

Planar Analytic Geometry

For Mathematics and Computer Science in English

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These notes represent an English translation of the analogous material originally written in 2018 by Lect. dr. Daniel Văcărețu in Romanian. Any typos or errors should be imputed to the translator and should be signaled at george.turcas.ubb@gmail.com.

Chapter 1

The various equations of the line

1.1 The line defined by a point and a director vector

Suppose $\mathcal{R}\{O; \vec{i}, \vec{j}\}$ is an orthogonal Cartesian system in the plane. Let d be having the non-zero vector \vec{d} of components $(p, q) \in \mathbb{R}^2$ as director vector. On the line d we consider a fixed point M_0 of coordinates (x_0, y_0) .

The vector form of the equation of d is

$$d : \vec{r}_M = \vec{r}_{M_0} + \lambda \cdot \vec{d}, \quad (1.1)$$

where $\lambda \in \mathbb{R}$ and \vec{r}_M, \vec{r}_{M_0} represent the position vectors of the points M and M_0 , respectively.

The equation above should be understood as follows. A point M in the plane belongs to the line d if and only if there exists $\lambda \in \mathbb{R}$ such that the position vector \vec{r}_M satisfies (1.1).

Proposition 1.1.1. *If the components p, q of \vec{d} are both non-zero, the Cartesian equation of the line d with respect to the system $\mathcal{R}\{O; \vec{i}, \vec{j}\}$ is*

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}. \quad (1.2)$$

Proof. Suppose M is an arbitrary point of coordinates (x, y) in the plane. The vector equation of the line d , namely $\vec{r}_M = \vec{r}_{M_0} + \lambda \vec{d}$ can be rewritten as

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + \lambda(p\vec{i} + q\vec{j}).$$

As the vectors \vec{i} and \vec{j} are linearly independent, the previous equality is equivalent to

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \end{cases} \quad (1.3)$$

We recognise (1.3) as the parametric equations of the line d . Under the given hypothesis, equating λ from the two relations we obtain the desired conclusion

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}.$$

□

Remark. If one of p or q is equal to zero, then we cannot divide by it. Suppose that $p = 0$. From the parametric equations (1.3) we deduce that $x = x_0$ and y can be anything (by varying $\lambda \in \mathbb{R}$, the variable y can attain every real value). The equation of the line is in this case $x = x_0$ and one should notice that this is a line parallel to the Oy axis. Similarly, if $q = 0$, then we obtain a line parallel to the Ox axis which has equation $y = y_0$.

Remark. If $p \neq 0$, i.e. if the line d is not parallel to Oy , then the equation of the line d can be written as $y - y_0 = \frac{q}{p}(x - x_0)$. Writing $m := \frac{q}{p}$, the equation of the line d takes the form

$$y - y_0 = m(x - x_0), \quad (1.4)$$

where m is called the slope (or the angular coefficient) of d . Note that $m = \tan \alpha$, where α represents the angle between the Ox axis and the line d .

1.2 The line defined by two distinct points

Let d be a line in the plane. We will describe its equation relative to the Cartesian system \mathcal{R} introduced in the previous section.

Suppose M_1 and M_2 are two fixed distinct points on the line d , such that their coordinates relative to \mathcal{R} are (x_1, y_1) and (x_2, y_2) , respectively. Notice that $\overrightarrow{M_1 M_2} = \vec{r}_{M_2} - \vec{r}_{M_1}$ can be chosen as a director vector of the line d . Using (1.1), the vector equation of d can be written as

$$\vec{r}_M = \vec{r}_{M_1} + \lambda(\vec{r}_{M_2} - \vec{r}_{M_1}).$$

If we write (x, y) for the coordinates of M , the previous equation can be rewritten as

$$x\vec{i} + y\vec{j} = x_1\vec{i} + y_1\vec{j} + \lambda(x_2\vec{i} + y_2\vec{j} - x_1\vec{i} - y_1\vec{j}),$$

which is equivalent to

$$\begin{cases} x = x_1 + \lambda(x_2 - x_1) \\ y = y_1 + \lambda(y_2 - y_1) \end{cases}.$$

If $x_1 \neq x_2$ and $y_2 \neq y_1$, then the above can be written as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad (1.5)$$

and the latter is often called *the equation of the line d determined by the points M_1 and M_2* .

