CALCULUS ON $\mathbb{R}$

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Contents

CHAPTER 1

Series of real numbers

1. Defintions and Terminology

The notion of a *s*eries of real numbers is a natural extension for a finite summation. The study of the series reduces to the study of particular sequences of real numbers, and determining the sum o a series is equivalent to computing the limit of a certain sequence.

1.1. General notions. This section contins the definitions for the following notions: series of real numbers, convergent series, divergent series and the sum of a series.

Definition 1.1.1 *Each ordered pair of two sequences of real numbers,* $((u_n),(s_n))_{n\in\mathbb{N}}$, *where* $(u_n)_{n\in\mathbb{N}}$ *is a given sequence, and*

$$s_n = u_1 + u_2 + \cdots + u_n, \text{ for all } n \in \mathbb{N}$$

is called a ***series of real numbers***. $\Diamond$

The usual notation for a series of real numbers $((u_n),(s_n))$ is

$$\sum_{n=1}^{\infty} u_n \quad \text{or} \quad \sum_{n\in\mathbb{N}} u_n \quad \text{or} \quad \sum_{n\geq 1} u_n \quad \text{or} \quad u_1 + u_2 + \ldots + u_n + \ldots$$

and, when there is no confusion, shorter

$$\sum u_n.$$

The real number u_n, $(n\in\mathbb{N})$ is called **the general term** of the series $\sum\limits_{n=1}^{\infty} u_n$, and the sequence (u_n) is called **the sequence of terms** of the series $\sum\limits_{n=1}^{\infty} u_n$. The real number s_n, $(n\in\mathbb{N})$ is called **the partial sum of degree** n of the series $\sum\limits_{n=1}^{\infty} u_n$, while (s_n) *is* **the sequence of the partial sums** of the series $\sum\limits_{n=1}^{\infty} u_n$.

Definition 1.1.2 *The series* $\sum\limits_{n=1}^{\infty} u_n = ((u_n), (s_n))$ *is said to be* ***convergent*** *if the sequence of its partial sums* (s_n)*, is convergent.*

A series is called ***divergent*** *if it is not convergent* . $\Diamond$

Definition 1.1.3

If the sequence of the partial sums (s_n) *of the series* $\sum\limits_{n=1}^{\infty} u_n = ((u_n), (s_n))$ *has the limit* $s \in \mathbb{R} \cup \{+\infty, -\infty\}$, *then this limit is caled* ***the sum*** *of the series* $\sum\limits_{n=1}^{\infty} u_n$*. Thus*

$$\sum_{n=1}^{\infty} u_n = \lim_{n\to\infty} s_n = s.$$

$\Diamond$

Example 1.1.4 The series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{1.1.1}$$

having as general term $u_n = \frac{1}{n(n+1)}$, $(n \in \mathbb{N})$ and the partial sum of degree $n \in \mathbb{N}$, equal to

$$s_n = u_1 + \cdots + u_n = \frac{1}{1 \cdot 2} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

Due to the fact that the sequence of the partial sums is convergent, the series is convergent as well, and it has the sum $\lim\limits_{n\to\infty} s_n = 1$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \ \Diamond$$

Example 1.1.5 A **geometric series** (of ratio q) has the following expression

$$\sum_{n=1}^{\infty} q^{n-1}, \tag{1.1.2}$$

where q is a fixed real number. The general term of (1.1.2) is

$$u_n = q^{n-1},$$

$(n \in \mathbb{N})$, and its partial sum of degree n is for all $n \in \mathbb{N}$

$$s_n = 1 + q + \cdots + q^{n-1} = \begin{cases} \dfrac{1-q^n}{1-q}, & \text{if } q \neq 1 \\ n, & \text{if } q = 1. \end{cases}$$

Therefore, the geometric series (1.1.2) if and only if $|q| < 1$. In this case, its sum is $1/(1-q)$. Hence

$$\sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q}.$$

If $q \geq 1$, then the geometric seris (1.1.2) is divergent and has the sum $+\infty$, so we write

$$\sum_{n=1}^{\infty} q^{n-1} = +\infty.$$

If $q \leq -1$, the geometric series (1.1.2) is divergent and does not have any sum. $\Diamond$

The study of a series reduces to two aspects:

1) Determining the nature of the series (convergent or divergnet).

2) If the series is convergent, determining the sum of the series.

In order to determine the nature of a series, there are several convergence/divergence criteria. The second aspect, with respect to the precise sum, is restricted to limited number of particul series, for which we can specify precisely the sum.

In the following we present some convergence/divergence criteria for series.

Theorem 1.1.6 (*Cauchy's general convergence criterion*) *The series* $\sum\limits_{n=1}^{\infty} u_n$ *is convergent if and only if* $\varepsilon > 0$ *există un număr natural* n_ε *cu proprietatea că oricare ar fi numerele naturale* n *şi* p *cu* $n \geq n_\varepsilon$ *avem*

$$\forall \varepsilon \in \mathbb{R}, \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon, \forall p \in \mathbb{N}, \quad |u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon.$$

Proof. Let $s_n = u_1 + \cdots + u_n$, for all $n \in \mathbb{N}$. Then the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent if and only if the sequence of the partial sums (s_n) is convergent, therefore, according to Cauchy's theorem (with respect to fundamental-Cauchy sequnces), this is furthere equaivalent to(s_n) being fundamental, meaning that

$$\forall \varepsilon > 0, \quad \exists n_\varepsilon \in \mathbb{N}, \text{ s.t. } \forall n \geq n_\varepsilon, \quad \forall p \in \mathbb{N}, \quad |s_{n+p} - s_n| < \varepsilon.$$

Due to the fact that

$$s_{n+p} - s_n = u_{n+1} + u_{n+2} + \cdots + u_{n+p}, \quad \forall n, p \in \mathbb{N},$$

we conclude the proof. $\blacksquare$

Example 1.1.7 The *harmonic seris* is stated as

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{1.1.3}$$

It is divergent, and has the sum $+\infty$.

Solution. Assume by contradiction that the harmonic series (1.1.3) is convegent. This means, according to Cauchy's general condensation criterion, (teorema 1.1.6), that for the particular $\varepsilon = 1/2 > 0$ there exists a natural number n_0, with the property that for all n and p natural number, such that $n \geq n_0$ it holds

$$\left| \frac{1}{n+1} + \cdots + \frac{1}{n+p} \right| < \frac{1}{2}.$$

By choosing $p = n = n_0 \in \mathbb{N}$, we get

$$\frac{1}{n_0+1} + \cdots + \frac{1}{n_0+n_0} < \frac{1}{2}. \tag{1.1.4}$$

Moreover, from $n_0 + k \leq n_0 + n_0$, for all $k \in \mathbb{N}$, $k \leq n_0$ we deduce that

$$\frac{1}{n_0+1} + \cdots + \frac{1}{n_0+n_0} \geq \frac{n_0}{2n_0} = \frac{1}{2}$$

hence the inequality (1.1.4) does not hold. This contradiction leads us to the conclusion that the harmonic series (1.1.3) is divergent. Due to the fact that the sequence of the partial sums (s_n) is increasing (in romanian este strict crescător), we have:

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

■

Example 1.1.8 The series

$$\sum_{n=1}^{\infty} \frac{\sin n}{2^n} \tag{1.1.5}$$

is convergent.

Solution. Let $u_n = (\sin n)/2^n$, for all $n \in \mathbb{N}$; then for each $n, p \in \mathbb{N}$ it hols

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| = \left| \frac{\sin(n+1)}{2^{n+1}} + \cdots + \frac{\sin(n+p)}{2^{n+p}} \right| \leq$$

$$\leq \frac{|\sin(n+1)|}{2^{n+1}} + \cdots + \frac{|\sin(n+p)|}{2^{n+p}} \leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p}} =$$

$$= \frac{1}{2^n}\left(1 - \frac{1}{2^p}\right) < \frac{1}{2^n}.$$

Let $\varepsilon > 0$ be randomly chosen. Due to the fact that the sequence $(1/2^n)$ has the limit 0, (by using the ε characterisation of the limi), there exists $n_\varepsilon \in \mathbb{N}$ such that $1/2^n < \varepsilon$, for all $n \geq n_\varepsilon$. Then

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \frac{1}{2^n} < \varepsilon,$$

for all $n, p \in \mathbb{N}$ with $n \geq n_\varepsilon$. As a consequnce, the series (1.1.5) is convergent. ■

Theorem 1.1.9 *If the series* $\sum\limits_{n=1}^{\infty} u_n$ *is convegent, then*

$$\lim_{n\to\infty} u_n = 0.$$

Proof. Let $\varepsilon > 0$ be randomly chosen. Then, according to Cauchy's general condensation criterion (teorema 1.1.6), there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon, \text{ for all } n, p \in \mathbb{N} \text{ with } n \geq n_\varepsilon.$$

Choose $p = 1$. Then $|u_{n+1}| < \varepsilon$, for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$, hence

$$|u_n| < \varepsilon, \text{ for all } n \in \mathbb{N}, n \geq n_\varepsilon + 1;$$

therefore

$$\lim_{n\to\infty} u_n = 0.$$

■

Remark 1.1.10 The mutual theorem for 1.1.9 (in Romanian, teorema reciprocă), is not usually true. This means that there are series $\sum\limits_{n=1}^{\infty} u_n$ having

$$\lim_{n\to\infty} u_n = 0,$$

and being divergent at the same time. For example, consider the harmonic series (1.1.3) which, as we have proved is divergent, event though

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Remark 1.1.11 The power of the previous theorem lies in the fact

$$\lim_{n\to\infty} u_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} u_n \text{ is divergent .}$$

This mean that if for a given series, it is easy to compute $\lim_{n\to\infty} u_n$, one should do it. If that limit is 0, the series should be investigated by other means, but if it is different from 0 we conclude immediately that we are dealing with a divergent series.

Theorem 1.1.12 *Let* $m \in \mathbb{N}$ *be such that* $m > 1$*. Then the sereis* $\sum\limits_{n=1}^{\infty} u_n$ *is convergent if and only if the series* $\sum\limits_{n=m}^{\infty} u_n$ *is convergent.*

Proof. We consider the sequnces of the partial sums corresponding to the two series: Let $s_n = u_1 + \cdots + u_n$, for all $n \in \mathbb{N}$ and $t_n = u_m + \cdots + u_n$, for all $n \in \mathbb{N}$ $n \geq m$. Then the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent if and only if the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent, which is equivalent to the sequence $(t_n)_{n\geq m}$ being convergent, fact further equivalent to the series $\sum\limits_{n=m}^{\infty} u_n$ being convergent. ■

Theorem 1.1.13 *If $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} v_n$ are two convergent series and a and b are real numbers, then the series $\sum\limits_{n=1}^{\infty} (au_n + bv_n)$ is convergent and has the sum*

$$a\sum_{n=1}^{\infty} u_n + b\sum_{n=1}^{\infty} v_n.$$

Proof. For each $n \in \mathbb{N}$ it holds:

$$\sum_{k=1}^{n} (au_k + bv_k) = a\left(\sum_{k=1}^{n} u_k\right) + b\left(\sum_{k=1}^{n} v_k\right),$$

so, according to the properties of the convergent sequences, we get the conclusion of the theorem. ■

Example 1.1.14 Due to the fact that the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$$

are convergent, having the sums 2 and 3/2, respectively, we deduce that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\cdot\frac{1}{2^{n-1}} - \frac{1}{3}\cdot\frac{1}{3^{n-1}}\right)$$

is convergent and has the sum $(1/2)\cdot 2 - (1/3)\cdot(3/2) = 1/2$. ◊

Definition 1.1.15 *Let $\sum\limits_{n=1}^{\infty} u_n$ be a convergent series having the sum s, n be a natural number, and $s_n = u_1 + \cdots + u_n$ be the partial sum of degree n n of the series $\sum\limits_{n=1}^{\infty} u_n$. The real number $r_n = s - s_n$ is called* ***the remainder of degree*** *n of the series $\sum\limits_{n=1}^{\infty} u_n$.* ◊

Theorem 1.1.16 *If a the series $\sum\limits_{n=1}^{\infty} u_n$ convergent, then the sequence (r_n) of the reminders of degree n, has the limit 0.*

Proof. Let $s = \sum\limits_{n=1}^{\infty} u_n$. Due to the fact the the sequence of the partial sums (s_n) of the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent as has the limit $s = \lim\limits_{n\to\infty} s_n$, and due to the fact that $r_n = s - s_n$, for all $n \in \mathbb{N}$, we conclude that (r_n) has the limit 0. ■

2. Series with positive terms

Remark 1.2.1 If $\sum\limits_{n=1}^{\infty} u_n$ is a convergent series of real numbers, then its attached sequence of partial sums (s_n) is bounded.

The mutual statement is not usually true, meaning that there are divergent series, having the sequence of the partial sums bounded. Such an example is the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}$ for which the general term of degree n for the partial sums is s_n, $(n \in \mathbb{N})$ equal to

$$s_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{dacă } n \text{ is odd.} \end{cases}$$

Obviously, the sequence (s_n) is bounded ($|s_n| \leq 1$, for all $n \in \mathbb{N}$) even though the seires $\sum\limits_{n=1}^{\infty}(-1)^{n-1}$ is divergent, due to the fact that the sequnce (şirul (s_n) does not have a limit).

