József Wildt International Mathematical Competition

The Edition XXIXth , 2019 [1]

The solution of the problems W.1 - W.70 must be mailed before 26. October 2019, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: benczemihaly@gmail.com; benczemihaly@yahoo.com

W1. The Pell numbers P_n satisfy $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Find

$$\sum_{n=1}^{\infty} \left(\arctan \frac{1}{P_{2n}} + \arctan \frac{1}{P_{2n+2}} \right) \arctan \frac{2}{P_{2n+1}}.$$

Ángel Plaza

W2. If $0 < a \leq c \leq b$ then:

$$\frac{(b^{30} - a^{30})(b^{30} - c^{30})}{36b^{10}} \leq \frac{(b^{25} - a^{25})(b^{25} - c^{25})}{25} \leq \frac{(b^{30} - a^{30})(b^{30} - c^{30})}{36(ac)^5}$$

Daniel Sitaru

W3. Compute

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x + 1 - x^2}{(1 + x \sin x)\sqrt{1 - x^2}} dx$$

D.M. Bătineţu-Giurgiu and Stanciu Neculai

W4. If $x, y, z, t > 1$ then:

$$\left(\log_{zxt} x \right)^2 + \left(\log_{xyt} y \right)^2 + \left(\log_{xyz} z \right)^2 + \left(\log_{yzt} t \right)^2 > \frac{1}{4}$$

Daniel Sitaru

W5. Let $n \geq 1$. Find a set of distincts real numbers $(x_j)_{1 \leq j \leq n}$ such that for any bijections $f : \{1; 2; ...; n\}^2 \to \{1; 2; ...; n\}^2$ the matrix $\left(x_{f(i,j)} \right)_{1 \leq i,j \leq n}$ is invertible.

Moubinool Omarjee

W6. Compute

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1 + \ln x)\cos x + x \sin x \ln x}{\cos^2 x + x^2 \ln^2 x} dx$$

D.M. Bătineţu-Giurgiu and Stanciu Neculai

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W7. If

$$\Omega_n = \sum_{k=1}^{n}\left(\int_{-\frac{1}{k}}^{\frac{1}{k}}(2x^{10}+3x^8+1)\cdot\cos^{-1}(kx)dx\right)$$

then find:

$$\Omega = \lim_{n\to\infty}(\Omega_n - \pi\cdot H_n)$$

Daniel Sitaru

W8. Let $(a_n)_{n\geq 1}$ be a positive real sequence given by $a_n = \sum\limits_{k=1}^{n}\frac{1}{k}$. Compute

$$\lim_{n\to\infty} e^{-2a_n}\sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2\right]$$

where we denote by $[x]$ the integer part of x.

D.M. Bătineţu-Giurgiu and Stanciu Neculai

W9. Let $\alpha > 0$ be a real number. Compute the limit of the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n = \begin{cases}\sum\limits_{k=1}^{n}\sinh\left(\dfrac{k}{n^2}\right), & n > \frac{1}{\alpha};\\ 0, & n \leq \frac{1}{\alpha}\end{cases}.$$

José Luis Díaz-Barrero

W10. If $si(x) = -\int_x^\infty\left(\frac{\sin t}{t}\right)dt; x > 0$ then:

$$\int_e^{e^2}\left(\frac{1}{x}(si(e^4x)-si(e^3x)\right)dx = \int_3^{e^4}\left(\frac{1}{x}(si(e^2x)-si(ex)\right)dx$$

Daniel Sitaru

W11. Let $(s_n)_{n\geq 1}$ be a sequence given by $s_n = -2\sqrt{n}+\sum\limits_{k=1}^{n}\frac{1}{\sqrt{k}}$ with $\lim\limits_{n\to\infty} s_n = s =$ Ioachimescu constant and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be a positive real sequences such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a \in R_+^*, \lim_{n\to\infty}\frac{b_{n+1}}{b_n\sqrt{n}} = b \in R_+^*$$