This should be understood as follows: An arbitrary point $M(x, y)$ belongs to the line d determined by $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ if and only if the coordinates x, y of M satisfy the equation (1.5).

Remark. If $x_2 = x_1$ the equation of d is $x = x_1$ which represents a line parallel to Oy . Similarly, if $y_2 = y_1$ the equation of d is $y = y_1$ and d is a line parallel to the Ox axis. When $x_2 \neq x_1$, the equation of the line d can be written as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

and comparing this to (1.4), we observe that the slope of the line d is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Remark. The equation of the line d given by two fixed points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ can be written using a determinant, namely

$$d : \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \quad (1.6)$$

Showing that (1.6) and (1.5) are equivalent can be done by transforming the determinant: Keep the third line constant in the determinant, subtract it from the first two lines and express the resulting determinant after the last column. The rest is left as an easy exercise for the reader.

Remark. Three given points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The latter is often mentioned as “the coliniarity condition of the points M_1 , M_2 and M_3 ” with respect to the Cartesian system \mathcal{R} .

The equation of d given by cuts. Suppose the line d intersects the Ox axis in the point $A(a, 0)$ and the Oy axis in the point $B(0, b)$. Using (1.6) we see that the equation of the line AB is

$$AB : \begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0,$$

which is equivalent to

$$AB : -bx - ay + ab = 0.$$

If $a, b \neq 0$, the equation can be written as

$$AB : \frac{x}{a} + \frac{y}{b} = 1,$$

and the latter is commonly called “the equation of the line AB through cuts.”

1.3 The angle between two given lines

By definition, the angle between two lines can be taken as the angle between two director vectors (or π minus the angle between two director vectors).

Proposition 1.3.1. Suppose the lines d_1 and d_2 have director vectors $\vec{d}_1(p_1, q_1)$ and $\vec{d}_2(p_2, q_2)$. Then

$$\cos(\angle(d_1, d_2)) = \pm \frac{p_1 p_2 + q_1 q_2}{\sqrt{p_1^2 + q_1^2} \cdot \sqrt{p_2^2 + q_2^2}}.$$

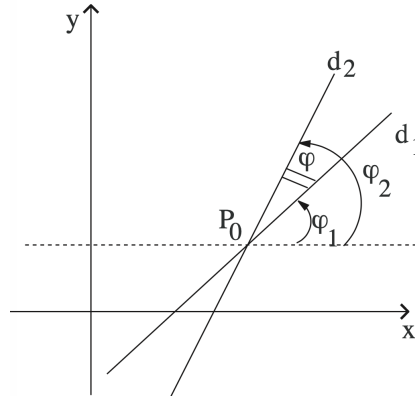
Proof. On one hand, we have $\vec{d}_1 \cdot \vec{d}_2 = \|\vec{d}_1\| \cdot \|\vec{d}_2\| \cos(\angle(\vec{d}_1, \vec{d}_2))$.

On the other hand, using the algebraic form of the dot product we can compute $\vec{d}_1 \cdot \vec{d}_2 = p_1 p_2 + q_1 q_2$. Moreover, $\|\vec{d}_1\| = \sqrt{p_1^2 + q_1^2}$, $\|\vec{d}_2\| = \sqrt{p_2^2 + q_2^2}$.

