Nonconvex second-order damped gradient systems and metastability

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find minimizers u^* of $E(\cdot)$

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- Gradient Flows $\dfrac{du}{dt} = -\nabla E(u)$,
- Second order Gradient Systems $\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} = -\nabla E(u)$ or
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AIM: discuss second order dynamical systems associated to a nonconvex, noncoercive functional E on an infinite dimensional space X.

A nonconvex functional in image processing

Samson et. al.[1] (cf. also Aubert and Kornprobst[2]) have proposed a Mumford-Shah-type nonconvex, noncoercive functional that can achieve both image classification and restoration simultaneously

$$E_\varepsilon(u) = \int_\Omega (u - u_0)^2 \, dx + \varepsilon \int_\Omega \varphi(|\nabla u|) \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx$$

where u_0 is the image to be restored and classified, $\varepsilon > 0$ is a small parameter and $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(w) = \frac{w^2}{w^2+1}$ while W is the double-well potential $W : \mathbb{R} \to \mathbb{R}$, $W(u) = \frac{1}{4}\left(u^2 - 1\right)^2$.

[1] C. Samson, L. Blanc-Féraud, G. Aubert, J. Zerubia, *A variational model for image classification and restoration*, IEEE Trans. Pattern Anal. Mach. Intell. 22 (2000), no. 5, 460-472.

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Why not $\|u_x\|^2$?

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Why not $\|u_x\|^2$? Not edge-preserving (too much smoothing).

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Semilinear wave equations $u_{tt} + u_t = u_{xx} + f(u)$: classical results of Haraux & Jendoubi[3]

If f analytic then it satisfies the Lojasiewicz inequality.
Solutions satisfy the energy balance equation (with $F' = f$)

$$\frac{d}{dt}\left(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|u_x\|^2 + \int_\Omega F(u)dx\right) = -\|v\|^2.$$

Theorem (Haraux-Jendoubi, '01)

Assume that $f : \mathbb{R} \to \mathbb{R}$ is analytic, with f'' is bounded on $(-\beta, \beta)\forall\beta > 0$ and let u be a solution such that

$$\bigcup_{t\geq 1}[u(t), v(t)] \quad \textit{rel. compact in} \quad H^2(\Omega) \times H^1(\Omega)$$

then there exits a stationary point u^* $T > 0$ large enough such that for all $t \geq T$

$$\|u(t) - u^*\|_{H^1} \leq Ct^{\theta/(1-2\theta)} \quad \textit{if} \quad 0 < \theta < \frac{1}{2}$$
$$\|u(t) - u^*\|_{H^1} \leq Ce^{-\omega t} \quad \textit{if} \quad \theta = \frac{1}{2}.$$

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[3] A. Haraux, M. Jendoubi, *Decay estimates for some evolution equations with an analytic nonlinrearity*, Asympt. Anal. 26(2001), 21-36.

A semiliner wave equation for the image processing functional

The semilinear equation associated to

$$E(u) = \int_\Omega \varphi\left(u_x\right) dx + \int_\Omega W\left(u\right) dx$$

is

$$u_{tt} + u_t = \varphi''(u_x) u_{xx} + u - u^3,$$

with the energy-dissipation equation

$$\frac{d}{dt}\left(\frac{1}{2}\left\|v\right\|^2 + \int_\Omega \frac{u_x^2}{1+u_x^2} dx + \int_\Omega W(u) dx\right) = -\left\|v\right\|^2.$$

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NO... even well-posedness fails. Discontinuities develop in finite time.

Possible remedies/Alternative approaches

- weakening the solution concept (?!?)

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- modifying the energy functional by adding some higher order regularization (???)
- changing the damping (more/stronger damping)
- both changing the damping and adding a higher order regularization to the functional

Approach I: stronger damping + h.o. regularization

We consider

$$E(u)_\varepsilon = \frac{\varepsilon}{2}\|u_{xx}\|^2 + \int_\Omega \varphi(u_x)\,dx + \int_\Omega W(u)\,dx$$

and the higher order, damped wave equation

$$u_{tt} = u_{txx} - u_{xxxx} + \varphi''(u_x)u_{xx} + u - u^3.$$

Energy balance now reads

$$\frac{d}{dt}\left(\frac{1}{2}\|v\|^2 + \frac{\varepsilon}{2}\|u_{xx}\|^2 + \int_\Omega \frac{u_x^2}{1+u_x^2}dx + \int_\Omega W(u)dx\right) = -\|v_x\|^2.$$

Approach I (continued): rigorous results based on semigroup methods

This approach has a series of advantages:

- the linear part of the equation generates an analytic, immediately compact semigroup (see Engel & Nagel)
- global solutions exist in $H_0^2(\Omega) \times L^2(\Omega)$
- with improved regularity $(u(t), v(t)) \in H^4(\Omega) \cap H_0^2(\Omega) \times H_0^2(\Omega)$ for all $t > 0$
- all trajectories are relatively compact (by a result of Pazy)
- all trajectories converge to equilibrium (by LaSalle's Invariance Principle)
  and furthermore
- the model admits an ε-independent $\|u_{xx}\|$-estimate, on bounded time intervals $[0, T]$, such that solutions converge (as $\varepsilon \to 0$) to weak solutions of the $\varepsilon = 0$-model (by Aubin's Lemma)

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- the $t \to \infty$ and $\varepsilon \to 0$ limits do not commute.

Approach II: strong damping

We consider te unperturbed energy

$$E(u) = \int_\Omega \varphi\left(u_x\right) dx + \int_\Omega W\left(u\right) dx$$

and the damped wave equation with a strong damping

$$u_{tt} = u_{txx} + \varphi''(u_x)u_{xx} + u - u^3.$$

Energy balance now reads

$$\frac{d}{dt}\left(\frac{1}{2}\left\|v\right\|^2 + \int_\Omega \frac{u_x^2}{1+u_x^2} dx + \int_\Omega W(u) dx\right) = -\left\|v_x\right\|^2.$$

Approach II (continued): rigorous results

Thes analysis of this model is much more involved than what we had previously:

- local solutions exist in $H_0^2(\Omega) \times L^2(\Omega)$ (by semigroup methods or by the fixed point theorem of Krasnoselskii for the sum of two operators[4])
- global existence in $H_0^2(\Omega) \times L^2(\Omega)$ follows from a new $\|u_{xx}\|$-estimate however
- no improved regularity or compactness is present
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Work in progress: energy decay estimates along bounded trajectories.

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Returning to the classical semilinear wave equation: Metastability and the role of small coefficients

Let us return to the classical semilinear wave equation, but with a small coefficient $\varepsilon \ll 1$

$$u_{tt} = -u_t + \varepsilon^2 u_{xx} + u - u^3.$$

There exists a manifold $\mathcal{M}$ of initial data, in the state space, for which theorem of Haraux and Jendoubi still holds but with a very large T

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This phenomenon is called Metastability[5]

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