Remark 1.2.2 Each series with positive terms $\sum\limits_{n=1}^{\infty} u_n$ has the property that its attached sequence of partial sums (s_n) is inscreasing; due to this, the boundedness of (s_n) is equivalent to its convergence.

This sections continues with convergence criteria for series with positive terms.

Definition 1.2.3 *A* ***series with positive terms*** *is a series* $\sum\limits_{n=1}^{\infty} u_n$ *with the property that* $u_n > 0$ *for all* $n \in \mathbb{N}$. $\lozenge$

Theorem 1.2.4 *If* $\sum\limits_{n=1}^{\infty} u_n$ *is a series with positive terms, then*

1^0 The series $\sum\limits_{n=1}^{\infty} u_n$ *has a sum, and*

$$\sum_{n=1}^{\infty} u_n = \sup\left\{\sum_{k=1}^{n} u_k : \ n \in \mathbb{N}\right\}.$$

2^0 The series $\sum\limits_{n=1}^{\infty} u_n$ *is convergent if and only if the sequence* $\left(s_n = \sum\limits_{k=1}^{n} u_k\right)$ *of the partial sums is bounded.*

Proof. For each $n \in \mathbb{N}$ we set

$$s_n := \sum_{k=1}^{n} u_k.$$

1^0 The sequence (s_n) is increasing, so, according to Weirstrass' theorem with respect to monotonic sequences, statement 1^0 is proved.

2^0 If the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent, than the sequence of the partial sumes (s_n) is convergent, and therefore bounded. The necessity is thus proved.

For the sufficiency, we know that the sequence (s_n) is bounded, (and we already know it is monotonic). Thus, it becomes convergent, consequently the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent. ■

Theorem 1.2.5 (the first comparison criterion) *If $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} v_n$ are two series with positive terms with the property that there exists $a > 0$ and $n_0 \in \mathbb{N}$ such that*

$$u_n \leq a v_n \text{ for all } n \in \mathbb{N},\ n \geq n_0, \tag{1.2.6}$$

then:

1^0 *If the series $\sum\limits_{n=1}^{\infty} v_n$ is convergent, then the seires $\sum\limits_{n=1}^{\infty} u_n$ is convergent too.*

2^0 *If the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent, then the series $\sum\limits_{n=1}^{\infty} v_n$ is divergent too.*

Proof. For each $n \in \mathbb{N}$, let $s_n = u_1 + ... + u_n$ and $t_n = v_1 + ... + v_n$; then, from (1.2.6) it holds

$$s_n \leq s_{n_0} + a\left(v_{n_0+1} + ... + v_n\right), \text{ for all } n \in \mathbb{N},\ n \geq n_0. \tag{1.2.7}$$

1^0 If the series $\sum\limits_{n=1}^{\infty} v_n$ is convergent, then the sequence (t_n) is bounded, consequently there exists a real number $M > 0$ such that $t_n \leq M$, for all $n \in \mathbb{N}$. From (1.2.7) we deduce that for all $n \in \mathbb{N}$, $n \geq n_0$ the following inequalities hold

$$s_n \leq s_{n_0} + a\left(t_n - t_{n_0}\right) \leq s_{n_0} + at_n - at_{n_0} \leq s_{n_0} + at_n \leq s_{n_0} + aM,$$

implying that the sequence (s_n) is bounded. Then, according to Theorem 1.2.4, the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 Assume that the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. If the series $\sum\limits_{n=1}^{\infty} v_n$ were convergent, then, according to statement 1^0, the series $\sum\limits_{n=1}^{\infty} u_n$ would be convergent, which contradicts the hypothesis that the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. Therefore, the series $\sum\limits_{n=1}^{\infty} v_n$ is divergent. ■

Example 1.2.6 The series $\sum\limits_{n=1}^{\infty} n^{-1/2}$ is divergent. Indeed, the inequality $\sqrt{n} \leq n$ true for all $n \in \mathbb{N}$, leads to the conclusionthat $n^{-1} \leq n^{-1/2}$, for all $n \in \mathbb{N}$. Due to the fact that the harmonic series $\sum\limits_{n=1}^{\infty} n^{-1}$ is divergent, according to Theorem 1.2.4, statement 2^0, we conclude that the series $\sum\limits_{n=1}^{\infty} n^{-1/2}$ is divergent. ♢

Theorem 1.2.7 (the second comparison criterion for series) *If* $\sum_{n=1}^{\infty} u_n$ *and* $\sum_{n=1}^{\infty} v_n$ *are series with positive terms with the property that there exists*

$$\lim_{n\to\infty} \frac{u_n}{v_n} \in [0,+\infty]\,, \tag{1.2.8}$$

then

1^0 *If*

$$\lim_{n\to\infty} \frac{u_n}{v_n} \in]0,+\infty[,$$

then the series $\sum_{n=1}^{\infty} u_n$ *şi* $\sum_{n=1}^{\infty} v_n$ *have the same nature.*

2^0 *If*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 0,$$

then:

a) If the series $\sum_{n=1}^{\infty} v_n$ *is convergent, then the series* $\sum_{n=1}^{\infty} u_n$ *is convergent too.*

b) If the series $\sum_{n=1}^{\infty} u_n$ *is divergent, then the series* $\sum_{n=1}^{\infty} v_n$ *is divergent too.*

3^0 *If*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = +\infty,$$

then:

a) If the series $\sum_{n=1}^{\infty} u_n$ *is convergent, then the series* $\sum_{n=1}^{\infty} v_n$ *is convergent too.*

b) If the series $\sum_{n=1}^{\infty} v_n$ *is divergent, then the series* $\sum_{n=1}^{\infty} u_n$ *is divergent too.*

Proof. 1^0 Let $a := \lim_{n\to\infty} (u_n/v_n) \in]0,+\infty[$; then there exists a natural number n_0 such that

$$\left|\frac{u_n}{v_n} - a\right| < \frac{a}{2}, \text{ for all } n \in \mathbb{N},\, n \geq n_0,$$

implying that

$$v_n \leq (2/a)\, u_n, \text{ for all } n \in \mathbb{N},\, n \geq n_0 \tag{1.2.9}$$

and

$$u_n \leq (3a/2)\, v_n, \text{ for all } n \in \mathbb{N},\, n \geq n_0. \tag{1.2.10}$$

If the series $\sum_{n=1}^{\infty} u_n$ is converent, then, according to the first comparison criterion (teorema 1.2.5), which can be applied because $(1.2.9)$, we get that the series $\sum_{n=1}^{\infty} v_n$ is convergent.

If the series $\sum_{n=1}^{\infty} v_n$ is convergent, then, due to the fact that (1.2.10) , according to the first comparion criterion (theorem 1.2.5) if follows that the series $\sum_{n=1}^{\infty} u_n$ is convergent.

2^0 If $\lim_{n\to\infty} (u_n/v_n) = 0$, then there exists a natural number n_0 such that $u_n/v_n < 1$, for all $n \in \mathbb{N}$, $n \geq n_0$, implying that

$$u_n \leq v_n, \text{ for all } n \in \mathbb{N}, n \geq n_0.$$

We apply then the first comparison criterion and the conclusion follows immediately.

3^0 If $\lim_{n\to\infty} (u_n/v_n) = +\infty$, then there exists a natural number n_0 such that $u_n/v_n > 1$, for all $n \in \mathbb{N}$, $n \geq n_0$, implying that

$$v_n \leq u_n, \text{ for all } n \in \mathbb{N}, n \geq n_0.$$

We apply then the first comparison criterion and the conclusion follows immediately. ■

Example 1.2.8 The series $\sum_{n=1}^{\infty} n^{-2}$ is convergent. Indeed

$$\lim_{n\to\infty} \frac{n^2}{n\,(n+1)} = 1 \in]0, +\infty[,$$

we deduce that the series $\sum_{n=1}^{\infty} n^{-2}$ and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ have the same nature. Due to the fact that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent (see example 1.1.4), we get that the series $\sum_{n=1}^{\infty} n^{-2}$ is convergent. ♢

Theorem 1.2.9 (al the third comparison crieterion) *If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are series with positive terms such that there exists $n_0 \in \mathbb{N}$ such that:*

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}, \text{ for all } n \in \mathbb{N},\ n \geq n_0, \tag{1.2.11}$$

then:

1^0 *If the series $\sum_{n=1}^{\infty} v_n$ is converent, then the series $\sum_{n=1}^{\infty} u_n$ is converent too.*

2^0 *If the series $\sum_{n=1}^{\infty} u_n$ is divergent, then the series $\sum_{n=1}^{\infty} v_n$ is divergent too.*

Proof. Let $n \in \mathbb{N}$, $n \geq n_0 + 1$; then from (1.2.11) we obtain succesively:

$$\frac{u_{n_0+1}}{u_{n_0}} \leq \frac{v_{n_0+1}}{v_{n_0}}$$

$$\cdot \quad \cdot \quad \cdot$$

$$\frac{u_n}{u_{n-1}} \leq \frac{v_n}{v_{n-1}},$$

from where, by multiplying on both sides, we obtain

$$\frac{u_n}{u_{n_0}} \leq \frac{v_n}{v_{n_0}}.$$

Thus

$$u_n \leq \frac{u_{n_0}}{v_{n_0}} v_n, \text{ for all } n \in \mathbb{N}, n \geq n_0.$$

By applying the first comparison criterion (teorema 1.2.5). the theorem is porved. ■

Theorem 1.2.10 (Cauchy's condensation criterion) *Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms with the property that the sequence of the terms of the series, (u_n), is decreasing. Then the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} 2^n u_{2^n}$ have the same nature.*

Proof. Let $s_n := u_1 + u_2 + ... + u_n$ be the partial sum of degree $n \in \mathbb{N}$ of the series $\sum_{n=1}^{\infty} u_n$ and let $S_n := 2u_2 + 2^2 u_{2^2} + ... + 2^n u_{2^n}$ be the partial sum of degree $n \in \mathbb{N}$ of the series $\sum_{n=1}^{\infty} 2^n u_{2^n}$.

Assume that the series $\sum_{n=1}^{\infty} 2^n u_{2^n}$ is convergent; then the sequence (S_n) of the partial sums is bounde, consequently there exists a real nuber $M > 0$ such that

$$0 \leq S_n \leq M, \text{ for all } n \in \mathbb{N}.$$

In order to prove for the series $\sum_{n=1}^{\infty} u_n$ to be convergent, based on Theorem 1.2.4, if suffices to prove that the sequence (s_n) of the partial sums is bounded. Due to the fact that the series $\sum_{n=1}^{\infty} u_n$ is with positive terms, from $n \leq 2^{n+1} - 1$, $(n \in \mathbb{N})$ we deduce that

$$s_n \leq s_{2^{n+1}-1} = u_1 + (u_2 + u_3) + (u_4 + \cdots + u_7) +$$

$$+ (u_{2^n} + u_{2^n+1} + \cdots + u_{2^{n+1}-1}).$$

Due to the fact that the sequence (u_n) is decreasing, it follows that

$$u_{2^k} > u_{2^k+1} > \cdots > u_{2^{k+1}-1}, \text{ for all } k \in \mathbb{N}$$

therefore s_n we can conlude that

$$s_n \leq s_{2^{n+1}-1} \leq u_1 + 2 \cdot u_2 + 2^2 \cdot u_{2^2} + \cdots + 2^n \cdot u_{2^n} =$$

$$= u_1 + S_n \leq u_1 + M.$$

Thus, the sequence (s_n) is bounded, therefore the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Assume now that the series $\sum_{n=1}^{\infty} u_n$ is convergent; then the sequence of the partial sums, (s_n), of the series $\sum_{n=1}^{\infty} u_n$ is bounde, consequently, there exists a real number $M > 0$ such that $0 \leq s_n \leq M$, for all $n \in \mathbb{N}$. In order to prove that the series $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ is convergent, it sufficies to prove that the sequence (S_n) is bounded. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} s_{2^n} &= u_1 + u_2 + (u_3 + u_4) + (u_5 + u_6 + u_7 + u_8) + \cdots + \\ &\quad + (u_{2^{n-1}+1} + \cdots + u_{2^n}) \geq \\ &\geq u_1 + u_2 + 2u_{2^2} + 2^2 u_{2^3} + \cdots + 2^{n-1} u_{2^n} \geq \\ &\geq u_1 + \frac{1}{2} S_n \geq \frac{1}{2} S_n, \end{aligned}$$

consequently, the following inequalities hold

$$S_n \leq 2 s_{2^n} \leq 2M.$$

Thus the sequence (S_n) is bounded and thus the series $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ is convergent. ■

Example 1.2.11 The series

$$\sum_{n=1}^{\infty} \frac{1}{n^a}, \text{ with } a \in \mathbb{R},$$

called **the generalized harmonic series**,is divergent for $a \leq 1$ and convergent for $a > 1$.