The desired conclusion follows easily. \square

Let d_1 and d_2 be two concurrent lines given by their reduced equations:

$$d_1 : y = m_1 x + n_1 \quad \text{and} \quad d_2 : y = m_2 x + n_2.$$



Recall that in this case the coefficients m_1 and m_2 represent the slopes of d_1 and d_2 respectively and we have that $m_1 = \tan \varphi_1$ and $m_2 = \tan \varphi_2$. One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$ and that $\varphi_2 \geq \varphi_1$, so we have $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

The angle φ determined by d_1 and d_2 can be found by first computing

$$\tan \varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan \varphi_2 - \tan \varphi_1}{1 + \tan \varphi_1 \tan \varphi_2},$$

hence

$$\tan \varphi = \frac{m_2 - m_1}{1 + m_1 m_2}. \quad (1.7)$$

and then applying the inverse tangent function.

Remark. The lines d_1 and d_2 are parallel if and only if $\tan \varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \quad (1.8)$$

Remark. The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \quad (1.9)$$

The parallelism and perpendicularity conditions can be easily expressed in terms of director vectors, as remarked below.

Remark. If the lines d_1 and d_2 have director vectors $\bar{d}_1(p_1, q_1)$ and $\bar{d}_2(p_2, q_2)$, then d_1 and d_2 are parallel if and only if $\bar{d}_1 \parallel \bar{d}_2$, which holds if and only if

$$\exists \lambda \in \mathbb{R}^* \text{ s.t. } \bar{d}_1 = \lambda \bar{d}_2 \iff \exists \lambda \in \mathbb{R}^* \text{ s.t. } p_1 = \lambda p_2 \text{ and } q_1 = \lambda q_2.$$

Remark. If the lines d_1 and d_2 have director vectors $\bar{d}_1(p_1, q_1)$ and $\bar{d}_2(p_2, q_2)$, then d_1 and d_2 are perpendicular if and only if $\bar{d}_1 \cdot \bar{d}_2 = 0$, which holds if and only if

$$p_1 p_2 + q_1 q_2 = 0.$$

1.4 The distance between a point and a line

Suppose $M_0(x_0, y_0)$ is a given point and d a line. The general Cartesian equation of the line d can be written as

$$d : ax + by + c = 0, \quad (1.10)$$

where a, b, c are fixed real numbers with $a^2 + b^2 > 0$.

Remark. All the different forms in which the equation of a line was presented previously can be reduced (using simple arithmetic) to such a general Cartesian equation.

Proposition 1.4.1. *The distance between the point M_0 and the line d is equal to*

$$d(M_0, d) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

Proof. It is easy to deduce that the vector $\bar{d}(-b, a)$ is a director vector of the line d . Let us write the equation of the line d' which passes through M_0 and is perpendicular to d .

We can choose $\bar{d}'(a, b)$ as director vector for d' since clearly $\bar{d}' \cdot \bar{d} = 0$, so the two vectors are perpendicular. The parametric equations of the line d' are

$$d' : \begin{cases} x = x_0 + a \cdot t \\ y = y_0 + b \cdot t \end{cases}, \text{ where } t \in \mathbb{R}. \quad (1.11)$$

To determine the intersection between d and d' we replace x and y from (1.11) in (1.10) and find the parameter t . This gives

$$a(x_0 + a \cdot t) + b(y_0 + b \cdot t) + c = 0,$$

hence

$$t = \frac{-ax_0 - by_0 - c}{a^2 + b^2},$$

which gives rise to the point $M_d \left(x_0 + a \cdot \frac{-ax_0 - by_0 - c}{a^2 + b^2}, y_0 + b \cdot \frac{-ax_0 - by_0 - c}{a^2 + b^2} \right) \in d \cap d'$.

The sought-after distance is

$$d(M_0, d) = d(M_0, M_d) = \sqrt{a^2 \cdot \left(\frac{-ax_0 - by_0 - c}{a^2 + b^2} \right)^2 + b^2 \cdot \left(\frac{-ax_0 - by_0 - c}{a^2 + b^2} \right)^2},$$

and the conclusion follows after some simple computations. \square

As an application, we derive a formula for the area of a triangle determined by three points.

