\pi}{6}$; $\boxed{\text{E}}$ $\frac{\pi}{4}$.

3. (5 points) Let X be a set with 3 elements and Y a set with 2 elements. The number of functions $f : X \to Y$ is:

$\boxed{\text{A}}$ C_3^2; $\boxed{\text{B}}$ A_3^2 $\boxed{\text{C}}$ 0; $\boxed{\text{D}}$ 8; $\boxed{\text{E}}$ 9.

4. (5 points) Let $x_1, x_2 \in \mathbb{C}$ be the solutions of equation $x^2 - 2x + 3 = 0$. The value of $x_1^2 + x_2^2$ is:

$\boxed{\text{A}}$ 3; $\boxed{\text{B}}$ $\frac{2}{3}$; $\boxed{\text{C}}$ $\frac{3}{2}$; $\boxed{\text{D}}$ 1; $\boxed{\text{E}}$ -2.

5. (5 points) Let $a \in \mathbb{R}^*$ and the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, given by $f(x) = \dfrac{\sqrt{x^2+a^2}}{x-1}$. Decide which of the following are true:

$\boxed{\text{A}}$ the graph of f has 2 asymptotes; $\boxed{\text{B}}$ the graph of f has 3 asymptotes;

$\boxed{\text{C}}$ on $(1, \infty)$, f is decreasing; $\boxed{\text{D}}$ on $(1, \infty)$, f is increasing;

$\boxed{\text{E}}$ on $(0, \infty) \setminus \{1\}$, f is monotone.

6. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 e^x$. The second order derivative of the function f is:

$\boxed{\text{A}}$ $f''(x) = (x^2 + 2x + 2)e^x$; $\boxed{\text{B}}$ $f''(x) = (x^2 + 4x + 2)e^x$; $\boxed{\text{C}}$ $f''(x) = 2e^x$;

$\boxed{\text{D}}$ $f''(x) = (2x^2 + x + 2)e^x$; $\boxed{\text{E}}$ $f''(x) = 2(x^2 + x + 1)e^x$.

PART B

1. (10 points) Consider the triangle $\triangle ABC$ and the points $N \in (AB)$, $P \in (BC)$ and $Q \in (AC)$ such that $AN = 2BN$, $BP = 2CP$ and $CQ = 2AQ$. Express the vectors $\overrightarrow{NP}$, $\overrightarrow{PQ}$ and $\overrightarrow{QN}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

2. a) (10 points) Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. Compute $A + A^2 + \cdots + A^{2018}$.

b) (15 points) On $\mathbb{R}$ let „$*$" be the operation defined by $x * y = xy + ax + ay + b$. Find $a, b \in \mathbb{Z}$, $|a| \le 2$ and $|b| \le 2$, such that $(\mathbb{R}, *)$ is a monoid.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \min\{x, x^2, x^3\}$.

a) (15 points) Draw the graph of the function f.

b) (10 points) Compute $\int_{1/2}^{2} f(x)dx$.

NOTE:

All subjects are mandatory. Grading starts at 10 points.
Working time is 3 hours.

Răspunsuri şi soluţii

PARTEA A

1. $\boxed{\text{B}}$, $\boxed{\text{C}}$; **2.** $\boxed{\text{B}}$, $\boxed{\text{D}}$; **3.** $\boxed{\text{D}}$; **4.** $\boxed{\text{E}}$; **5.** $\boxed{\text{B}}$, $\boxed{\text{C}}$; **6.** $\boxed{\text{B}}$.