Compute

$$\lim_{n\to\infty}\left(1+e^{s_n}-e^{s_{n+1}}\right)^{\sqrt[n]{a_n b_n}}$$

D.M. Bătineţu-Giurgiu and Stanciu Neculai

W12. If $0 < a < b$ then:

$$\frac{\int_a^{\frac{a+b}{2}}(\tan^{-1}t)dt}{\int_a^b(\tan^{-1}t)dt} < \frac{1}{2}$$

Daniel Sitaru

W13. Let a, b and c be complex numbers such that $abc = 1$. Find the value of the cubic root of

$$\begin{vmatrix} b+n^3c & n(c-b) & n^2(b-c) \\ n^2(c-a) & c+n^3a & n(a-c) \\ n(b-a) & n^2(a-b) & a+n^3b \end{vmatrix}$$

José Luis Díaz-Barrero

W14. If $a, b, c > 0; ab + bc + ca = 3$ then:

$$4(\tan^{-1} 2)(\tan^{-1}(\sqrt[3]{abc})) \leq \pi \tan^{-1}(1 + \sqrt[3]{abc})$$

Daniel Sitaru

W15. It is possible to partition the set $\{100, 101, \ldots, 1000\}$ into two subsets so that for any two distinct elements x and y belonging to the same subset $\sqrt[3]{x+y}$ is irrational?

José Luis Díaz-Barrero

W16. If $f : [a, b] \to (0, \infty); 0 < a \leq b; f$ derivable; f' continuous then:

$$\int_a^b \frac{f'(x)\sqrt{f(x)}}{f^3(x) + 1} dx \leq \tan^{-1}\left(\frac{f(b) - f(a)}{1 + f(a)f(b)}\right)$$

Daniel Sitaru

W17. Let $f_n = \left(1 + \frac{1}{n}\right)^n ((2n-1)!!F_n)^{1/n}$. Find $\lim\limits_{n\to\infty} (f_{n+1} - f_n)$ where F_n denotes the nth Fibonacci number (given by $F_0 = 0$, $F_1 = 1$, and by $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$.

Ángel Plaza

W18. Let $\{c_k\}_{k\geq 1}$ be a sequence with $0 \leq c_k \leq 1$, $c_1 \neq 0$, $\alpha > 1$. Let $C_n = c_1 + \ldots + c_n$. Prove

$$\lim_{n\to\infty} \frac{C_1^\alpha + \ldots + C_n^\alpha}{(C_1 + \ldots + C_n)^\alpha} = 0$$

Perfetti Paolo

W19. Let $\{F_n\}_{n\in Z}$ and $\{L_n\}_{n\in Z}$ denote the Fibonacci and Lucas numbers, respectively, given by

$$F_{n+1} = F_n + F_{n-1} \text{ and } L_{n+1} = L_n + L_{n-1} \text{ for all } n \geq 1,$$

with $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. Prove that for integers $n \geq 1$ and $j \geq 0$

(i) $\sum_{k=1}^n F_{k\pm j}L_{k\mp j} = F_{2n+1} - 1 + \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\pm j}F_{\pm 2j}, & \text{if } n \text{ is odd} \end{cases}$.

(ii) $\sum_{k=1}^n F_{k+j}F_{k-j}L_{k+j}L_{k-j} = \frac{F_{4n+2}-1-nL_{4j}}{5}$.

Ángel Plaza

W20. i). Let G be a $(4, 4)$ unoriented graph, 2-regulate, containing a cycle with the length 3. Find the characteristic polynomial $P_G(\lambda)$, its spectrum $Spec(G)$ and draw the graph G.
ii). Let G' be another 2-regulate graph, having its characteristic polynomial $P_{G'}(\lambda) = \lambda^4 - 4\lambda^2 + \alpha$, $\alpha \in R$. Find the spectrum Spec(G') and draw the graph G'.
iii). Are the graphs G and G' cospectral or isomorphic?

Laurenţiu Modan

W21. Let f be a continuously differentiable function on $[0,1]$ and $m \in \mathbb{N}$. Let $A = f(1)$ and let $B = \int_0^1 x^{-\frac{1}{m}} f(x)dx$. Calculate

$$\lim_{n\to\infty} n(\int_0^1 f(x)dx - \sum_{k=1}^{n} (\frac{k^m}{n^m} - \frac{(k-1)^m}{n^m}) f(\frac{(k-1)^m}{n^m}))$$

in terms of A and B.