Solution. If $a \leq 0$, then the sequence of the terms of the series (n^{-a}) does not have the limit 0, therefore the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^a}$ is divergent. If $a > 0$, then the sequence of the tersm of the series (n^{-a}) is decreasing and has the limit zero, thus we may apply the Cauchy's condensation criterion, obtaining that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^a}$ and $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^a}$ have the same nature. Due to the fact that $2^n \frac{1}{(2^n)^a} = \left(\frac{1}{2^{a-1}}\right)^n$, for all $n \in \mathbb{N}$, we deduce that the series $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^a}$ turns out to be a geometric series, $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2^{a-1}}\right)^n$, which is divergent for $a \leq 1$ and convergent for $a > 1$. Consequently, the sereis $\sum\limits_{n=1}^{\infty} \frac{1}{n^a}$ is divergent for $a \leq 1$ and convergent for $a > 1$. ■

Theorem 1.2.12 (D'Alembert's quotient criterion) *Let $\sum\limits_{n=1}^{\infty} u_n$ be a series with positive terms.*

1^0 *If there exists a real number $q \in [0, 1[$ and a natural number n_0 such that:*

$$\frac{u_{n+1}}{u_n} \leq q \textit{ for all } n \in \mathbb{N},\, n \geq n_0,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 *If there exists a natural number n_0 such that:*

$$\frac{u_{n+1}}{u_n} \geq 1 \textit{ for all } n \in \mathbb{N},\, n \geq n_0,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is divergent.*

Proof. 1^0 We apply the third comparison criterion for series with positive terems (teorema 1.2.9, statement 1^0), by choosing $v_n := q^{n-1}$, for all $n \in \mathbb{N}$. It holds

$$\frac{u_{n+1}}{u_n} \leq q = \frac{v_{n+1}}{v_n}, \text{ for all } n \in \mathbb{N},\ n \geq n_0,$$

and the series $\sum\limits_{n=1}^{\infty} v_n$ is convergent, consequently, the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 From $u_{n+1}/u_n \geq 1$ it follows that $u_{n+1} \geq u_n$, for all $n \in \mathbb{N}$, $n \geq n_0$, therefore, the sequence (u_n) does not have the limit 0; thus, according to Theorem 1.1.9, the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. ■

Theorem 1.2.13 (the consequence of the quotient criterion) *Let* $\sum\limits_{n=1}^{\infty} u_n$ *be a series with positive terms, for which there exists* $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$.

1^0 *If*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is convergent.*

2^0 *If*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is divergent.*

Proof. Let $a := \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$. It is clear that $a \geq 0$.

1^0 Since $a \in [0, 1[$ it follows that there exists a real number $q \in]a, 1[$. Then, from $a \in]a-1, q[$ it follows that there exists a natural number n_0 such that

$$\frac{u_{n+1}}{u_n} \in]a-1, q[, \text{ for all } n \in \mathbb{N},\ n \geq n_0.$$

It follows that

$$\frac{u_{n+1}}{u_n} \leq q, \text{ for all } n \in \mathbb{N},\ n \geq n_0.$$

By applying the quotient criterion, we get that the series $\sum\limits_{n=1}^{\infty} u_n$ is convegent.

2^0 If $1 < a$, then there exists a natural number n_0 such that

$$\frac{u_{n+1}}{u_n} \geq 1, \text{ for all } n \in \mathbb{N},\ n \geq n_0.$$

By applying the quotient criterion, we get that the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. ■

Example 1.2.14 The series

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} \tag{1.2.12}$$

is convergent.

Solution. We have that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{27} < 1,$$

thus, accordingly to the consequence of the quotient criterion, the series (1.2.12) is convergent. ■

Remark 1.2.15 Consider the series $\sum_{n=1}^{\infty} u_n$. If there exists $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ and it is equal to 1, then the consequence of the quotient criterion cannot be applies. The series $\sum_{n=1}^{\infty} u_n$ could be either convergent or divergent. There are series either convergent or divergent with the property that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. For example, consider the series $\sum_{n=1}^{\infty} n^{-1}$ and $\sum_{n=1}^{\infty} n^{-2}$. In both cases $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, but the first one is divergent (see example 1.1.4) while the second one is convergent (see example 1.2.8). ◊

Theorem 1.2.16 (Cauchy's root criterion) *Let* $\sum_{n=1}^{\infty} u_n$ *be a series with positive terms.*

1^0 *If there exists a real number* $q \in [0, 1[$ *and a natural number* n_0 *such that*

$$\sqrt[n]{u_n} \le q, \ \textit{for all } n \in \mathbb{N},\ n \ge n_0, \tag{1.2.13}$$

then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

2^0 *If there exists the natural number* n_0 *such that*

$$\sqrt[n]{u_n} \ge 1, \ \textit{for all } n \in \mathbb{N},\ n \ge n_0, \tag{1.2.14}$$

then the series $\sum_{n=1}^{\infty} u_n$ *is divergent.*

Proof. 1^0 Assume that there exists $q \in [0, 1[$ and $n_0 \in \mathbb{N}$ such that (1.2.13) holds. Then

$$u_n \le q^n, \ \text{for all } n \in \mathbb{N},\ n \ge n_0.$$

We apply the first comparison criterion (theorem 1.2.5, statement 1^0), by considering $v_n := q^{n-1}$, for all $n \in \mathbb{N}$ and $a := q$. Due to the fact that the series $\sum_{n=1}^{\infty} q^{n-1}$ is convergent, we get that the series $\sum_{n=1}^{\infty} u_n$ is convergent.

2^0 From (1.2.14) we deduce that $u_n \geq 1$, for all $n \in \mathbb{N}$, $n \geq n_0$, consequently, the sequence (u_n) does not have the limit 0. Then, according to the theorem 1.1.9, the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. ■

Theorem 1.2.17 (the consequence of the root criterion) *Let $\sum\limits_{n=1}^{\infty} u_n$ be a series with positive terms, for which there exists $\lim\limits_{n\to\infty} \sqrt[n]{u_n}$.*

1^0 *If*

$$\lim_{n\to\infty} \sqrt[n]{u_n} < 1,$$

atunci seria $\sum\limits_{n=1}^{\infty} u_n$ este convergentă.

2^0 *If*

$$\lim_{n\to\infty} \sqrt[n]{u_n} > 1,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Proof. Let $a := \lim\limits_{n\to\infty} \sqrt[n]{u_n}$. It holds $a \geq 0$.

1^0 Due to the fact that $a \in [0,1[$ we conclude that there exists the real number $q \in]a,1[$. Then, from $a \in]a-1,q[$ it follows that there exists the natural number n_0 such that

$$\sqrt[n]{u_n} \in]a-1,q[, \text{ for all } n \in \mathbb{N},\, n \geq n_0.$$

It follows that

$$\sqrt[n]{u_n} \leq q, \text{ for all } n \in \mathbb{N},\, n \geq n_0.$$

By applying the root criterion we obtain that the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 If $1 < a$, then there exists a natural number n_0 such that

$$\sqrt[n]{u_n} \geq 1, \text{ for all } n \in \mathbb{N},\, n \geq n_0.$$

By applying the root criterion we obtain that the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. ■

Example 1.2.18 The series

$$\sum_{n=1}^{\infty} \left(\sqrt[3]{n^3+3n^2+1} - \sqrt[3]{n^3-n^2+1}\right)^n. \tag{1.2.15}$$

is convergent.

Solution. It holds

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left(\sqrt[3]{n^3+3n^2+1} - \sqrt[3]{n^3-n^2+1}\right) = \frac{4}{3} > 1.$$

and thus, according tot the consequence of the root criterion, the series (1.2.15) is divergent. ■

Remark 1.2.19 Having a series with positive terms $\sum\limits_{n=1}^{\infty} u_n$ for which the limit $\lim\limits_{n\to\infty} \sqrt[n]{u_n}$ exists and is equal to 1, the consequence of the root criterion does not decide weather the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent or divergent. There are series, either convergent, or divergent for which $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = 1$. For exameple, given the series $\sum\limits_{n=1}^{\infty} n^{-1}$ and $\sum\limits_{n=1}^{\infty} n^{-2}$ it holds, în ambele cazuri, $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = 1$. The first one is divergent (see example 1.1.4) and the second one is convergent (see example 1.2.8). ◊

Theorem 1.2.20 (Kummer's criterion) *Let* $\sum\limits_{n=1}^{\infty} u_n$ *be a series with positive terms.*

1^0 *If there exists a sequence of real positive numbers* $(a_n)_{n\in\mathbb{N}}$ *, de numere reale pozitive, there exists a real number* $r > 0$ *and there exists a natural number* n_0 *such that*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} \geq r, \text{ for all } n \in \mathbb{N},\ n \geq n_0, \tag{1.2.16}$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is convergent.*

2^0 *If there exists a sequence of real positive numbers* (a_n) *, such that the series* $\sum\limits_{n=1}^{\infty} \frac{1}{a_n}$ *is divergent and there exists a natural number* n_0 *such that*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} \leq 0, \text{ for all } n \in \mathbb{N},\ n \geq n_0, \tag{1.2.17}$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is divergent.*

Proof. For each natural number n, we denote by

$$s_n := u_1 + u_2 + ... + u_n$$

the partial sum of degree n of the series $\sum\limits_{n=1}^{\infty} u_n$.

1^0 Assume that there exists the sequence of real positive numbers (a_n), there exists a real number $r > 0$ and there exists a natural number n_0 such that (1.2.16). We notice that the relation (1.2.16) is equivalent to

$$a_n u_n - a_{n+1} u_{n+1} \geq r u_{n+1}, \text{ for all } n \in \mathbb{N},\ n \geq n_0. \tag{1.2.18}$$

Let $n \in \mathbb{N}$, $n \geq n_0 + 1$; then , from(1.2.18) we obtain:

$$a_{n_0} u_{n_0} - a_{n_0+1} u_{n_0+1} \geq r u_{n_0+1},$$

$$. \quad . \quad .$$

$$a_{n-1} u_{n-1} - a_n u_n \geq r u_n,$$

thus, by adding up, we obtain

$$a_{n_0}u_{n_0} - a_n u_n \geq r(u_{n_0+1} + \cdots + u_n).$$

This leads us to the conclusion that for all natural $n \geq n_0$ it holds

$$s_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n_0} u_k + \sum_{k=n_0+1}^{n} u_k \leq s_{n_0} + \frac{1}{r}\left(a_{n_0 u_{n_0}} - a_n u_n\right) \leq$$

$$\leq s_{n_0} + \frac{1}{r} a_{n_0} u_{n_0},$$

therefore the sequence (s_n) of the partial sums of the series $\sum_{n=1}^{\infty} u_n$ is bounded. According to theorem 1.2.4, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

2^0 Assume that there exists a sequence of positive real numbers (a_n), with the property that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent and there exists a natural number n_0 such that (1.2.17) holds. Obviously (1.2.17) is equivalent to

$$\frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} \leq \frac{u_{n+1}}{u_n}, \text{ for all } n \in \mathbb{N},\, n \geq n_0.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent, according to the third comparison criterion, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

■

Theorem 1.2.21 (Raabe-Duhamel's criterion) *Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms.*

1^0 *If there exists a real number $q > 1$ and a natural number n_0 such that*

(1.2.19) $$n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq q \text{ for all } n \in \mathbb{N},\, n \geq n_0,$$

then the series $\sum_{n=1}^{\infty} u_n$ is convergent.

2^0 *If there exists a natural number n_0 such that*

(1.2.20) $$n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq 1 \text{ for all } n \in \mathbb{N},\, n \geq n_0,$$

then the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Proof. According to Kummer's criterion (teorema 1.2.20) consider $a_n := n$, oricare ar fi $n \in \mathbb{N}$; then we get

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} = n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1.$$

1^0 If we take $r := q-1 > 0$, then, întrucât (1.2.16) is equivalent to (1.2.19) , we deduce that $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 Since the series $\sum\limits_{n=1}^{\infty} n^{-1}$ is divergent and (1.2.17) is equivalent to (1.2.20) , we get that the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. ■

Theorem 1.2.22 (The consequence of Raabe-Duhamel's criterion) *Let $\sum\limits_{n=1}^{\infty} u_n$ be a series with positive terms, for which there exits the limit*

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right).$$

1^0 *If*

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 *If*

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Proof. Let

$$b := \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right).$$

1^0 From $b > 1$ we conclude that there exists a real number $q \in]1, b[$. Then $b \in]q, b+1[$ the existence of a natural number n_0 such that

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \in]q, b+1[, \text{ for all } n \in \mathbb{N},\ n \geq n_0,$$

thus (1.2.19) holds. By applying the Raabe-Duhamel criterion we get that the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

2^0 If $b < 1$, then there exits a natural number n_0 such that (1.2.20) holds. By applying the Raabe-Duhamel criterion we get that the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent. ■

Example 1.2.23 The series

$$\sum_{n=1}^{\infty} \frac{n!}{a(a+1)\cdots(a+n-1)}, \text{ unde } a > 0,$$

is convergent if and only if $a > 2$.