Proposition 1.4.2. *Let $M_i(x_i, y_i)$, where $i \in \{1, 2, 3\}$ be three distinct points. The area of the triangle $M_1M_2M_3$ is*

$$\text{Area}[M_1M_2M_3] = \frac{1}{2} \cdot \left| \begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array} \right|.$$

Proof. The equation of the line M_2M_3 is

$$M_2M_3 : \left| \begin{array}{ccc} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array} \right| = 0$$

which is equivalent to

$$(y_2 - y_3)x - (x_2 - x_3)y + x_2y_3 - x_3y_2 = 0$$

The area of the triangle can be computed as

$$\text{Area}[M_1M_2M_3] = \frac{1}{2} |M_2M_3| \cdot d(M_1, M_2M_3)$$

and using the formula derived in the previous proposition we get

$$\text{Area}[M_1M_2M_3] = \frac{1}{2} \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} \cdot \frac{|(y_2 - y_3)x_1 - (x_2 - x_3)y_1 + x_2y_3 - x_3y_2|}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}.$$

We obtain that

$$\text{Area}[M_1M_2M_3] = \frac{1}{2} \cdot |(y_2 - y_3)x_1 - (x_2 - x_3)y_1 + x_2y_3 - x_3y_2| = \left| \frac{1}{2} \cdot \begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array} \right|,$$

as desired. □

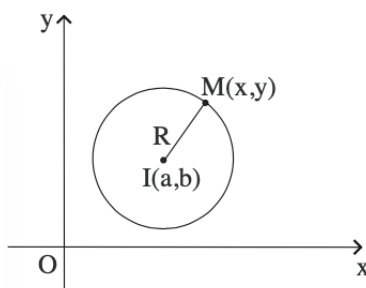
Chapter 2

Conic sections

2.1 The circle

A *circle* is a closed plane curve, defined as the geometric locus of the points at a given distance R from a point I . The point I is the *center* of the circle and the number R is the *radius* of the circle. We shall denote the circle of center I and radius R by $\mathcal{C}(I, R)$.

In order to determine the equation of the circle, suppose that xOy is an associated Cartesian system of coordinates in the plane, and $I(a, b)$. An arbitrary point $M(x, y)$ belongs to $\mathcal{C}(I, R)$ if and only if $|MI| = R$.



Hence, $\sqrt{(x - a)^2 + (y - b)^2} = R$, or

$$(x - a)^2 + (y - b)^2 = R^2. \quad (2.1)$$

The equation (2.1) represents the equation of the circle centered at $I(a, b)$ and of radius R .

Remark. In a Cartesian system of coordinates, the locus of points $M(x, y)$ satisfying the equation

$$x^2 + y^2 + mx + ny + p = 0, \text{ where } m, n, p \in \mathbb{R}$$

is either a circle, a point or the empty set.

Exercise. Complete the perfect squares in the equation to justify the remark above.

2.2 The ellipse

Definition. An *ellipse* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant sum.

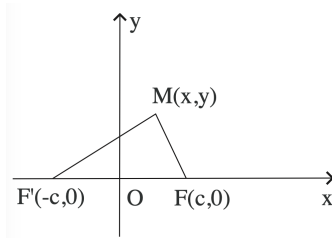
The two fixed points are called the *foci* of the ellipse and the distance between the foci is the *focal distance*.

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$

If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may choose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$ as in the diagram below.



Remark that, by triangle inequality, we have $|MF| + |MF'| > |FF'|$, hence $2a > 2c$.