Li Yin

W22. Let A and B the series:

$$A = \sum_{n>0} \frac{C_{2n}^1}{C_{2n}^0 + C_{2n}^1 + ... + C_{2n}^{2n}}, \quad B = \sum_{n>0} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{5}{2}\right)}.$$

Study if $\frac{A}{B}$ is irrational number.

Laurenţiu Modan

W23. If b, c are the legs, and a is the hypotenuse of a right triangle, prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 5 + 3\sqrt{2}$$

Ovidiu Pop

W24. If $a, b, c > 0$, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a+b}{a+b+2c} + \frac{b+c}{b+c+2a} + \frac{c+a}{c+a+2b}$$

Ovidiu Pop

W25. Let $x_i, y_i, z_i, \omega_i \in R^+, i = 1, 2 \cdots n$, such that

$$\sum_{i=1}^{n} x_i = nx, \sum_{i=1}^{n} y_i = ny, \sum_{i=1}^{n} \omega_i = n\omega$$

$$\Gamma(z_i) \geqslant \Gamma(\omega_i), \sum_{i=1}^{n} \Gamma(z_i) = n\Gamma^*(z).$$

Then

$$\sum_{i=1}^{n} \frac{(\Gamma(x_i)+\Gamma(y_i))^2}{\Gamma(z_i)-\Gamma(\omega_i)} \geqslant n\frac{(\Gamma(x)+\Gamma(y))^2}{\Gamma^*(z)-\Gamma(\omega)}.$$

Li Yin

W26. Let $n \in N, n \geq 2, a_1, a_2, ..., a_n \in R$ and $a_n = \max\{a_1, a_2, ..., a_n\}$
a). If $t_k, t'_k \in R, k \in \{1, 2, ..., n\}, t_k \leq t'_k$, for any $k \in \{1, 2, ..., n-1\}$ and

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} t'_k$$

prove that

$$\sum_{k=1}^{n} t_k a_k \geq \sum_{k=1}^{n} t'_k a_k$$

b). If $b_k, c_k \in R_+^*$, $k \in \{1, 2, ..., n\}, b_k \leq c_k$ for any $k \in \{1, 2, ..., k-1\}$ and

$$b_1 \cdot b_2 \cdot \ldots \cdot b_n = c_1 \cdot c_2 \cdot \ldots \cdot c_n$$

prove that

$$\prod_{k=1}^{n} b_k^{a_k} \geq \prod_{k=1}^{n} c_k^{a_k}$$

Ovidiu Pop and Petru Braica

W27. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(-x) + \int_0^x t f(x-t) \mathrm{d}t = x, \quad \forall x \in \mathbb{R}.$$

Ovidiu Furdui and Alina Sîntămărian

W28. In a room, we have 2019 aligned switches, connected to 2019 light bulbs, all initially switched on. Then, 2019 people enter the room one by one, performing the operation: The first, uses all the switches; the second, every second switch; the third, every third switch, and so on. How many lightbulbs remain switched on, after all the people entered ?

Ovidiu Bagdasar

W29. Prove that

$$\int_0^{\infty} e^{3t} \frac{4e^{4t}(3t-1) + 2e^{2t}(15t-17) + 18(1-t)}{(1+e^{4t}-e^{2t})^2} dt = 12 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 10$$

Perfetti Paolo

W30. a). Prove that

$$\lim_{n\to\infty} \left(n + \frac{1}{4} - \zeta(3) - \zeta(5) - \cdots - \zeta(2n+1) \right) = 0.$$

b). Calculate

$$\sum_{n=1}^{\infty} \left(n + \frac{1}{4} - \zeta(3) - \zeta(5) - \cdots - \zeta(2n+1) \right).$$