Solution. We have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

therefore, according to the consequence of the quotient criterion, we cannot state the nature of the series Due to the fact that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = a - 1,$$

according to the consequence of the Raabe-Duhamel' criterion, if $a > 2$, the series is convergent, and if $a < 2$ the series is divergent. If $a = 2$, then the series becomes $\sum\limits_{n=1}^{\infty} \frac{1}{n+1}$ which is divergent. This means that the given series is convergent if and only if $a > 2$. ■

3. Exercises to solve - series

Exercise 1.3.1 Compute the sums of the following geometric series:

$$a)\sum_{n\geq 3} \frac{3}{5^n},\quad b)\sum_{n\geq 4} \frac{2^{n-3}+(-3)^{n+3}}{5^n},\quad c)\sum_{n\geq 5} e^n,\quad d)\sum_{n\geq 2}\left(-\frac{1}{\pi}\right)^n\quad e)\sum_{n\geq 3}(-3)^n.$$

Exercise 1.3.2 Compute the sums of the following telescopic series:

$$a)\sum_{n\geq 1} \frac{1}{4n^2-1},\quad b)\sum_{n\geq 1} \frac{1}{\sqrt{n}+\sqrt{n+1}},\quad c)\sum_{n\geq 5} \frac{1}{n(n+1)(n+2)}$$

$$d)\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right),\quad e)\sum_{n\geq 2} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)}.$$

Exercise 1.3.3 Determine the nature of the following series:

$$a)\sum_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}},\quad b)\sum_{n\geq 1} \frac{1}{\sqrt[n]{n}},\quad c)\sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}},\quad d)\sum_{n\geq 1}\left(1+\frac{1}{n}\right)^n.$$

Exercise 1.3.4 Determine the nature of the following series:

$$a)\sum_{n\geq 1} \frac{2^n+3^n}{5^n},\quad b)\sum_{n\geq 1} \frac{2^n}{3^n+5^n}.$$

Exercise 1.3.5 Determine the nature of the following series

$$a) \sum_{n\geq 1} \frac{1}{2n-1}, \quad b) \sum_{n\geq 1} \frac{1}{(2n-1)^2}, \quad c) \sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}, \quad d) \sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}.$$

Exercise 1.3.6 Determine the nature of the following series:

$$a) \sum_{n\geq 1} \frac{100^n}{n!}, \quad b) \sum_{n\geq 1} \frac{2^n n!}{n^n}, \quad c) \sum_{n\geq 1} \frac{3^n n!}{n^n}, \quad d) \sum_{n\geq 1} \frac{(n!)^2}{2^{n^2}}, \quad e) \sum_{n\geq 1} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}.$$

Exercise 1.3.7 Determine, depending on the values of the parameter $a > 0$, the nature of the following series:

$$a) \sum_{n\geq 1} \frac{a^n}{n^n}, \quad b) \sum_{n\geq 1} \left(\frac{n^2+n+1}{n^2} a\right)^n, \quad c) \sum_{n\geq 1} \frac{3^n}{2^n+a^n}.$$

Example 1.3.1 For each $a, b > 0$, study the nature of the series:

$$a) \sum_{n=1}^{\infty} \frac{a^n}{a^n+b^n}; \quad b) \sum_{n=1}^{\infty} \frac{2^n}{a^n+b^n}; \quad c) \sum_{n=1}^{\infty} \frac{a^n b^n}{a^n+b^n};$$

$$d) \sum_{n=1}^{\infty} \frac{(2a+1)(3a+1)\cdots(na+1)}{(2b+1)(3b+1)\cdots(nb+1)}.$$

Example 1.3.2 Determine the nature of the series:

$$a) \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1}; \quad b) \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}; \quad c) \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{n}{e}\right)^n.$$

Example 1.3.3 For each $a : 0$ determine the nature of the series:

$$a) \sum_{n=1}^{\infty} \frac{n!}{a(a+1)\ldots(a+n)}; \quad b) \sum_{n=1}^{\infty} a^{-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)}; \quad c) \sum_{n=1}^{\infty} \frac{a^n \cdot n!}{n^n}.$$

Remark 1.3.4 For further detalis go to [**5**].

CHAPTER 2

Taylor's Formula

1. Taylor's Polynomial: definition, properties

Taylor's formula, mainly used in approxiamting functions by the means of the polynomials, is one of the most important formulae in mathematics..

Definition 2.1.1 *Let D be a nonempty subest of $\mathbb{R}$, $x_0 \in D$ and $f : D \to \mathbb{R}$ be a n times differentiable function at x_0. The (polynomial) function $T_{n;x_0}f : \mathbb{R} \to \mathbb{R}$ defined by*

$$\left(T_{n;x_0}f\right)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$\left(T_{n;x_0}f\right)(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ...$$
$$... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \text{ for all } x \in \mathbb{R},$$

*is called **Taylor's Polynomial of order n attached to the function f, centered at the point x_0**.* $\lozenge$

Remark 2.1.2 Taylor's polynomial of order n, has the degree at most n.$\lozenge$

Example 2.1.3 For the exponential function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \exp x, \text{ for all } x \in \mathbb{R},$$

it holds

$$f^{(k)}(x) = \exp x, \text{ for all } x \in \mathbb{R} \text{ şi } k \in \mathbb{N}.$$

Taylor's polynomial of order n attached to the exponential function $\exp : \mathbb{R} \to \mathbb{R}$, centered at the point $x_0 = 0$ is

$$\left(T_{n;0}\exp\right)(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n,$$

for all $x \in \mathbb{R}$. $\lozenge$

Remark 2.1.4 We notice the foolowing:

- the domain of definition of Taylor's polynomial is $\mathbb{R}$, in contrast to the domain of definition for the function f, which sometimes is much smaller;

- being a polynomial function $T_{n;x_0}f$ is indefinite differentiable on $\mathbb{R}$ şi, for all $x \in \mathbb{R}$, so it holds

$$(T_{n;x_0}f)'(x) = f^{(1)}(x_0) + \frac{f^{(2)}(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{(n-1)!}(x-x_0)^{n-1} =$$
$$= (T_{n-1;x_0}f')(x),$$

$$(T_{n;x_0}f)''(x) = f^{(2)}(x_0) + \frac{f^{(3)}(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{(n-2)!}(x-x_0)^{n-2} =$$
$$= (T_{n-2;x_0}f'')(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(T_{n;x_0}f)^{(n-1)}(x) = f^{(n-1)}(x_0) + \frac{f^{(n)}(x_0)}{1!}(x-x_0) =$$
$$= \left(T_{1;x_0}f^{(n-1)}\right)(x),$$
$$(T_{n;x_0}f)^{(n)}(x) = f^{(n)}(x_0) = \left(T_{0;x_0}f^{(n)}\right)(x),$$
$$(T_{n;x_0}f)^{(k)}(x) = 0, \text{ for all } k \in \mathbb{N},\ k \geq n+1.$$

This leads to

$$(T_{n;x_0}f)^{(k)}(x_0) = f^{(k)}(x_0), \text{ for all } k \in \{0,1,\cdots,n\}$$

şi

$$(T_{n;x_0}f)^{(k)}(x_0) = 0, \text{ for all } k \in \mathbb{N},\ k \geq n+1.$$

Remark 2.1.5 At the point x_0, not only the value of Taylors' polynomial of order n attached to the function f, centered at x_0, but also the values of its derivatives, up to the order n, coincide to the values of the function f, and its derivatives up to the order n, respectivley .

2. Taylor's Formula

Definition 2.2.1 *Let D be a nonempty subset of $\mathbb{R}$, $x_0 \in D$ and $f : D \to \mathbb{R}$ be a function n time differentiable at the point x_0. The function $R_{n;x_0}f : D \to \mathbb{R}$ defined by*

$$(R_{n;x_0}f)(x) = f(x) - (T_{n;x_0}f)(x), \text{ for all } x \in D$$

*is called **Taylor's remainder of order** n **attached to the function** f **centered at** x_0.*

When the form of the reminder is given by a certain computational expression, the following

$$f = T_{n;x_0}f + R_{n;x_0}f,$$

is called ***Taylor'a formula of order*** n ***attaced to the funciton*** f ***centered at*** x_0. *In this case* $R_{n;x_0}f$ *is called the reminder of order* n *for Taylor's formula.* $\Diamond$

Remark 2.2.2 Because f and $T_{n;x_0}f$ are n times differentiable at x_0, it follows that the reminder $R_{n;x_0}f = f - T_{n;x_0}f$ is a n times differentiable function at x_0, and

$$(R_{n;x_0}f)^{(k)}(x_0) = 0, \text{ for all } k \in \{0, 1, \cdots, n\}.$$

Remark 2.2.3 The function $R_{n;x_0}f : D \to \mathbb{R}$ being differentiable at x_0 is obviously continuous at x_0 and thus there exists

$$\lim_{x \to x_0} (R_{n;x_0}f)(x) = (R_{n;x_0}f)(x_0) = 0.$$

Therefore, for all$\varepsilon > 0$ there exits $\delta > 0$ such that for all $x \in D$ for which $|x - x_0| < \delta$ it holds

$$|f(x) - (T_{n;x_0}f)(x)| < \varepsilon.$$

As a consequnce, through the values of $x \in D$, close enough to x_0, the value of $f(x)$ may be approximated by $(T_{n;x_0}f)(x)$.

In the following we give a characterization of the reminder.

Theorem 2.2.4 *Let* I *be an interval in* $\mathbb{R}$, $x_0 \in I$ *and* $f : I \to \mathbb{R}$ *be a* n *times differentiable function at* x_0. *Then*

$$\lim_{x \to x_0} \frac{(R_{n;x_0}f)(x)}{(x - x_0)^n} = 0.$$

Proof. By applying l'Hôpital' theorem $n-1$ time, and tacking into account that

$$\lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} = f^{(n)}(x_0),$$

we get

$$\lim_{x \to x_0} \frac{(R_{n;x_0}f)(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{f(x) - (T_{n;x_0}f)(x)}{(x - x_0)^n} =$$

$$= \lim_{x \to x_0} \frac{f'(x) - (T_{n;x_0}f)'(x)}{n(x - x_0)^{n-1}} = \cdots$$

$$\cdots = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - (T_{n;x_0}f)^{(n-1)}(x)}{n!(x - x_0)} =$$

$$= \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x - x_0)}{n!(x - x_0)} =$$

$$= \frac{1}{n!} \lim_{x \to x_0} \left[\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} - f^{(n)}(x_0) \right] = 0.$$

$\blacksquare$

Remark 2.2.5 If we denote by $\alpha_{n;x_0}f : I \to \mathbb{R}$ the function defined by

$$(\alpha_{n;x_0}f)(x) = \begin{cases} \dfrac{(R_{n;x_0}f)(x)}{(x-x_0)^n}, & \text{dacă } x \in I\backslash\{x_0\} \\ 0, & \text{dacă } x = x_0, \end{cases}$$

then, from theorem 2.2.4 it follows that the function $\alpha_{n;x_0}f$ is continuous at x_0. Moreover, for each $x \in I$ it holds:

$$f(x) = (T_{n;x_0}f)(x) + (x-x_0)^n (\alpha_{n;x_0}f)(x). \ \Diamond$$

Example 2.2.6 For the exponential function, $\exp : \mathbb{R} \to \mathbb{R}$, Taylor-Young's formula, for $x_0 = 0$, is:

$$\exp x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + x^n (\alpha_{n;0}f)(x),$$

for all $x \in \mathbb{R}$, where

$$\lim_{x\to 0} (\alpha_{n;0}f)(x) = (\alpha_{n;0}f)(0) = 0. \ \Diamond$$

According to theorem 2.2.4, if I is an interval in $\mathbb{R}$, $x_0 \in I$ and $f : I \to \mathbb{R}$ is a n times differentiable function at x_0, then, for all $x \in I$, it holds

(2.2.21) $$f(x) = (T_{n;x_0}f)(x) + o((x-x_0)^n) \text{ pentru } x \to x_0.$$

We obtain the following result.

Theorem 2.2.7 *Let I be an interval in $\mathbb{R}$, $x_0 \in I$ and $f : I \to \mathbb{R}$ be a function. If f is n times differentiable at x_0, then for all $x \in I$, the equality (2.2.21) holds.*

The relation (2.2.21) is calledc *Taylor's formula with the reminder in Peano's form.* $\Diamond$

3. Different Forms of the Reminder in Taylor's Formula

Theorem 2.3.1 (Taylor′s theorem) *Let I be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ be a $n+1$ differentiable function on I, $x_0 \in I$ and $p \in \mathbb{N}$. Then for each $x \in I\backslash\{x_0\}$, there exists at least one point c strictly between x and x_0 such that*

$$f(x) = (T_{n;x_0}f)(x) + (R_{n;x_0}f)(x),$$

where

(2.3.22) $$(R_{n;x_0}f)(x) = \frac{(x-x_0)^p (x-c)^{n-p+1}}{n!p} f^{(n+1)}(c). \ \Diamond$$

Proof.

We start by proving that the reminder $R_{n;x_0}f$ in Taylor's formula may be written as

$$(R_{n;x_0}f)(x) = (x - x_0)^p K,$$

where $p \in \mathbb{N}$ şi $K \in \mathbb{R}$.

Let I be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ be a $n+1$ differentiable function on$I$, p be a natural number, and x and x_0 be two distinct point in I. Let $K \in \mathbb{R}$ be such that it holds

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + ...$$
$$... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + (x - x_0)^p K.$$

The function $\varphi : I \to \mathbb{R}$, defined for all $t \in I$, by

$$\varphi(t) = f(t) + \frac{f^{(1)}(t)}{1!}(x - t) + \frac{f^{(2)}(t)}{2!}(x - t)^2 + ... + \frac{f^{(n)}(t)}{n!}(x - t)^n +$$
$$+ (x - t)^p K,$$

is differentiable on I.

Since $\varphi(x_0) = \varphi(x) = f(x)$, we deduce that the function φ satisfies the hypotheses of Rolle's theorem on the closed interval having as bounds x_0 and x; then there exits at least one point c strictly between x_0 and x such that $\varphi'(c) = 0$. Since

$$\varphi'(t) = \frac{(x - t)^n}{n!} f^{(n+1)}(t) - p(x - t)^{p-1} K, \text{ for all } t \in I, .$$

the equality $\varphi'(c) = 0$ becomes

$$\frac{(x - c)^n}{n!} f^{(n+1)}(c) - p(x - c)^{p-1} K = 0,$$

from which we get

$$K = \frac{(x - c)^{n-p+1}}{n!p} f^{(n+1)}(c).$$

Consequently, the reminder $R_{n;x_0}f$ has the following form

$$(R_{n;x_0}f)(x) = \frac{(x - x_0)^p (x - x_0)^{n-p+1}}{n!p} f^{(n+1)}(c).$$

■

The general form of the reminder from (2.3.22), was obtained independently by both **Schlömilch** and **Roche**, this is why this form of the reminder (2.3.22) is called the **Schlömilch-Roche reminder**.