Let us determine the equation of an ellipse. Starting with the definition, $|MF| + |MF'| = 2a$, or

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a.$$

This is equivalent to

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

and

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2.$$

One obtains

$$a\sqrt{(x + c)^2 + y^2} = cx + a^2,$$

which gives

$$a^2(x^2 + 2xc + c^2) + a^2y^2 = c^2x^2 + 2a^2cx + a^2,$$

or

$$(a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) = 0.$$

Denoting $a^2 - c^2 = b^2$ (possible, since $a > c$), one has

$$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$

Dividing by a^2b^2 , one obtains the equation of the ellipse

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.} \quad (2.2)$$

Remark. The equation (2.2) is equivalent to

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}; \quad x = \pm \frac{a}{b} \sqrt{b^2 - y^2},$$

which means that the ellipse is symmetric with respect to both Ox and Oy . In fact, the line FF' , determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its *center*.

In order to sketch the graph of the ellipse, remark that is it enough to represent the function

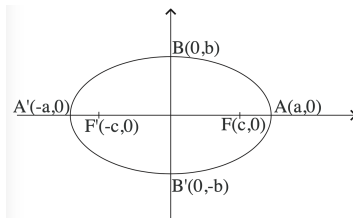
$$f : [-a, a] \rightarrow \mathbb{R}, \quad f(x) = \frac{b}{a} \sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to Ox . One has

$$f'(x) = -\frac{bx}{a\sqrt{a^2 - x^2}}, \quad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$		+	+	+	0
$f(x)$	0		\nearrow	b	\searrow
$f''(x)$		-	-	-	-

The graph of the ellipse is therefore



Remark. In particular, if $a = b$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, one obtains the equation $x^2 + y^2 - a^2 = 0$ of the circle centered at the origin and of radius a . This happens when $c = 0$, i.e. when the foci coincide, so that the circle may be seen as an ellipse whose foci are identical.

Remark. All the considerations can be done in a similar way, by taking the foci of the ellipse on Oy . One obtains a similar equation for such an ellipse.

2.3 The hyperbola

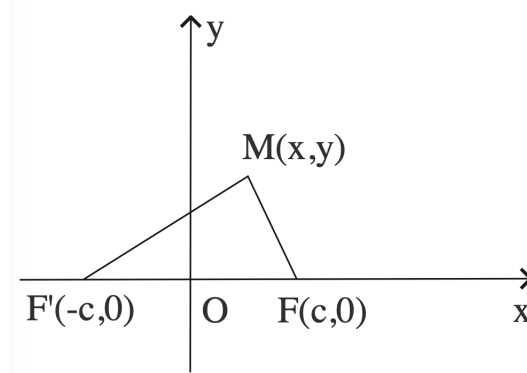
Definition. A *hyperbola* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant difference.

The two fixed points are called the *foci* of the hyperbola, and the distance between the foci is the *focal distance*.

Denote by F and F' the foci of the hyperbola and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point on the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Let us choose a Cartesian system of coordinates, having the center at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.



Note that in the triangle $\triangle MFF'$, $||MF| - |MF'|| < |FF'|$, so $a < c$.

The metric relation $|MF| - |MF'| = \pm 2a$ becomes

$$\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = \pm 2a,$$

or

$$\sqrt{(x - c)^2 + y^2} = \pm 2a + \sqrt{(x + c)^2 + y^2}.$$

This is

$$\begin{aligned} x^2 - 2cx + c^2 + y^2 &= 4a^2 \pm 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \iff \\ &\iff cx + a^2 = \pm a\sqrt{(x + c)^2 + y^2} \iff \\ &\iff c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \iff \\ &\iff (c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0. \end{aligned}$$

Denote $c^2 - a^2 = b^2$ (this is possible, since $c > a$). One then obtains the equation of the hyperbola

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0.} \quad (2.3)$$

Remark. The equation (2.3), commonly called the canonical equation of the hyperbola, is equivalent to

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}; \quad x = \pm \frac{a}{b}\sqrt{y^2 + b^2}.$$

Then, the coordinate axes are axes of symmetry for the hyperbola. Their intersection point is the *center* of the hyperbola.

To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \rightarrow \mathbb{R}, \quad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

by taking into account that the hyperbola is symmetrical with respect to Ox .