Ovidiu Furdui and Alina Sîntămărian

W31. Let $a, b \in \Gamma, a < b$ and the differentiable function $f : [a,b] \to \Gamma$, such that $f(a) = a$ and $f(b) = b$. Prove that

$$\int_a^b (f'(x))^2 \, dx \geq b - a$$

Dorin Mărghidanu

W32. Let u_k, v_k, a_k and b_k be non-negative real sequences such as $u_k > a_k$ and $v_k > b_k$, where $k = 1, 2, \ldots, n$. If $0 < m_1 \leq u_k \leq M_1$ and $0 < m_2 \leq v_k \leq M_2$, then

$$\sum_{k=1}^{n} (\ell u_k v_k - a_k b_k) \geq \left(\sum_{k=1}^{n} \left(u_k^2 - a_k^2 \right) \right)^{1/2} \left(\sum_{k=1}^{n} \left(v_k^2 - b_k^2 \right) \right)^{1/2}, \qquad (1.1)$$

where

$$\ell = \frac{M_1M_2 + m_1m_2}{2(m_1M_1m_2M_2)^{1/2}}. \tag{1.2}$$

Chang-Jian Zhao and Mihály Bencze

W33. Let $0 < \frac{1}{q} \le \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let u_k, v_k, a_k and b_k be non-negative real sequences such as $u_k^2 > a_k^p$ and $v_k > b_k^q$, where $k = 1, 2, \dots, n$. If $0 < m_1 \le u_k \le M_1$ and $0 < m_2 \le v_k \le M_2$, then

$$\left(\sum_{k=1}^{n}\left(\ell^p(u_k+v_k)^2 - (a_k+b_k)^p\right)\right)^{1/p} \ge$$
$$\ge \left(\sum_{k=1}^{n}(u_k^2 - a_k^p)\right)^{1/p} + \left(\sum_{k=1}^{n}(v_k^2 - b_k^p)\right)^{1/p}, \tag{1.5}$$

where ℓ is as in (1.2).

Chang-Jian Zhao and Mihály Bencze

W34. Let a, b, c be positive real numbers and let $m, n\,(m \ge n)$ be positive integers. Prove that

$$\frac{a^{n-1}b^{n-1}c^{m-n-1}}{a^{m+n} + b^{m+n} + a^nb^nc^{m-n}} + \frac{b^{n-1}c^{n-1}a^{m-n-1}}{b^{m+n} + c^{m+n} + b^nc^na^{m-n}} +$$
$$+\frac{c^{n-1}a^{n-1}b^{m-n-1}}{c^{m+n} + a^{m+n} + a^na^nb^{m-n}} \le \frac{1}{abc}$$

Dorin Mărghidanu and Kunihiko Chikaya

W35. Calculate

$$\lim_{n\to\infty} \frac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}.$$

Arkady Alt

W36. For any $a, b, c > 0$ and for any $n \in N^*$, prove the inequality

$$(a-b)\left(\frac{a}{b}\right)^n + (b-c)\left(\frac{b}{c}\right)^n + (c-a)\left(\frac{c}{a}\right)^n \ge (a-b)\frac{a}{b} + (b-c)\frac{b}{c} + (c-a)\frac{c}{a}$$

Dorin Mărghidanu

W37. For real $a > 1$ find

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=2}^{n}\left(a - a^{1/k}\right)}.$$

Arkady Alt

W38. Let a, b, c be the sides of an acute triangle ΔABC, then for any $x, y, z \ge 0$, such that

$$xy + yz + zx = 1,$$

holds inequality:

$$a^2x + b^2y + c^2z \ge 4F,$$

where F is the area of the triangle ΔABC.

Arkady Alt

W39. Let u, v, w complex numbers such that:

$$u+v+w=1,\ u^2+v^2+w^2=3, uvw=1$$

Prove that
a). u, v, w are distinct numbers two by two
b). if $S^{(k)} := u^k+v^k+w^k$, then $S^{(k)}$ is an odd natural number
c). the expression

$$\frac{u^{2n+1}-v^{2n+1}}{u-v}+\frac{v^{2n+1}-w^{2n+1}}{v-w}+\frac{w^{2n+1}-u^{2n+1}}{w-u}$$

is an integer number.

Dorin Mărghidanu

W40. Let f_n be $n-th$ Fibonacci number defined by recurrence $f_{n+1}-f_n-f_{n-1}=0, n\in\mathbb{N}$ and initial conditions $f_0=0, f_1=1$. Prove that for any $n\in\mathbb{N}$

$$(n-1)(n+1)(2nf_{n+1}-(n+6)f_n)$$

is divisible by 150 for any $n\in\mathbb{N}$.