PARTEA B

1. Avem $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$. Rezultă că:

$$\overrightarrow{NP} = \overrightarrow{NB} + \overrightarrow{BP} = \frac{1}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AB} + \frac{2}{3}(\overrightarrow{AC} - \overrightarrow{AB}) = -\frac{1}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{AC},$$
$$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CQ} = \frac{1}{3}\overrightarrow{BC} - \frac{2}{3}\overrightarrow{AC} = \frac{1}{3}(\overrightarrow{AC} - \overrightarrow{AB}) - \frac{2}{3}\overrightarrow{AC} = -\frac{1}{3}\overrightarrow{AB} - \frac{1}{3}\overrightarrow{AC},$$
$$\overrightarrow{QN} = \overrightarrow{AN} + \overrightarrow{QA} = \frac{2}{3}\overrightarrow{AB} - \frac{1}{3}\overrightarrow{AC}.$$

Având exprimaţi 2 dintre cei 3 vectori în funcţie de $\overrightarrow{AB}$ şi $\overrightarrow{AC}$, al treilea se poate determina şi din egalitatea $\overrightarrow{NP} + \overrightarrow{PQ} + \overrightarrow{QN} = \overrightarrow{0}$.

2. a) Avem $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$, $A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -A$, $A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$, deci $A + A^2 + A^3 + A^4 = O_2$. Inductiv, $A^{4k+1} + A^{4k+2} + A^{4k+3} + A^{4k+4} = O_2$ pentru orice $k \in \mathbb{N}$. Rezultă că $A + A^2 + \cdots + A^{2018} = (A + A^2 + \cdots + A^{2016}) + A^{2017} + A^{2018} = A^{2017} + A^{2018} = A + A^2 = A - I_2 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$.

b) $(\mathbb{R}, *)$ este monoid dacă şi numai dacă legea de compoziţie „$*$” este asociativă şi există element neutru relativ la „$*$”. Pentru ca „$*$” să fie asociativă trebuie să avem $(x*y)*z = x*(y*z)$ pentru orice $x, y, z \in \mathbb{R}$. Folosind definiţia legii de compoziţie „$*$”, condiţia are loc dacă şi numai dacă $(a+b)z + a^2x = (a+b)x + a^2z$ pentru orice $x, z \in \mathbb{R}$ dacă şi numai dacă $(z-x)(a^2-a-b) = 0$ pentru orice $x, z \in \mathbb{R}$. Astfel avem $a^2 - a - b = 0$. Deoarece căutăm $a, b \in \mathbb{Z}$ cu $|a| \leq 2$ şi $|b| \leq 2$, rezultă că $(a, b) \in \{(-1, 2), (0, 0), (1, 0), (2, 2)\}$.

Presupunem că există element neutru $e \in \mathbb{R}$ relativ la „$*$”, deci $e * x = x$ pentru orice $x \in \mathbb{R}$ (legea este evident comutativă). Condiţia are loc dacă şi numai dacă $(e + a - 1)x + ae + b = 0$ pentru orice $x \in \mathbb{R}$. Astfel avem $e + a - 1 = 0$ şi $ae + b = 0$, de unde $e = 1 - a$ şi $a^2 - a - b = 0$. Rezultă că valorile $a, b \in \mathbb{Z}$ cu $|a| \leq 2$ şi $|b| \leq 2$ pentru care există element neutru sunt aceleaşi cu cele pentru care legea este asociativă.

În concluzie, avem $(a, b) \in \{(-1, 2), (0, 0), (1, 0), (2, 2)\}$.

3. a) Numerele $-1, 0, 1$ sunt rădăcini pentru cel puţin una dintre ecuaţiile $x = x^2$, $x = x^3$, $x^2 = x^3$. Comparând cantităţile x, x^2, x^3 succesiv pe intervalele $(-\infty, -1]$, $(-1, 0]$, $(0, 1]$, $(1, \infty)$, obţinem:

$$f(x) = \begin{cases} x^3, & \text{dacă } x \in (-\infty, -1] \cup [0, 1], \\ x, & \text{dacă } x \in (-1, 0) \cup (1, \infty). \end{cases}$$

Trasarea graficului lui f se bazează pe trasarea graficelor funcţiilor $x \mapsto x$ şi $x \mapsto x^3$.

b) Avem:

$$\int_{1/2}^{2} f(x)dx = \int_{1/2}^{1} x^3\,dx + \int_{1}^{2} x\,dx = \frac{111}{64}.$$