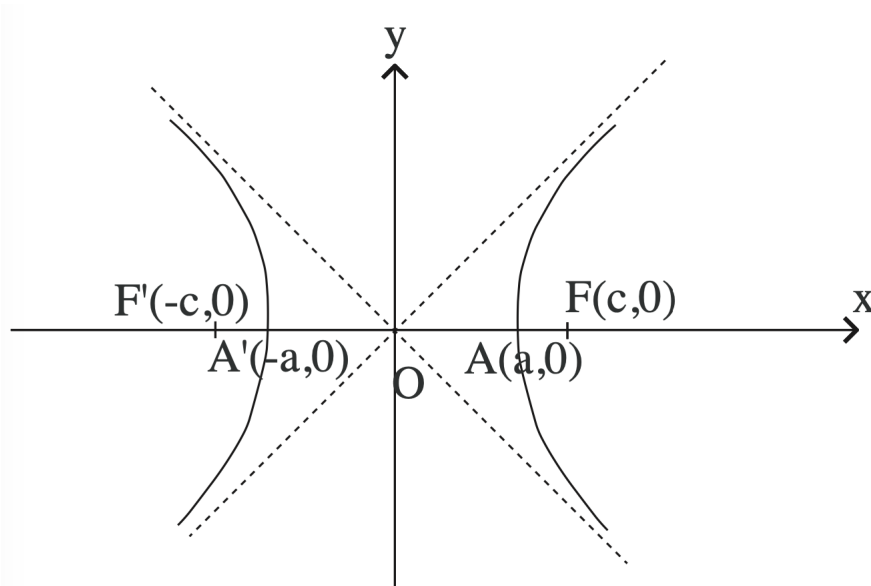
Since $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \frac{b}{a}$ and $\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = -\frac{b}{a}$, it follows that $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are asymptotes of f .

One has, also,

$$f'(x) = \frac{bx}{a\sqrt{x^2 - a^2}}, \quad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	-	-	-	/	/	/	+
$f(x)$	∞		\searrow	0	/	/	\nearrow
$f''(x)$	-	-	-	/	/	/	-

and hence the graph of the hyperbola is



Remarks. If $a = b$, the equation of the hyperbola becomes $x^2 - y^2 = a^2$. In this case, the asymptotes are the bisectors of the system of coordinates and one deals with an *equilateral* hyperbola.

As in the case of an ellipse, one can consider the hyperbola having the foci on Oy .

2.4 The parabola

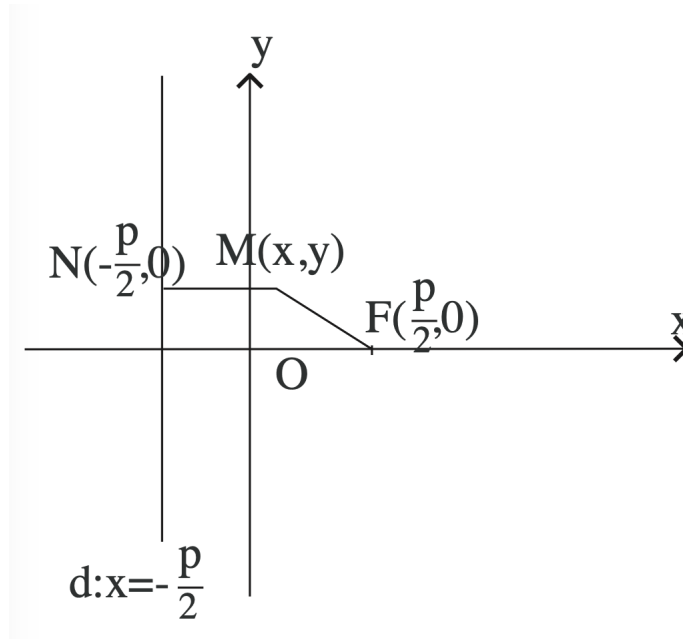
Definition. The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F .

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy , in which $F\left(\frac{p}{2}, 0\right)$ and $d: x = -\frac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on d .