Laagrange and Cauchy had obtained previously, two particular cases of the general form.

Cuachy's reminder was:

$$(R_{n;x_0}f)(x) = \frac{(x - x_0)(x - c)^n}{n!} f^{(n+1)}(c), \tag{2.3.23}$$

and is exactly Schlömilch-Roche's reminder considered for $p = 1$.

Lagrange obtained the form:

$$(2.3.24) \qquad (R_{n;x_0} f)(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

and is exactly Schlömilch-Roche's reminder considered for $p = n + 1$.

When f is a polynomial function of order n, for all $x_0 \in \mathbb{R}$,

$$(R_{n;x_0} f)(x) = 0, \text{ for all } x \in \mathbb{R}.$$

This was the case studied by Tylor. By custom all the above studied cases are named "*Taylor's formula*" except the one for: $0 \in I$ and $x_0 = 0$. This was studied by Maclaurin. We consider the following definition

Definition 2.3.2 *Taylor's formula of order n attached to the function f, centered at $x_0 = 0$, with the Lagrange reminder is called* ***Maclaurin's formula***.(*1698 - 1746*). $\diamondsuit$

Example 2.3.3 For the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ Maclaurin's formula is

$$\exp x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + (R_{n;0} f)(x),$$

where

$$(R_{n;0} \exp)(x) = \frac{x^{n+1}}{(n+1)!} \exp(c), \quad \text{cu } |c| < |x|.$$

It holds

$$|(R_{n;0} \exp)(x)| = \frac{|x|^{n+1}}{(n+1)!} \exp(c) < \frac{|x|^{n+1}}{(n+1)!} \exp|x|, \ x \in \mathbb{R}.$$

Since for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \exp|x| = 0,$$

we deduce that the series

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots,$$

is convergent for all $x \in \mathbb{R}$, and its sum is $\exp x$, meaning that

$$\exp x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots, \text{ for all } x \in \mathbb{R}. \ \diamondsuit$$

Similarly we obtain for all $a > 0$, $a \neq 1$,

$$a^x = 1 + \frac{\ln a}{1!}x + \frac{\ln^2 a}{2!}x^2 + \cdots + \frac{\ln^n a}{n!}x^n + \cdots, \ x \in \mathbb{R}. \ \diamondsuit$$

Because in 2.3.1, c is strictly between x and x_0, we conclude that there exits the number

$$\theta = \frac{c - x_0}{x - x_0} \in]0, 1[$$

and

$$c = x_0 + \theta (x - x_0).$$

Then the reminder $R_{n;x_0} f$ can be expressed as:

(2.3.25) $$(R_{n;x_0} f)(x) = \frac{(x - x_0)^{n+1} (1 - \theta)^{n-p+1}}{n! p} f^{(n+1)} (x_0 + \theta (x - x_0)),$$

(Schlömilch – Roche)

(2.3.26) $$(R_{n;x_0} f)(x) = \frac{(x - x_0)^{n+1} (1 - \theta)^{n}}{n!} f^{(n+1)} (x_0 + \theta (x - x_0)) \quad \text{(Cauchy)}$$

(2.3.27) $$(R_{n;x_0} f)(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)} (x_0 + \theta (x - x_0)) \quad \text{(Lagrange).}$$

This was exactly the proof for the following theorem.

Theorem 2.3.4 *Let I be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ be a $n+1$ differntiable function on I, $x_0 \in I$ and $p \in \mathbb{N}$. Then for each $x \in I \backslash \{x_0\}$, there exits at least a number $\theta \in]0, 1[$ such that it holds*

$$f(x) = (T_{n;x_0} f)(x) + (R_{n;x_0} f)(x),$$

where $(R_{n;x_0} f)(x)$ is given by $(2.3.25)$.

If $p = 1$, we obtain $(2.3.26)$, and if $p = n + 1$ then $(R_{n;x_0} f)(x)$ is given by $(2.3.27)$. $\Diamond$

Example 2.3.5 For the function$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \exp x, \text{ for all } x \in \mathbb{R},$$

Maclaurin's formula is

$$\exp x = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n + (R_{n;0} f)(x),$$

where

$$R_{n;0} f(x) = \frac{x^{n+1}}{(n+1)!} \exp(\theta x), \quad \theta \in]0, 1[, \ x \in \mathbb{R}. \ \Diamond$$

4. Exercises to be Solved

Example 2.4.1 Write Taylor's polynomial of order $n = 2m-1$ attached to the function, $\sin : \mathbb{R} \to \mathbb{R}$, centered at the point $x_0 = 0$.

Example 2.4.2 Write Taylor's polynomial of order $n = 2m$ attached to the function $\cos : \mathbb{R} \to \mathbb{R}$ centered at the point $x_0 = 0$.

Example 2.4.3 Write Maclaurin's formula of order n for the function $\sin : \mathbb{R} \to \mathbb{R}$.

Example 2.4.4 Write Maclaurin's formula of order n for the function $\cos : \mathbb{R} \to \mathbb{R}$.

Example 2.4.5 Write Maclaurin's formula of order n for the function $f :]-1,+\infty[\to \mathbb{R}$ defined by

$$f(x) = \ln(1+x), \text{ for all } x \in]-1,+\infty[.$$

Example 2.4.6 Write Maclaurin's formula of order n for the function $f :]-1,+\infty[\to \mathbb{R}$ defined by

$$f(x) = (1+x)^r, \text{ oricare ar fi } x \in]-1,+\infty[,$$

where $r \in \mathbb{R}$.

Example 2.4.7 Let $f :]0,+\infty[\to \mathbb{R}$ be a function defined by $f(x) = 1/x$, for all $x \in]0,+\infty[$. Write Taylor's formula of order n attached to the function f centered at $x_0 = 1$.

Example 2.4.8 Write Maclaurin's formula of order n attached to the following functions, by using when necessary the following formula for the computing the n-th derivative of a product of functions

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) \cdot g^{(k)} :$$

a) $f :]-1,+\infty[\to \mathbb{R}$ definită prin $f(x) = x\ln(1+x)$, for all $x \in]-1,+\infty[$;

b) $f :]-\infty,1[\to \mathbb{R}$ defined by $f(x) = x\ln(1-x)$, for all $x \in]-\infty,1[$;

c) $f :]-1,1[\to \mathbb{R}$ defined by $f(x) = \sqrt{3x+4}$, for all $x \in]-1,1[$;

d) $f :]-1/2,+\infty[\to \mathbb{R}$ defined by $f(x) = 1/\sqrt{2x+1}$, for all $x \in]-1/2,+\infty[$.

Example 2.4.9 Write Taylor's formula of order n, attached to the function f, centered at x_0, for:

a) $f :]0,+\infty[\to \mathbb{R}$ defined by $f(x) = 1/x$, for all $x \in]0,+\infty[$ and $x_0 = 2$;

b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x-1)$, for all $x \in \mathbb{R}$ and $x_0 = 1$.

Remark 2.4.10 For futher details see [**5**] and [**2**].

CHAPTER 3

The Riemann Integral

1. Partitions of a compact interval

Definition 3.1.1 *Let $a, b \in \mathbb{R}$ cu $a < b$. A* ***partition of the interval*** $[a,b]$ *is each ordered system*

$$\Delta = (x_0, x_1, ..., x_p)$$

of $p+1$ points $x_0, x_1, ..., x_p$ from $[a,b]$ with the property that

$$a = x_0 < x_1 < \cdots < x_{p-1} < x_p = b. \ \square$$

If $\Delta = (x_0, x_1, ..., x_p)$ is a partition of the interval $[a,b]$, then $x_0, x_1, ..., x_p$ are called **the points of the partition** Δ.

We denote by $\mathrm{Div}\,[a,b]$ the set of all partitions of the interval $[a,b]$, thus

$$\mathrm{Div}\,[a,b] = \{\Delta : \ \Delta \text{ is a partition of the interval } [a,b]\}.$$

If $\Delta = (x_0, x_1, ..., x_p)$ is a partition of the interval $[a,b]$, then the number

$$\|\Delta\| = \max\{x_1 - x_0, x_2 - x_1, ..., x_p - x_{p-1}\}$$

is called **the norm of the partition** Δ.

Example 3.1.2 The systems

$$\Delta^1 = (0,1), \quad \Delta^2 = (0, 1/3, 1), \quad \Delta^3 = (0, 1/4, 1/2, 3/4, 1)$$

are partitions of the interval $[0,1]$. They have the norms

$$\left\|\Delta^1\right\| = 1, \ \left\|\Delta^2\right\| = 2/3, \ \left\|\Delta^3\right\| = 1/4. \ \square$$

Theorem 3.1.3 *Let $a, b \in \mathbb{R}$ be such that $a < b$. For each real number $\varepsilon > 0$ there exits at least a patition Δ of the interval $[a,b]$ with the property that $\|\Delta\| < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and p be a real number such that $(b-a)/p < \varepsilon$. If $h = (b-a)/p$, then the ordered system

$$\Delta = (a, a+h, a+2h, \cdots, a+(p-1)h, b)$$

is a partition of the interval $[a,b]$. Moreover $\|\Delta\| = h < \varepsilon$. ∎

Definition 3.1.4 *Let $a, b \in \mathbb{R}$ be such that $a < b$ and $\Delta = (x_0, x_1, ..., x_p)$ and $\Delta' = \left(x'_0, x'_1, \cdot \cdot \cdot, x'_q\right)$ be two partitions of the interval $[a, b]$. The partitio Δ is said to be* ***finer*** *then the partition Δ' and we write this as $\Delta \supseteq \Delta'$ (or $\Delta' \subseteq \Delta$) if*

$$\{x'_0, x'_1, \cdot \cdot \cdot, x'_q\} \subseteq \{x_0, x_1, \cdot \cdot \cdot, x_p\}. \ \Diamond$$

The following theorem states that the finner the norm of the partition is, the more points the partition has

Theorem 3.1.5 *Let $a, b \in \mathbb{R}$ be such that $a < b$ and Δ and Δ' be two partitions of the interval $[a, b]$. If the partition Δ is finer than the partition Δ', then $\|\Delta\| \leq \|\Delta'\|$.*

Proof. It follows immediately form the definitions. ■

Remark 3.1.6 If Δ, $\Delta' \in \mathrm{Div}\,[a, b]$, then from $\|\Delta\| \leq \|\Delta'\|$ **does not** usually follow that $\Delta' \subseteq \Delta$. $\Diamond$

Definition 3.1.7 *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $\Delta' = \left(x'_0, x'_1, \cdot \cdot \cdot, x'_p\right)$ and $\Delta'' = \left(x''_0, x''_1, ..., x''_q\right)$ are partitions of the interval $[a, b]$, then the partition $\Delta = (x_0, x_1, \cdot \cdot \cdot, x_r)$ of the interval $[a, b]$ whose points are $\{x'_0, x'_1, ..., x'_p\} \cup \{x''_0, x''_1, \cdots, x''_q\}$, taken strictly increazing is called* ***the reunion*** *of Δ' with Δ'' and is denoted by $\Delta' \cup \Delta''$.* $\Diamond$

Theorem 3.1.8 *Let $a, b \in \mathbb{R}$ and $a < b$. If Δ' and Δ'' are partitions of the interval $[a, b]$, then*

1^0 $\Delta' \cup \Delta'' \supseteq \Delta'$ *şi* $\Delta' \cup \Delta'' \supseteq \Delta''$.

2^0 $\|\Delta' \cup \Delta''\| \leq \|\Delta'\|$ *and* $\|\Delta' \cup \Delta''\| \, \|\Delta''\|$.