Arkady Alt

W41. For $n\in\mathbb{N}$, consider in $\mathbb{R}^3$ the regular tetrahedron with vertices $O(0,0,0)$, $A(n,9n,4n)$, $B(9n,4n,n)$ and $C(4n,n,9n)$. Show that the number N of points (x,y,z), $(x,y,z\in\mathbb{Z})$ inside or on the boundary of the tetrahedron $OABC$ is given by

$$N=\frac{343n^3}{3}+\frac{35}{2}n^2+\frac{7}{6}n+1.$$

Eugen J. Ionascu

W42. For p, q, l strictly positive real numbers, consider the following problem: for $y\geq 0$ fixed, determine the values $x\geq 0$ such that $x^p-lx^q\leq y$. Denote by $S(y)$ the set of solutions of this problem. Prove that if one has $p<q$, $\varepsilon\in(0, l^{\frac{1}{p-q}})$, $0\leq x\leq\varepsilon$ and $x\in S(y)$, then

$$x\leq ky^{\delta},\ \text{where } k=\varepsilon(\varepsilon^p-l\varepsilon^q)^{-\frac{1}{p}} \text{ and } \delta=\frac{1}{p}.$$

József Kolumbán

W43. Consider the sequence of polynomials $P_0(x)=2$, $P_1(x)=x$ and $P_n(x)=xP_{n-1}(x)-P_{n-2}(x)$ for $n\geq 2$. Let x_n be the greatest zero of P_n in the the interval $|x|\leq 2$. Show that

$$\lim_{n\to\infty} n^2\left(4-2\pi+n^2\int_{x_n}^{2}P_n(x)dx\right)=2\pi-4-\frac{\pi^3}{12}.$$

Eugen J. Ionascu

W44. We consider a natural number $n, n\geq 2$ and the matrices

$$A=\begin{pmatrix} 1 & 2 & 3 & \ldots & n\\ n & 1 & 2 & \ldots & n-1\\ n-1 & n & 1 & \ldots & n-2\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 2 & 3 & 4 & \ldots & 1 \end{pmatrix}$$

Show that:

$$\varepsilon^n \det\left(I_n - A^{2n}\right) + \varepsilon^{n-1} \det\left(\varepsilon \cdot I_n - A^{2n}\right) + \varepsilon^{n-2} \det\left(\varepsilon^2 \cdot I_n - A^{2n}\right) + ...$$

$$+ \det\left(\varepsilon^n \cdot I_n - A^{2n}\right) =$$

$$= n\left(-1\right)^{n-1} \left[\frac{n^n\left(n+1\right)}{2}\right]^{2n^2-4n} \left(1 + \left(n+1\right)^{2n} \left(2n + \left(-1\right)^n \left(\begin{array}{c} 2n \\ n \end{array}\right)\right)\right)$$

where $\varepsilon \in C \backslash R$, $\varepsilon^{n+1} = 1$.

Stănescu Florin

W45. Consider the complex numbers $a_1, a_2, ..., a_n$, $n \geq 2$. Which have the following properties:
a). $|a_i| = 1, (\forall)\, i = \overline{1,n}$;   b). $\sum\limits_{k=1}^{n} \arg a_k \leq \pi$
Show that the inequality

$$\left(n^2 \cot\left(\frac{\pi}{2n}\right)\right)^{-1} \left|\sum_{k=0}^{n} \left(-1\right)^k \left[3n^2 - \left(8k+5\right)n + 4k\left(k+1\right)\right] \sigma_k\right| \geq$$

$$\geq \sqrt{\left(1 + \frac{1}{n}\right)^2 \cot^2\left(\frac{\pi}{2n}\right)} + 16 \left|\sum_{k=0}^{n} \left(-1\right)^k \cdot \sigma_k\right|,$$

where $\sigma_0 = 1, \sigma_k = \sum\limits_{1 \leq i_1 < i_2 < ... < i_k \leq n} a_{i_1} a_{i_2} ... a_{i_k}, (\forall)\, k = \overline{1,n}$.