Thus, the coordinates of a point of the parabola verify

$$\begin{aligned} \sqrt{\left(x + \frac{p}{2}\right)^2 + 0} &= \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} \Leftrightarrow \\ \Leftrightarrow \left(x + \frac{p}{2}\right)^2 &= \left(x - \frac{p}{2}\right)^2 + y^2 \Leftrightarrow \\ \Leftrightarrow x^2 + px + \frac{p^2}{4} &= x^2 - px + \frac{p^2}{4} + y^2, \end{aligned}$$

and the equation of the parabola is

$$\boxed{y^2 = 2px.} \tag{2.4}$$

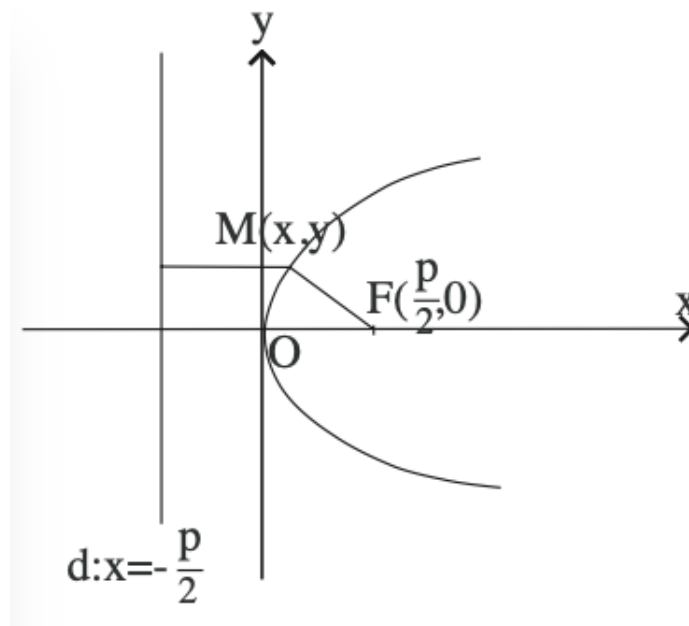
Remark. The equation (2.4), commonly called the canonical equation of the parabola, is equivalent to $y = \pm\sqrt{2px}$, so that the parabola is symmetrical with respect to Ox .

Representing the graph of the function $f : [0, \infty) \rightarrow [0, \infty)$ and using the symmetry of the curve with respect to Ox , one obtains the graph of the parabola. One has

$$f'(x) = \frac{p}{\sqrt{2px}}; \quad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0	∞
$f'(x)$		+
$f(x)$	0	\nearrow
$f''(x)$	-	-

One obtains the graph



The tangent to the parabola in a given point. Let $\mathcal{P} : y^2 = 2px$ be a parabola and $M_0(x_0, y_0)$ be a point of \mathcal{P} . Suppose that $y_0 > 0$, so that the point M_0 belongs to the graph of the function $f : [0, \infty) \rightarrow [0, \infty)$, $f(x) = \sqrt{2px}$. The slope of the tangent at M_0 to the curve is

$$f'(x_0) = \frac{p}{\sqrt{2px_0}} = \frac{p}{y_0}.$$

A similar computation leads to the angular coefficient of the tangent for $y_0 < 0$, which is still $\frac{p}{y_0}$.