Proof. Is clear ■

Definition 3.1.9 *Let $a, b \in \mathbb{R}$ with $a < b$ and $\Delta = (x_0, x_1, ..., x_p) \in Div[a, b]$. A* ***system of intermediate points attaced to the partion*** *Δ is a random system $\xi = (\xi_1, \xi_2, ..., \xi_p)$ of p points $\xi_1, \xi_2, ..., \xi_p \in [a, b]$ which satisfy the realtions*

$$x_{i-1} \leq \xi_i \leq x_i, \ \textit{oricare ar fi } i \in \{1, ..., p\}. \ \Diamond$$

We will denote by $\mathrm{Pi}\,(\Delta)$ the set of all systems of intermediate points attaced to the partition Δ, thus

$$\mathrm{Pi}\,(\Delta) = \{\xi : \ \xi \text{ is a system of intermediate points attached to the partition } \Delta\}.$$

2. The Riemann Integral

Definition 3.2.1 *Let $a,b\in\mathbb{R}$ with $a<b$, $\Delta=(x_0,x_1,...,x_p)$ be a partition of $[a,b]$, $\xi=(\xi_1,\xi_2,...,\xi_p)$ be a system of intermediate points attached to the partition Δ and let $f:[a,b]\to\mathbb{R}$ be a function. The real number*

$$\sigma(f;\Delta,\xi)=\sum_{i=1}^{p} f(\xi_i)(x_i-x_{i-1})$$

is called ***the Riemann sum attache to the function*** f ***the partition*** Δ ***and the system of intermediate points*** ξ. $\square$

Definition 3.2.2 *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$. The function f is said to be* ***Riemann integrable*** *on $[a,b]$ (or, simplyer* ***integrable****) if for each sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions $\Delta^n\in\mathrm{Div}[a,b]$, $(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ for each sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n\in\mathrm{Pi}(\Delta^n)$, $(n\in\mathbb{N})$, the sequence $(\sigma(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of the Riemann sums $\sigma(f;\Delta^n,\xi^n)$, $(n\in\mathbb{N})$ is convergent.* $\square$

Theorem 3.2.3 *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$. The function f is Riemann integrable on $[a,b]$ if and only if there exists a real number I such that for each sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions $\Delta^n\in\mathrm{Div}[a,b]$, $(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ and for each sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n\in\mathrm{Pi}(\Delta^n)$, $(n\in\mathbb{N})$, the sequence $(\sigma(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of the Riemann sums $\sigma(f;\Delta^n,\xi^n)$, $(n\in\mathbb{N})$ has the limit I.*

Proof. Necessity. Let $\left(\widetilde{\Delta}^n\right)_{n\in\mathbb{N}}$ be a sequence of partitions having as general term

$$\widetilde{\Delta}^n=(a,a+h,a+2h,...a+(n-1)h,b),\ \ (n\in\mathbb{N})$$

and let $(\xi^n)_{n\in\mathbb{N}}$ be the sequence with the general term

$$\widetilde{\xi}^n=(a,a+h,a+2h,...a+(n-1)h),\ \ (n\in\mathbb{N})$$

where

$$h:=\frac{b-a}{n}.$$

For each $n\in\mathbb{N}$ it holds:

$$\widetilde{\Delta}^n\in\mathrm{Div}[a,b],\quad \|\Delta^n\|=\frac{(b-a)}{n}\ \text{ and }\ \widetilde{\xi}^n\in\mathrm{Pi}\left(\widetilde{\Delta}^n\right).$$

Then the sequence $\left(\sigma\left(f;\widetilde{\Delta}^n,\widetilde{\xi}^n\right)\right)_{n\in\mathbb{N}}$ is convergent; let $I\in\mathbb{R}$ be the limit of the sequence $\left(\sigma\left(f;\widetilde{\Delta}^n,\widetilde{\xi}^n\right)\right)_{n\in\mathbb{N}}$.

We will prove that for each sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions of the interval $[a,b]$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ and for each sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n\in\mathrm{Pi}(\Delta^n)$, $(n\in\mathbb{N})$, the sequence $(\sigma(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of the Riemann sums $\sigma(f;\Delta^n,\xi^n)$, $(n\in\mathbb{N})$ has the limit I.

Consider $(\Delta^n)_{n\in\mathbb{N}}$ a sequence of partitions $\Delta^n \in \mathrm{Div}\,[a,b]\,,\,(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ and let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of systems $\xi^n \in \mathrm{Pi}\,(\Delta^n)\,,\,(n\in\mathbb{N})$. Then the sequences $(\underline{\Delta}^n)_{n\in\mathbb{N}}\,,\,\left(\underline{\xi}^n\right)_{n\in\mathbb{N}}\,,$ where

$$\underline{\Delta}^n=\begin{cases}\widetilde{\Delta}^k, & \text{if } n=2k\\ \Delta^k, & \text{if} n=2k+1,\end{cases}\qquad \underline{\xi}^n=\begin{cases}\widetilde{\xi}^k, & \text{if } n=2k\\ \xi^k, & \text{if } n=2k+1,\end{cases}$$

have the following propertis:

i) $\underline{\Delta}^n\in\mathrm{Div}\,[a,b],\ \underline{\xi}^n\in\mathrm{Pi}\,(\underline{\Delta}^n)\,,$ for all $n\in\mathbb{N}$;

ii) $\lim\limits_{n\to\infty}\|\underline{\Delta}^n\|=0.$

We know from the hypothesis that $\left(\sigma\left(f;\underline{\Delta}^n,\underline{\xi}^n\right)\right)_{n\in\mathbb{N}}$ is convergent. Let $\underline{I}$ be its limit. Taking into account that the sequence $\left(\sigma\left(f;\widetilde{\Delta}^n,\widetilde{\xi}^n\right)\right)_{n\in\mathbb{N}}$ is a subsequence of the convergent sequence $\left(\sigma\left(f;\underline{\Delta}^n,\underline{\xi}^n\right)\right)_{n\in\mathbb{N}}\,,$ we deduce that $\underline{I}=I$. Intrucât $(\sigma\,(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ is a subsequence of the convergent sequence $\left(\sigma\left(f;\underline{\Delta}^n,\underline{\xi}^n\right)\right)_{n\in\mathbb{N}}$, we obtain that the sequence $(\sigma\,(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ has the limit I.

Sufficiency It follows from the defintion. ■

Theorem 3.2.4 (the uniqueness of the integral) *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$. Then there exits at most a real number I with the property that each sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions $\Delta^n\in\mathrm{Div}\,[a,b]\,,\,(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ and for each sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n\in\mathrm{Pi}\,(\Delta^n)\,,\,(n\in\mathbb{N})$, the sequence $(\sigma\,(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of Riemann sums $\sigma\,(f;\Delta^n,\xi^n)\,,\,(n\in\mathbb{N})$ has the limit I.* □

In conclusion, for a function $f:[a,b]\to\mathbb{R}$ we have one of the following situations:

a) there exists a real number I with the property that for each sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions $\Delta^n\in\mathrm{Div}\,[a,b]\,,\,(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ and for each sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n\in\mathrm{Pi}\,(\Delta^n)\,,\,(n\in\mathbb{N})$, şirul $(\sigma\,(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of the Riemann sums $\sigma\,(f;\Delta^n,\xi^n)\,,$ $(n\in\mathbb{N})$ has the limit I.

In this case, according to theorem 3.2.4, the real number I is unique and is called the **Riemann integral of the function f on the interval** $[a,b]$ and it will be denoted by

$$I:=\int_a^b f\,(x)\,\mathrm{d}x.$$

b) There does not exist a real number I with the property that for each sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions $\Delta^n\in\mathrm{Div}\,[a,b]\,,\,(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|\Delta^n\|=0$ and each sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n\in\mathrm{Pi}\,(\Delta^n)\,,\,(n\in\mathbb{N})$, the sequence $(\sigma\,(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of the Riemann sums $\sigma\,(f;\Delta^n,\xi^n)\,,\,(n\in\mathbb{N})$ has the limti I. In this case the function f is not Riemann integrable on $[a,b]$. In conclusion, a function $f:[a,b]\to\mathbb{R}$ is not Riemann integrable on $[a,b]$ if and only if for each real number I and for eacg sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions

$\Delta^n \in \mathrm{Div}\,[a,b]$, $(n \in \mathbb{N})$ with $\lim\limits_{n\to\infty} \|\Delta^n\| = 0$ and a sequence $(\xi^n)_{n\in\mathbb{N}}$ of systems $\xi^n \in \mathrm{Pi}\,(\Delta^n)$, $(n \in \mathbb{N})$, with the property that the sequence $(\sigma(f;\Delta^n,\xi^n))_{n\in\mathbb{N}}$ of the Riemann sums $\sigma(f;\Delta^n,\xi^n)$, $(n \in \mathbb{N})$ does not have the limit I.

3. Antiderivatives

The concept of the **antiderivatives** makes the connection between two key conceps of Calculus, namely: **the derivative** and **the integral**.

Definition 3.3.1 *Let D be a nonempty subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ be a function and I a nonempty subset of D. We say that the function f **has antiderivatives** on I if there exits a function $F : I \to \mathbb{R}$ such that:*

$i)$ the function F is differntiable on I;

$ii)$ $F'(x) = f(x)$, for all $x \in I$.

*If the function f has antiderivatives on its entire domain D, we just say that f **has primitives**.*□

Example 3.3.2 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$, for all $x \in \mathbb{R}$, has primitives on $\mathbb{R}$ because the differentiable function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2/2$, for all $x \in \mathbb{R}$, has the property that $F' = f$. □

Definition 3.3.3 *Let D be a nonempty subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ be a function, and I be a nonempty subset of D. It is called an **antiderivative of the function** f on the set I each function $F : I \to \mathbb{R}$ who satisfies the following propertis:*

$i)$ the function F is differentiable on I;

$ii)$ $F'(x) = f(x)$, for all $x \in I$.

If F is an antiderivative of f on the entire domain D of f, then we just say that F is an antiderivative of the function f. □

Theorem 3.3.4 *Let I be an interval on $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. If $F_1 : I \to \mathbb{R}$ şi $F_2 : I \to \mathbb{R}$ are two antiderivatives of the function f on I, then there exists a real number c such that*

$$F_2(x) = F_1(x) + c, \text{ for all } x \in I.$$

(*Each two distinct antiderivatives differ through a constant*).

Proof. The functions F_1 and F_2 being antiderivatives of the function f, are differentiable, and $F_1' = F_2' = f$, hence

$$(F_2 - F_1)' = F_2' - F_1' = 0.$$

The differentiable function $F_2 - F_1$ having the derivative 0 on the interval I, is constant on this interval. Consequently, there exists a real number c such that

$$F_2(x) - F_1(x) = c, \text{ for all } x \in I.$$

■

Remark 3.3.5 In theorem 3.3.4, the hypothesis that the set I is an interval is fundamental. Indeed, for the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by

$$f(x) = 0, \text{ for all } x \in \mathbb{R}\backslash\{0\},$$

the functions $F_1, F_2 : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by

$$F_1(x) = 0, \text{ for all } x \in \mathbb{R}\backslash\{0\},$$

$$F_2(x) = \begin{cases} 0, & \text{dacă } x < 0 \\ 1, & \text{dacă } x > 0, \end{cases},$$

respectively, are antiderivatives of the function f on $\mathbb{R}\backslash\{0\}$. We notice that there exits no $c \in \mathbb{R}$ to satisfy $F_2(x) = F_1(x) + c$, for all $x \in \mathbb{R}\backslash\{0\}$. We underline the fact that $\mathbb{R}\backslash\{0\}$ is not an interval. □

Definition 3.3.6 *Let I be an internval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function that has antiderivatives on I. The set of all antiderivatives of f on the interval I is called the* ***indefinite integral*** *of the function f on the interval I and is denoted by*

$$\int f(x)\,\mathrm{d}x, \quad x \in I.$$

The operation of computing the antiderivative of f is called ***integration****.*

Let I be an interval in $\mathbb{R}$ and $\mathfrak{F}(I;\mathbb{R})$ be the set of all functions defined on I with values in $\mathbb{R}$. If $\mathcal{G}$ amd $\mathcal{H}$ are nonempty subsets of $\mathfrak{F}(I,\mathbb{R})$ and a is a real number, then:

$$\mathcal{G} + \mathcal{H} = \{f : I \to \mathbb{R}\text{: exists } g \in \mathcal{G} \text{ and } h \in \mathcal{H} \text{ such that } \ f = g + h\},$$

şi

$$a\mathcal{G} = \{f : I \to \mathbb{R}\text{: exists } g \in \mathcal{G} \text{ such that } \ f = ag\}.$$

If $\mathcal{G}$ has a single function g_0, meaning that $\mathcal{G} = \{g_0\}$, then, instead of $\mathcal{G}+\mathcal{H} = \{g_0\}+\mathcal{H}$ we simply write $g_0 + \mathcal{H}$.

We will denote by $\mathcal{C}$ the set of all constant functions defined on I with values in $\mathbb{R}$, meaning

$$\mathcal{C} = \{f : I \to \mathbb{R} : \text{ există } c \in \mathbb{R} \text{ such that } f(x) = c, \text{ for all } x \in I\}.$$

We immediately notice that:

a) $\mathcal{C} + \mathcal{C} = \mathcal{C}$;

$b)$ $a\mathcal{C} = \mathcal{C}$, for all $a \in \mathbb{R}$, $a \neq 0$,
meaning that the sum of two constant functions is a constant function as well, and a constant function multiplied by a real number is also a constant function.