Stănescu Florin

W46. Let $x, y, z > 0$ such that $x^2 + y^2 + z^2 = 3$. Then

$$x^3 \arctan\frac{1}{x} + y^3 \arctan\frac{1}{y} + z^3 \arctan\frac{1}{z} < \frac{\pi\sqrt{3}}{2}$$

Marian Cucoaneş and Marius Drăgan

W47. a). If $a, b, c, d > 0$, show inequality:

$$arctg^2\left(\frac{ad - bc}{ac + bd}\right) \geq 2\left(1 - \frac{ac + bd}{\sqrt{\left(a^2 + b^2\right)\left(c^2 + d^2\right)}}\right)$$

b). Calculate

$$\lim_{n\to\infty} n^\alpha \left(n - \sum_{k=1}^{n} \frac{n^2 + k^2 - k}{\sqrt{\left(n^2 + k^2\right)\left(n^2 + \left(k-1\right)^2\right)}}\right),$$

where $\alpha \in R$.

Stănescu Florin

W48. Let $f : (0, +\infty) \to R$ a convex function and $\alpha, \beta, \gamma > 0$. Then

$$\frac{1}{6\alpha}\int_0^{6\alpha} f(x)dx + \frac{1}{6\beta}\int_0^{6\beta} f(x)dx + \frac{1}{6\gamma}\int_0^{6\gamma} f(x)dx \geq$$

$$\geq \frac{1}{3\alpha + 2\beta + \gamma}\int_0^{3\alpha+2\beta+\gamma} f(x)dx + \frac{1}{\alpha + 3\beta + 2\gamma}\int_0^{\alpha+3\beta+2\gamma} f(x)dx +$$

$$+\frac{1}{2\alpha+\beta+3\gamma}\int_0^{2\alpha+\beta+3\gamma} f(x)dx \qquad (1)$$

Marius Drăgan

W49. Let $a, b, c \in (0, +\infty)$. Then the following inequality is true:

$$\sqrt{(a+b)(b+c)}+\sqrt{(b+c)(c+a)}+\sqrt{(c+a)(a+b)}+a+b+c \leq$$

$$\leq (ab+bc+ca)\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)$$

Mihály Bencze and Marius Drăgan

W50. Let $x, y, z > 0$, $\lambda \in (-\infty, 0) \cup (1, +\infty)$ such that $x+y+z=1$. Then

$$\sum x^\lambda y^\lambda \sum \frac{1}{(x+y)^{2\lambda}} \geq 9\left(\frac{1}{4}-\frac{1}{9}\sum\frac{1}{(x+1)^2}\right)^\lambda$$

Marius Drăgan and Sorin Rădulescu

W51. Let a, b, c, d, e be real strictly positive real numbers such that $abcde = 1$. Then is true the following inequality:

$$\frac{de}{a(b+1)}+\frac{ea}{b(c+1)}+\frac{ab}{c(d+1)}+\frac{bc}{d(e+1)}+\frac{cd}{e(a+1)} \geq \frac{5}{2}$$

Mihály Bencze and Marius Drăgan

W52. Let $f : R \to R$ a periodic and continue function with period T and $F : R \to R$ antiderivative of f. Then

$$\int_0^T \left[F(nx)-F(x)-f(x)\frac{(n-1)T}{2}\right]dx = 0$$

Marius Drăgan and Mihály Bencze

W53. Compute

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt[n+k+1]{n+1}-\sqrt[n+k]{n}}{\sqrt[n+k]{n+1}-\sqrt[n+k]{n}}$$

Marius Drăgan

W54. Let $x_1, x_2, ..., x_n$ be a positive numbers, $k \geq 1$. Then the following inequality is true:

$$(x_1^k+x_2^k+...+x_n^k)^{k+1} \geq (x_1^{k+1}+x_2^{k+1}+...+x_n^{k+1})^k + 2\left(\sum_{1\leq i<j\leq n} x_i^k x_j\right)^k$$