The equation of the tangent at M_0 to \mathcal{P} is

$$y - y_0 = f'(x_0)(x - x_0),$$

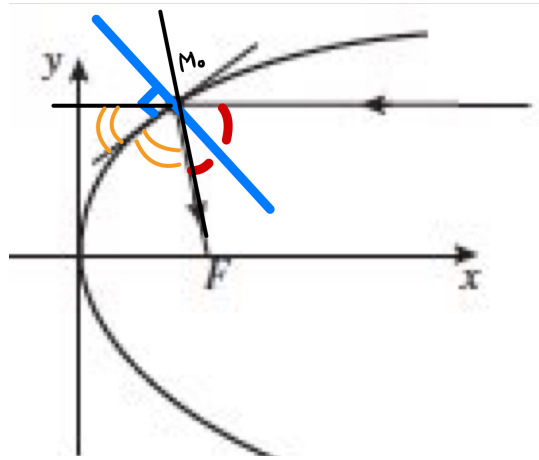
or, replacing $f'(x_0)$,

$$\begin{aligned} y - y_0 &= \frac{p}{y_0}(x - x_0) \Leftrightarrow \\ \Leftrightarrow yy_0 - y_0^2 &= p(x - x_0) \Leftrightarrow \\ yy_0 - 2px_0 &= p(x - x_0), \end{aligned}$$

hence the equation of the tangent is

$$\boxed{yy_0 = p(x + x_0)}. \quad (2.5)$$

Theorem 2.4.1 (The optical property of the parabola). *The tangent line and the normal line to the parabola in a given point $M_0(x_0, y_0)$ on the parabola are the bisectors of the angle between the focal radius M_0F and the parallel line to Ox which passes through M_0 .*



Proof. Let $B(-\frac{p}{2}, y_0)$ be the intersection point between the parallel to Ox through M_0 and the director line $d: x = -\frac{p}{2}$ of the parabola.

From the definition of the parabola, it follows immediately that the triangle $\triangle M_0BF$ is isosceles. To prove that the tangent in M_0 to the parabola is the bisector of the angle BM_0F it is sufficient to show that the tangent is the median corresponding to BF in this triangle.

The equation of the tangent in M_0 to the parabola is (see (2.5))

$$yy_0 = p(x + x_0).$$

The midpoint of BF has coordinates $(0, \frac{y_0}{2})$, and it follows immediately that this point verifies the equation of the tangent. The conclusion follows. \square

2.5 Proposed problems

1. Consider the points $A(\alpha, 0)$ and $B(0, \beta)$ in a Cartesian system xOy .
 - a) Compute the length of the segment AB and the coordinates of its midpoint.
 - b) Derive the equation of the circumscribed circle of the triangle AOB , where O is the origin of the system.
 - c) Suppose that the points $A(\alpha, 0), B(0, \beta)$ are variable meanwhile the length of the segment AB is fixed. Fix a point M on the segment $[AB]$. Determine the geometric locus described by M when $A(\alpha, 0), B(0, \beta)$ are varying. Describe the particular case in which M is the midpoint of $[AB]$.
2. Consider the parabola given by the equation $y^2 = 2px$.
 - a) Write the equation of the tangent in $M_0(x_0, y_0)$ to the parabola, when M_0 is a point on the parabola.
 - b) Consider $P(x_p, y_p)$ a point in the plane. Find the coordinates of the projection of P on the tangent line mentioned in the previous item.
 - c) Find the geometric locus of the projections from the focus F of the parabola to the tangents in the points on the parabola (the tangents are the ones varying here).
3. Consider the parabolas given by equations $y^2 = 2px$ and $y^2 = 2qx$, where $0 < q < p$. A variable tangent to the second parabola intersects the first one in the points M_1 and M_2 . Determine the geometric locus of the midpoint of the segment $[M_1M_2]$.
4. Consider a Cartesian system xOy and a point P situated on the first bisector of the coordinate axes. Let P_1 and P_2 be the orthogonal projections of P on the Ox and the Oy axes, respectively. A variable line d passes through the point P and intersects Ox in M and Oy in N respectively. Prove that for any such line d , the three lines MP_2 , NP_1 and the perpendicular on d which passes through the origin O are concurrent.
5. Let ABC be a triangle and denote by M the midpoint of the side $[BC]$. Consider N , a point on the line AB such that $A \in (BN)$. We denote by P and Q the projections of the orthocenter H of ABC on the bisectors of the angles $\angle BAC, \angle CAN$. Show that the points M, P and Q are collinear.

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