Let us recall that if $F_0 : I \to \mathbb{R}$ is an antiderivative of the function $f : I \to \mathbb{R}$ on the interval $I \subseteq \mathbb{R}$, then for any other antiderivative $F : I \to \mathbb{R}$ of f on I is of the form $F = F_0 + c$, where $c : I \to \mathbb{R}$ is a constant function, therefore belonging to $c \in \mathcal{C}$. Then

$$\int f(x)\mathrm{d}x = \{F \in \mathfrak{F}(I, \mathbb{R}) : \ F \text{ is an antiderivative of } f \text{ on } I\} =$$

$$= \{F_0 + c : \ c \in \mathcal{C}\} = F_0 + \mathcal{C}.$$

Remark 3.3.7 Let $f : I \to \mathbb{R}$ be a function that has antiderivatives on I and let $F_0 : I \to \mathbb{R}$ be an antiderivative of f I. By considering the remark 3.3.5, we obtain

$$\int f(x)\mathrm{d}x = \{ \ F : I \to \mathbb{R} : \ F \text{ is an antiderivative of the function } f\} = F_0 + \mathcal{C}.$$

It follows that

$$\int f(x)\mathrm{d}x + \mathcal{C} = (F_0 + \mathcal{C}) + \mathcal{C} = F_0 + (\mathcal{C} + \mathcal{C}) = F_0 + \mathcal{C},$$

hence

$$\int f(x)\mathrm{d}x + \mathcal{C} = \int f(x)\mathrm{d}x.$$

Remark 3.3.8 If the function $f : I \to \mathbb{R}$ has antiderivatives on the interval I and $F : I \to \mathbb{R}$ is an antiderivative of the function f on I, then

$$\int f(x)\mathrm{d}x = F + \mathcal{C}$$

or

$$\int F'(x)\mathrm{d}x = F + \mathcal{C}.$$

3.1. Antiderivatives of Continuous Functions. In the following we prove that continuous functions have antiderivatives.

Theorem 3.3.9 *Let I be an interval in $\mathbb{R}$, $x_0 \in I$ and $f : I \to \mathbb{R}$ be a locally Riemann integrable function on I. If the function f is continuous at the point x_0, then for all $a \in I$, the function $F : I \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\,dt, \ \textit{for all } x \in I,$$

is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. Obviously $F(a) = 0$. Let $\varepsilon > 0$. Since the function f is continuous x_0, there exits a real number $\delta > 0$ such that for all $t \in I$ with $|t - x_0| < \delta$ it holds

$$|f(t) - f(x_0)| < \varepsilon/2,$$

or, equivalently,

$$f(x_0) - \frac{\varepsilon}{2} < f(t) < f(x_0) + \frac{\varepsilon}{2}.$$

Let $x \in I\backslash\{x_0\}$ with $|x - x_0| < \delta$. We distinguish two cases:

Case 1: $x > x_0$; then, for all $t \in [x_0, x]$, it holds

$$f(x_0) - \frac{\varepsilon}{2} < f(t) < f(x_0) + \frac{\varepsilon}{2},$$

and thus

$$\int_{x_0}^{x} \left(f(x_0) - \frac{\varepsilon}{2}\right) \mathrm{d}t \leq \int_{x_0}^{x} f(t)\, \mathrm{d}t \leq \int_{x_0}^{x} \left(f(x_0) + \frac{\varepsilon}{2}\right) \mathrm{d}t,$$

thus is follows that

$$\left(f(x_0) - \frac{\varepsilon}{2}\right)(x - x_0) \leq F(x) - F(x_0) \leq \left(f(x_0) + \frac{\varepsilon}{2}\right)(x - x_0),$$

or, equivalently

$$f(x_0) - \frac{\varepsilon}{2} \leq \frac{F(x) - F(x_0)}{x - x_0} \leq f(x_0) + \frac{\varepsilon}{2}.$$

In conclusion

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon$$

Case 2: $x > x_0$; then for each $t \in [x_0, x]$, it holds

$$f(x_0) - \frac{\varepsilon}{2} < f(t) < f(x_0) + \frac{\varepsilon}{2},$$

and thus

$$\int_{x}^{x_0} \left(f(x_0) - \frac{\varepsilon}{2}\right) \mathrm{d}t \leq \int_{x}^{x_0} f(t)\, \mathrm{d}t \leq \int_{x}^{x_0} \left(f(x_0) + \frac{\varepsilon}{2}\right) \mathrm{d}t,$$

hence

$$\left(f(x_0) - \frac{\varepsilon}{2}\right)(x_0 - x) \leq F(x_0) - F(x) \leq \left(f(x_0) + \frac{\varepsilon}{2}\right)(x_0 - x),$$

or, equivalently,

$$f(x_0) - \frac{\varepsilon}{2} \leq \frac{F(x) - F(x_0)}{x - x_0} \leq f(x_0) + \frac{\varepsilon}{2}.$$

In conclusion

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon$$

This means that for all $x \in I\backslash\{x_0\}$ cu $|x - x_0| < \delta$ it holds

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon.$$

It follows that there exist

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0),$$

thus F is differntiable at the point x_0 and $F'(x_0)=f(x_0)$. ■

Remark 3.3.10 If the function F from theorem 3.3.9 is differentiable at x_0, it does not follow that the function f is continuous at the point x_0. Indeed, the functon $f:[0,1]\to\mathbb{R}$ defined by $f(x)=\lfloor x\rfloor$, for all $x\in[0,1]$, is not continuous at $x_0=1$, while the function $F:[0,1]\to\mathbb{R}$ defined by

$$F(x)=\int_0^x f(t)\mathrm{d}t=\int_0^x 0\mathrm{d}t=0,\ \text{for all } x\in[0,1],$$

is differentiable at the point1. ♢

Theorem 3.3.11 (*the theorem of existence of antiderivatives for continuous functions*) *Let I be an interval in $\mathbb{R}$, $a\in I$ and $f:I\to\mathbb{R}$. If the function f is continuous on the interval I, then the function $F:I\to\mathbb{R}$ defined by prin*

$$F(x)=\int_a^x f(t)\mathrm{d}t,\ \textit{for all } x\in I,$$

is an antiderivative of f on I, satisfying the property that $F(a)=0$.

Proof. Theorem 3.3.9 should be applied. ■

Theorem 3.3.12 (*the representation theorem for antiderivatives of continuous functions*) *Let I be an interval from $\mathbb{R}$, $a\in I$ and $f:I\to\mathbb{R}$ be a continuous function on I. If $F:I\to\mathbb{R}$ is an antiderivative of the function f on I with the property that $F(a)=0$, then*

$$F(x)=\int_a^x f(t)\mathrm{d}t,\ \textit{for all } x\in I.$$

Proof. According to the existence theorem for antiderivatives of continuous functions (teorema 3.3.11), the function $F_1:I\to\mathbb{R}$ defined by

$$F_1(x)=\int_a^x f(t)\mathrm{d}t,\ \text{for all } x\in I,$$

is an antiderivative of the function f on I. Then there exits $c\in\mathbb{R}$ such that $F(x)=F_1(x)+c$, for all $x\in I$. Since $F(a)=F_1(a)=0$, we deduce that $c=0$ and the theorem is proved. ■

Theorem 3.3.13 *Let I be an interval from $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a locally integrable function on I. If the function f is bounded on I, for all $a \in I$, the function $F : I \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\mathrm{d}t, \text{ for all } x \in I,$$

is Lipshcitz on I.

Proof. The function f is bounded I, then there exits a real number $M > 0$ such that

$$|f(t)| \leq M, \text{ for all } x \in I.$$

Therefore, for all $u, v \in I$, it holds

$$|F(u) - F(v)| = \left|\int_u^v f(t)\,\mathrm{d}t\right| \leq \left|\int_u^v |f(t)|\,\mathrm{d}t\right| \leq M\,|u - v|,$$

consequently, the function F Lipschitz. ■

4. The Leibniz-Newton Formula

Theorem 3.4.1 (teorema lui Leibniz – Newton) *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function. If*

(*i*) *the function f is Riemann integrable on $[a, b]$;*

(*ii*) *the function f has antiderivatives on $[a, b]$,*

then for each antiderivative $F : [a, b] \to \mathbb{R}$ of the function f the following equality holds:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{3.4.28}$$

Proof. Let $(\Delta^n)_{n\in\mathbb{N}}$ be a sequence of partitions $\Delta^n = (x_0^n, \cdots, x_{p_n}^n)$ of the interval $[a, b]$ such that $\lim\limits_{n\to\infty} \|\Delta^n\| = 0$. According to Cauchy's theorem applied to the function F on the interval $[x_{i-1}^n, x_i^n]$, $(n \in \mathbb{N})$ we conclude that there exits for each natural number n and for each $i \in \{1, \cdots, p_n\}$, a point $\xi_i^n \in]x_{i-1}^n, x_i^n[$ with the property that

$$F(x_i^n) - F\left(x_{i-1}^n\right) = F'(\xi_i^n)\left(x_i^n - x_{i-1}^n\right).$$

From the hypothesis $F'(x) = f(x)$, for all $x \in [a, b]$, we get

$$F(x_i^n) - F\left(x_{i-1}^n\right) = f(\xi_i^n)\left(x_i^n - x_{i-1}^n\right),$$

for all natural numbers n and for all $i \in \{1, \cdots, p_n\}$.

Obviously, for each natural number n it holds $\xi^n = \left(\xi_1^n, \cdots, \xi_{p_n}^n\right) \in \mathrm{Pi}(\Delta^n)$. Since

$$\sigma(f; \Delta^n, \xi^n) = \sum_{i=1}^{p_n} f(\xi_i^n)\left(x_i^n - x_{i-1}^n\right) = \sum_{i=1}^{p_n} F(x_i^n) - F\left(x_{i-1}^n\right) =$$
$$= F(b) - F(a), \text{ for all } n \in \mathbb{N},$$

because

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sigma\left(f;\Delta^n,\xi^n\right),$$

we get

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

■

Notation: Instead of $F(b) - F(a)$ we usually denote:

$$F(x)|_a^b \qquad \text{sau} \qquad [F(x)]_a^b$$

which is read $F(x)$ *taken between* a *and* b.

The equality (3.4.28) is called **the Leibniz-Newton fromula**.

Example 3.4.2 The funciton $f : [1,2] \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x(x+1)},\ \text{for all } x \in [1,2],$$

is continuous on $[1,2]$. Then the function f is Riemann integrable on $[1,2]$. On the other side, the function f has antiderivatives on the interval $[1,2]$ and $F : [1,2] \to \mathbb{R}$ defined by

$$F(x) = \ln x - \ln(x+1),\ \text{for all } x \in [1,2],$$

is an antiderivative of the function f on $[1,2]$. Acoording to Leibniz-Newton's formula (teorema 3.4.1), we get

$$\int_1^2 \frac{1}{x(x+1)}\mathrm{d}x = [\ln x - \ln(x+1)]_1^2 = \ln\frac{4}{3}. \ \square$$

5. Computational Methods

5.1. The Side Integrations. By using the formula for determining the derivative for the product of two differentiable functions and the result that each continuous function on an interval has antiderivatives, we obtain the following theorem:

Theorem 3.5.1 (the side integration formula) *Let I be an interval from $\mathbb{R}$ şi $f, g : I \to \mathbb{R}$. If:*

(i) the functions f and g are differentiable on I,

(ii) the derivatives f' and g' are continous on I,

then the functions fg' and $f'g$ have antiderivatives on I and the following equality holds:

$$\int (fg')(x)\mathrm{d}x = fg - \int (f'g)(x)\mathrm{d}x.$$

(*the side integration formula*)

Remark 3.5.2 Briefly, the side integration formula is written as

$$\int fg' = fg - \int f'g.$$

Example 3.5.3 Compute the integral

$$\int x \ln x \mathrm{d}x, \;\; x \in]0, +\infty[;$$

Solution. We consider the functions $f, g :]0, +\infty[\to \mathbb{R}$ defined by

$$f(x) = \ln x, \quad g'(x) = x, \;\; \text{for all } x \in]0, +\infty[.$$

We deduce that $g(x) = \dfrac{x^2}{2}$, for all $x \in]0, +\infty[$. By applying the side integration formula we get

$$\begin{aligned} \int x \ln x \mathrm{d}x &= \frac{x^2}{x} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \mathrm{d}x = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \mathrm{d}x = \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + \mathcal{C}, \;\; x \in]0, +\infty[. \end{aligned}$$

■

5.2. The Variable Change in Integration. The change of variable in the integral relies on the formula for the composition of functions.

Theorem 3.5.4 (*the first variable change*) *Let I and J be two intervals from $\mathbb{R}$ and $f : J \to \mathbb{R}$ and $u : I \to \mathbb{R}$ be two functions. Then:*

(i) $u(I) \subseteq J$;

(ii) *the function u is differentiable on I;*

(iii) *the function f has antiderivatives on J,*

and the function $(f \circ u) u'$ has antiderivatives on I.

Moreover, if $F : J \to \mathbb{R}$ is an antiderivative of the function f on J, then the function $F \circ u$ is an antiderivative of $(f \circ u) u'$ on I and the following equality holds:

$$\int f(u(x)) u'(x) \mathrm{d}x = F \circ u + \mathcal{C}.$$

Remark 3.5.5 Let I be an interval on $\mathbb{R}$. In order to compute the antiderivatives of the function $g : I \to \mathbb{R}$, more precisely, in order to compute the integral

$$\int g(x) \mathrm{d}x,$$

by using the variable change method, one should follow the next steps: 1^0 We emphasize, in the formulation of g, a differentiable function $u : I \to \mathbb{R}$ and a function primitivabilă $f : u(I) \to \mathbb{R}$ such that $g(x) = f(u(x))\, u'(x)$, for all $x \in I$.