Marius Drăgan

W55. Let $f,g,h:[a,b]\to R$ be n positive numbers such that $\sum\limits_{i=1}^{n}\sqrt{a_i}=\sqrt{n}$. Then

$$\prod_{i=1}^{n-1}\left(1+\frac{1}{a_i}\right)^{a_{i+1}}\left(1+\frac{1}{a_n}\right)^{a_1}\geq 1+\frac{n}{\sum\limits_{i=1}^{n}a_i}$$

Marius Drăgan

W56. Let $f,g,h:[a,b]\,R$, three integrable functions such that:

$$\int_a^b fgdx=\int_a^b ghdx=\int_a^b hfdx=\int_a^b g^2dx\int_a^b h^2dx=1$$

Then

$$\int_a^b g^2dx=\int_a^b h^2dx=1$$

Marius Drăgan and Sorin Rădulescu

W57. Let be $x_1=\frac{1}{\sqrt[n+1]{n!}}$ and $x_2=\frac{1}{\sqrt[n+1]{(n-1)!}}$ for all $n\in N^*$ and $f:\left(\frac{1}{\sqrt[n+1]{(n+1)!}},1\right]\to R$ where

$$f(x)=\frac{n+1}{x\ln(n+1)!+(n+1)\ln(x^x)}.$$

Prove that the sequence $(a_n)_{n\geq 1}$ when $a_n=\int\limits_{x_1}^{x_2}f(x)\,dx$ is convergent and compute $\lim\limits_{n\to\infty}a_n$.

Ionel Tudor

W58. In the $[ABCD]$ tetrahedron having all the faces acute angled triangles, is denoted by $r_X,\ R_X$ the radius lengths of the circle inscribed and circumscribed respectively on the face opposite to the $X\in\{A,B,C,D\}$ peak, and with R the length of the radius of the sphere circumscribed to the tetrahedron. Show that inequality occurs

$$8R^2\geq(r_A+R_A)^2+(r_B+R_B)^2+(r_C+R_C)^2+(r_D+R_D)^2$$

Marius Olteanu

W59. In the any $[ABCD]$ tetrahedron we denote with α,β,γ the measures, in radians, of the angles of the three pairs of opposite edges and with r,R the lengths of the rays of the sphere inscribed and respectively circumscribed the tetrahedron. Demonstrate inequality

$$\left(\frac{3r}{R}\right)^3\leq\sin\frac{\alpha+\beta+\gamma}{3}$$

(A refinement of inequality $R\geq 3r$).

Marius Olteanu

W60. In all tetrahedron ABCD holds

1). $(n(n+2))^{\frac{1}{n}}\sum\left(\frac{(h_a-r)^2}{(h_a^n-r^n)\left(h_a^{n+2}-r^{n+2}\right)}\right)^{\frac{1}{n}}\leq\frac{1}{r^2}$

2). $(n(n+2))^{\frac{1}{n}}\sum\left(\frac{(2r_a-r)^2}{((2r_a)^n-r^n)\left((2r_a)^{n+2}-r^{n+2}\right)}\right)^{\frac{1}{n}}\leq\frac{1}{r^2}$
for all $n\in N^*$.

Mihály Bencze

W61. If $a,b,c\in R$ then

$$\sum\sqrt{(a+c)^2b^2+a^2c^2}+\sqrt{5}\left|\sum ab\right|\geq\sum\sqrt{(ab+2bc+ca)^2+(b+c)^2a^2}.$$

Mihály Bencze

W62. Prove that

$$\int_0^{\frac{\pi}{2}}(\cos x)^{1+\sqrt{2n+1}}dx\leq\frac{2^{n-1}n!\sqrt{\pi}}{\sqrt{2(2n+1)!}}$$

for all $n\in N^*$.