2^0 We determine an antiderivative $F : u(I) \to \mathbb{R}$ of the function f on $u(I)$, namely

$$\int f(t)\,\mathrm{d}t = F + \mathcal{C}.$$

3^0 An antiderivative of the function $g = (f \circ u)\, u'$ on I is $F \circ u$, namely

$$\int g(x)\,\mathrm{d}x = F \circ u + \mathcal{C},$$

or, equivalently

$$\int g(x)\,\mathrm{d}x = F(u(x)) + \mathcal{C},\ \ x \in I.$$

Example 3.5.6 Compute the integral

$$\int \cot x \mathrm{d}x,\ \ x \in]0, \pi[.$$

Solution. We have $I =]0, \pi[$ şi $g(x) = \cot x$, for all $x \in]0, \pi[$. Since

$$g(x) = \frac{1}{\sin x}(\sin x)',\ \text{ for all } x \in]0, \pi[,$$

we consider $u :]0, \pi[\to \mathbb{R}$ defined by $u(x) = \sin x$, for all $x \in]0, \pi[$ and $f :]0, +\infty[\to \mathbb{R}$ defined by $f(t) = 1/t$, for all $t \in]0, +\infty[$. Evident

$$g(x) = f(u(x))\, u'(x),\ \text{ for all } x \in]0, +\infty[.$$

An antiderivative of the function f on $]0, +\infty[$ is the function $F :]0, +\infty[\to \mathbb{R}$ defined by

$$F(t) = \ln t,\ \text{ for all } t \in]0, +\infty[,$$

namely

$$\int f(t)\,\mathrm{d}t = \int \frac{1}{t}\mathrm{d}t = \ln t + \mathcal{C},\quad t \in]0, +\infty[.$$

Then, an antiderivative of g pe $]0, +\infty[$ is $F \circ u$, therefore, we have

$$\int \cot x \mathrm{d}x = \ln|\sin x| + \mathcal{C},\quad x \in]0, \pi[.$$

■

Theorem 3.5.7 (*the second method of variable change*) *Let I and J be two intervals from $\mathbb{R}$ şi $f : I \to \mathbb{R}$ and $u : J \to I$ be two function. If:*

(i) *the function u is bijective;*

(ii) *the function u is differentiable on J and $u'(x) \neq 0$, for all $x \in J$;*

(iii) the function $h = (f \circ u)\, u'$ has antiderivatives on J,
then the function f has antiderivatives on I.

Moreover, if $H : J \to \mathbb{R}$ is an antiderivative of the function $h = (f \circ u)\, u'$ pe J, then the function $H \circ u^{-1}$ is an antiderivative of f on I, consequently, the following equality holds:

$$\int f(x)\,\mathrm{d}x = H \circ u^{-1} + \mathcal{C}.$$

Remark 3.5.8 Let I be an interval from $\mathbb{R}$. In order to compute the antiderivatives of a function for which we know that has antiderivatives $f : I \to \mathbb{R}$, namely, in order to compute the integral

$$\int f(x)\,\mathrm{d}x,$$

by using the variable change from theorem 3.5.7, one should follow the next three steps:

1^0 Emphasize an interval $J \subseteq \mathbb{R}$ and a function $u : J \to I$ bijective, differentiable on J and with a derivative different from zero on J (It is said that the function u^{-1} changes its x variable into the variable t).

2^0 Determine an antiderivative $H : J \to \mathbb{R}$ of the function $(f \circ u)\, u'$ pe J, adică

$$\int f(u(t))\,\mathrm{d}t = H + \mathcal{C}.$$

3^0 An antiderivative of the function f on I is $H \circ u^{-1}$, adică

$$\int f(x)\,\mathrm{d}x = H \circ u^{-1} + \mathcal{C},$$

or, equivalently,

$$\int f(x)\,\mathrm{d}x = H\left(u^{-1}(x)\right) + \mathcal{C},\ x \in I.$$

Example 3.5.9 Compute the integral

$$\int \frac{1}{\sin x}\mathrm{d}x,\ x \in]0, \pi[.$$

It holds $I :=]0, \pi[$. We consider the function $u :]0, +\infty[\to]0, \pi[$ defined by $u(t) = 2 \arctan t$, for all $t \in]0, +\infty[$. The function u bijective and differentiable.

Remark 3.5.10 Let I and J be two intervals from $\mathbb{R}$ and $f : J \to \mathbb{R}$ and $u : I \to J$ be two functions with the following properties:

(a) the function u is bijective, differntiable on I with continuous derivative and different from zero on I;

(b) the function f is continuous on J.

Let $F : J \to \mathbb{R}$ be an antiderivative of the function f on J (such an antiderivative exists because f is continuous on J).

From the first variable change method (theorem 3.5.4), the function $F \circ u$ is an antiderivative of the function $(f \circ u)\, u'$ on I.

Conversely, assume that $H = F \circ u$ is an antiderivative of the function $(f \circ u)\, u'$ on I. Then, according to the second method of variable change (theorem 3.5.7), the function $H \circ u^{-1} = F \circ u \circ u^{-1} = F$ is an antiderivative of f on J.

Consequently, from (a) and (b), the function $F : I \to \mathbb{R}$ is an antiderivative of the functin f on J if and only if the function $F \circ u$ is an antiderivative of the function $(f \circ u)\, u'$ on I. Hence, under the hypotheses (a) and (b), the two methods coincide.

In fact, there exists just one method of variable change, with multiple instances.

Instance 1. We must compute

$$\int f(x)\, \mathrm{d}x, \quad x \in I.$$

Then:

1^0 We emphasize in the formulation of f, a function $u : I \to \mathbb{R}$ and a function $g : u(I) \to \mathbb{R}$ which has antiderivatives, such that

$$f(x) = g(u(x))\, u'(x)\,, \text{ for all } x \in I.$$

2^0 We make the following formal substitions $u(x) := t$ şi $u'(x)\, dx := dt$; and we get the indefinite integral

$$\int g(t)\, \mathrm{d}t = G(t) + \mathcal{C},\ t \in u(I)\,.$$

3^0 We go back to the old variable x, by putting $t := u(x)$ in the expression of the antidrivative G; we get

$$\int f(x)\, \mathrm{d}x = G(u(x)) + \mathcal{C},\ x \in I.$$

Instance 2. We compute

$$\int f(x)\, \mathrm{d}x, \quad x \in I.$$

Then:

1^0 We emphasize an interval $J \subseteq \mathbb{R}$ and a function $u : J \to I$ bijective and differentiable.

2^0 We make the formal substitutions $x := u(t)$ and $dx := u'(t)\, dt$; and get the indefinite integral

$$\int f(u(t))\, u'(t)\, \mathrm{d}t,\ t \in J,$$

which can be computed. Let

$$\int f(u(t))\, u'(t)\, \mathrm{d}t = H(t) + \mathcal{C},\ t \in J.$$

3^0 We go back to the old variable x, by setting $t := u^{-1}(x)$ in the expresion of the antidrivative H; we get

$$\int f(x)\,\mathrm{d}x = H\left(u^{-1}(x)\right) + \mathcal{C},\ x \in I.$$

Instance 3. We compute

$$\int f(x)\,\mathrm{d}x, \quad x \in I.$$

Then:

1^0 We emphasize in the formulation of f an injective and differentiable function $u : I \to \mathbb{R}$ with $u^{-1} : u(I) \to I$ and a function $g : u(I) \to \mathbb{R}$ such that

$$f(x) = g(u(x)),\ \text{for all } x \in I.$$

2^0 We make the formal substitution $u(x) := t$ and $dx := \left(u^{-1}\right)'(t)\,dt$; we get the indefinite integral

$$\int g(t)\left(u^{-1}\right)'(t)\,\mathrm{d}t,\ t \in u(I),$$

that we have to compute. Let

$$\int g(t)\left(u^{-1}\right)'(t)\,\mathrm{d}t = F(t),\ t \in u(I),$$

3^0 Going back to the first variable x, by setting $t := u(x)$ in the expression of the antiderivative F; we get

$$\int f(x)\,\mathrm{d}x = G(u(x)) + \mathcal{C},\ x \in I.$$

The expressions of the functions u are impose by the particular instance of the function f.

Example 3.5.11 Compute

$$I = \int \frac{\mathrm{tg}x}{1+\mathrm{tg}x}\mathrm{d}x,\ x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).$$

We make the substitution $\tan x = t$, hence $x := \arctan t$ şi $dx := \dfrac{1}{1+t^2}$. We get

$$I = \int \frac{t}{(1+t)(1+t^2)}\mathrm{d}t = \frac{1}{2}\int\left(\frac{1}{t^2+1} + \frac{1}{t^2+1} - \frac{1}{t+1}\right)\mathrm{d}t =$$
$$= \frac{1}{4}\left(t^2+1\right) + \frac{1}{2}\arctan t - \frac{1}{2}\ln(t+1) + \mathcal{C},\ t \in (-1, +\infty).$$

Then

$$I = \int \frac{\tan x}{1+\tan x}\mathrm{d}x = \frac{1}{2}\left(x - \ln(\sin x + \cos x)\right) + \mathcal{C},\ x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).$$

Remark 3.5.12 There are computational rules for the antiderivatives, only when considering a rather restrict family of functions.

Remark 3.5.13 For further details see [**5**] and [**3**].

6. Exercise to be Solved

Example 3.6.1 Prove that the following functions $f : I \to \mathbb{R}$ have antiderivatives on the interval $I \subseteq \mathbb{R}$ and determine one antiderivative $F : I \to \mathbb{R}$ of the function f on the interval I, if:

a) $f(x) = x^2 + x$, for all $x \in I = \mathbb{R}$;

b) $f(x) = x^3 + 2x - 4$, for all $x \in I = \mathbb{R}$;

c) $f(x) = x(x+1)(x+2)$, oricare ar fi $x \in I = \mathbb{R}$;

d) $f(x) = 1/x$, for all $x \in I =]0, +\infty[$;

e) $f(x) = 1/x$, for all $x \in I =]-\infty, 0[$;

f) $f(x) = x^5 + 1/x$, for all $x \in I =]0, +\infty[$;

g) $f(x) = 1/x^2$, for all $x \in I =]0, +\infty[$;

h) $f(x) = 1/x^2$, for all $x \in I =]-\infty, 0[$.

Example 3.6.2 Compute:

a) $\displaystyle\int \frac{2x-1}{x^2-3x+2}\mathrm{d}x, \quad x \in]2, +\infty[$;

b) $\displaystyle\int \frac{4}{(x-1)(x+1)^2}\mathrm{d}x, \quad x > 1$;

c) $\displaystyle\int \frac{1}{x^3-x^4}\mathrm{d}x, \quad x > 1$;

d) $\displaystyle\int \frac{2x+5}{x^2+5x+10}, \ x \in \mathbb{R}$;

e) $\displaystyle\int \frac{1}{x^2+x+1}, \ x \in \mathbb{R}$.

Example 3.6.3 Compute:

a) $\displaystyle I = \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\mathrm{d}x, \quad x \in]0, +\infty[$;

b) $\displaystyle I = \int \frac{1}{x+\sqrt{x-1}}\mathrm{d}x, \quad x \in]1, +\infty[$.

Example 3.6.4 Compute:

a) $\displaystyle I = \int \frac{1}{1+\sqrt{x^2+2x-2}}\mathrm{d}x, \quad x \in]\sqrt{3}-1, +\infty[$;

b) $I = \int \frac{1}{(x+1)\sqrt{-4x^2-x+1}}\mathrm{d}x, \quad x \in]\frac{-1-\sqrt{17}}{8}, \frac{\sqrt{17}-1}{8}[.$

Example 3.6.5 Compute:

a) $\int_1^2 \frac{1}{x^3+x^2+x+1}\mathrm{d}x;$ *b*) $\int_1^3 \frac{1}{x(x^2+9)}\mathrm{d}x;$

c) $\int_{-1}^1 \frac{x^2+1}{x^4+1}\mathrm{d}x;$ *d*) $\int_{-1}^1 \frac{x}{x^2+x+1}\mathrm{d}x.$

Example 3.6.6 Compute:

a) $\int_{-3}^{-2} \frac{x}{(x+1)(x^2+3)}\mathrm{d}x;$ *b*) $\int_0^1 \frac{x+1}{(x^2+4x+5)^2}\mathrm{d}x;$

c) $\int_1^2 \frac{1}{x^3+x}\mathrm{d}x;$ *d*) $\int_0^2 \frac{x^3+2x^2+x+4}{(x+1)^2}\mathrm{d}x.$

Example 3.6.7 Compute:

a) $\int_0^1 \frac{1}{1+x^4}\mathrm{d}x;$ *b*) $\int_0^1 \frac{1}{(x+1)(x^2+4)}\mathrm{d}x;$

c) $\int_2^3 \frac{2x^3+x^2+2x-1}{x^4-1}\mathrm{d}x;$ *d*) $\int_0^1 \frac{x^3+2}{(x+1)^3}\mathrm{d}x.$

Example 3.6.8 Compute:

a) $\int_{-1}^1 \frac{1}{\sqrt{4-x^2}}\mathrm{d}x;$ *b*) $\int_0^1 \frac{1}{\sqrt{x^2+x+1}}\mathrm{d}x;$

c) $\int_{-1}^1 \frac{1}{\sqrt{4x^2+x+1}}\mathrm{d}x;$ *d*) $\int_2^3 \frac{x^2}{(x^2-1)\sqrt{x^2-1}}\mathrm{d}x.$

Example 3.6.9 Compute:

a) $\int_2^3 \sqrt{x^2+2x-7}\mathrm{d}x;$ *b*) $\int_0^1 \sqrt{6+4x-2x^2}\mathrm{d}x;$

c) $\int_0^{3/4} \frac{1}{(x+1)\sqrt{x^2+1}}\mathrm{d}x;$ *d*) $\int_2^3 \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x.$

Example 3.6.10 Prove that:

a) $2\sqrt{2} < \int_{-1}^1 \sqrt{x^2+4x+5}\mathrm{d}x < 2\sqrt{10};$

b) $\mathrm{e}^2(\mathrm{e}-1) < \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{x}{\ln x} \mathrm{d}x < \frac{\mathrm{e}^3}{2}(\mathrm{e}-1)$.

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Index