Mihály Bencze

W63. If $b_k\geq a_k\geq 0$ $(k=1,2,3)$ and $\alpha\geq 1$ then

$$(\alpha+3)\sum_{cyclic}(b_1-a_1)\cdot$$

$$\cdot\left((b_2+b_3)^{\alpha+2}+(a_2+a_3)^{\alpha+2}-(a_2+b_3)^{\alpha+1}-(a_3+b_2)^{\alpha+1}\right)\leq$$

$$\leq(\alpha+2)(\alpha+3)\sum_{cyclic}(b_1-a_1)(b_2-a_2)\left(b_3^{\alpha+1}-a_3^{\alpha+1}\right)+(b_3+b_2+a_1)^{\alpha+3}+$$

$$+(b_3+a_2+a_1)^{\alpha+3}+(a_3+b_2+a_1)^{\alpha+3}+(a_3+a_2+b_2)^{\alpha+3}-(b_3+b_2+a_1)^{\alpha+3}-$$

$$-(b_3+a_2+a_1)^{\alpha+3}-(a_3+b_2+b_1)^{\alpha+3}-(a_3+a_2+a_1)^{\alpha+3}$$

Mihály Bencze

W64. Prove that exist different natural numbers x,y,z,t for which

$$256\cdot 2019^{180n+1}=2\cdot x^9-2\cdot y^6+z^5-t^4$$

for all $n\in N$.

Mihály Bencze and Chang-Jian Zhao

W65. If $a,b,c\geq 1;y\geq x\geq 1;p,q,r>0$ then

$$\left(\frac{1+y\left(a^pb^qc^r\right)^{\frac{1}{p+q+r}}}{1+x\left(a^pb^qc^r\right)^{\frac{1}{p+q+r}}}\right)^{\frac{p+q+r}{\left(a^pb^qc^r\right)^{\frac{1}{p+q+r}}}}\left(\frac{1+ya}{1+xa}\right)^{\frac{p}{a}}\cdot\left(\frac{1+yb}{1+xb}\right)^{\frac{q}{b}}\left(\frac{1+yc}{1+xc}\right)^{\frac{r}{c}}\geq$$

$$\geq\prod\left(\frac{1+y\left(a^pb^q\right)^{\frac{1}{p+q}}}{1+x\left(a^pb^q\right)^{\frac{1}{p+q}}}\right)^{\frac{p+q}{\left(a^pb^q\right)^{\frac{1}{p+q}}}}$$

Mihály Bencze

W66. If $0 < a \leq b$ then

$$\frac{2}{\sqrt{3}} arctg \frac{2\left(b^2 - a^2\right)\sqrt{3}}{\left(a^2+2\right)\left(b^2+2\right)} \leq$$

$$\leq \int_a^b \frac{\left(x^2+1\right)\left(x^2+x+1\right)dx}{\left(x^3+x^2+1\right)\left(x^3+x+1\right)} \leq \frac{4}{\sqrt{3}} arctg \frac{(b-a)\sqrt{3}}{a+b+2(1+ab)}$$

Mihály Bencze

W67. Denote T the Toricelli point of the triangle ABC. Prove that

$$AB^2 \cdot BC^2 \cdot CA^2 \geq 3\left(TA^2 \cdot TB + TB^2 \cdot TC + TC^2 \cdot TA\right) \cdot$$

$$\cdot \left(TA \cdot TB^2 + TB \cdot TC^2 + TC \cdot TA^2\right)$$

Mihály Bencze

W68. In all tetrahedron ABCD holds

1). $\sum \frac{h_a - r}{h_a + r} \geq \sum \frac{h_a^t - r^t}{(h_a + r)^{tt}}$ 2). $\sum \frac{2r_a - r}{2r_a + r} \geq \sum \frac{(2r_a)^t - r^t}{(2r_a + r)^t}$
for all $t \in [0, 1]$.

Mihály Bencze

W69. Denote $\overline{w_a}, \overline{w_b}, \overline{w_c}$ the external angle-bisectors in triangle ABC, prove that

$$\sum \frac{1}{w_a} \leq \sqrt{\frac{\left(s^2 - r^2 - 4Rr\right)\left(8R^2 - s^2 - r^2 - 2Rr\right)}{8s^2R^2r}}$$

Mihály Bencze

W70. If $x \in \left(0, \frac{\pi}{2}\right)$ then

$$\left(\frac{\sin\left(\frac{\pi}{2}\sin x\right)}{\sin x}\right)^2 + \left(\frac{\sin\left(\frac{\pi}{2}\cos x\right)}{\cos x}\right)^2 \geq 3.$$

Mihály Bencze