Solving Multi-Leader-Follower Games

by Smoothing the Follower's Best Response

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Motivation

Toll Models

Toll Caps in Privatized Road Networks

by Harks, Schröder, and Vermeulen 2018

Electricity Markets

Nash equilibrium in a pay-as-bid electricity market

by Aussel, Bendotti, and Pištěk 2017

Analysis of m-stationary points to an epec modeling oligopolistic competition in an electricity spot market.

by Henrion, Outrata, and Surowiec 2012

- These models may include large amount of players
- Dynamic games get high dimensional
- Mean field limit transforms a high dimensional system of ODEs to a (nonlinear) PDE

Motivation: Example

Optimization and Mean Field Behavior for $N \to \infty$

for $i = 1, \dots, N$:

$$\min_{y_i, u_i} \int_0^T \phi(y_i, x) + \frac{\alpha}{2} {u_i}^2(t)\, \mathrm{d}t$$

s.t. $\dot{y}_i(t) = u_i(t), y_i(0) = y_{i,0}$

$\xrightarrow{\text{MF}}$

$$\min_{f,u} \int_0^T \int \left[\phi(y, x) + \frac{\alpha}{2} u^2\right] f \,\mathrm{d}y\,\mathrm{d}t$$

s.t. $\partial_t f + \nabla_y (f u) = 0$

$$f(0, y) = \frac{1}{N} \sum_{i=1}^{N} \delta(y - y_{i,0})$$

- Mean field limit can be applied on dynamic Multi-leader-follower games
- Relation between optimization and mean field limit
- Conditions for consistency

Existence (and Uniqueness) of equilibrium solutions for Multi-leader-follower games often even in finite dimensions unclear.

General Games

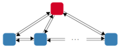
N-Person Game: $\Gamma = \{\theta_\nu, X_\nu\}_{\nu=1}^N$

- Set $\{1, \ldots, N\}$ of Players
- Strategy set $X_\nu \subseteq \mathbb{R}^{n_\nu}$ for each Player
- Pay-off function θ_ν for each player

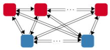
$$\theta_\nu : X_1 \times X_2 \times \ldots \times X_N \to \mathbb{R}$$

Optimization Problem

Nash Game



Stackelberg Game



Multi-Leader-Follower Game

Classification:

- Cooperative vs. Noncooperative Games
- Continuous vs. Discrete Games
- Complete vs. Incomplete Information

Nash Equilibrium (1950)

For all $\nu = 1, \ldots, N$

$$\theta_\nu(x_\nu^*, x_{-\nu}^*) \leq \theta_\nu(x_\nu, x_{-\nu}^*)$$

for all $x_\nu \in X_\nu$.

$x_{-\nu} = (x_1, \ldots, x_{\nu-1}, x_{\nu+1}, \ldots, x_N)$

Theoretical Results: Overview

For $\nu = 1, \dots, N$

MLFG

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + a^\top y$$
$$\text{s.t. } x_\nu \in X_\nu$$
$$\min_{y} \frac{1}{2} y^\top Q_y y - b(x)^\top y$$
$$\text{s.t. } y \geq l(x)$$

Convex NEP

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \sum_{i=1}^{m} a_i \max\{Q_y^{-1} b(x), l(x)\}_i$$
$\text{s.t. } x_\nu \in X_\nu$ convex

Smooth NEP(ε)

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \frac{1}{2} \sum_{i=1}^{m} a_i \left[\left(L^\top + Q_y^{-1} B^\top\right) x + \tilde{\phi}_\varepsilon \left(\left(L^\top - Q_y^{-1} B^\top\right) x \right) \right]_i$$
$\text{s.t. } x_\nu \in X_\nu$ convex

Theoretical Results: Existence

For $\nu = 1, \dots, N$

MLFG

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + a^\top y$$
$$\text{s.t. } x_\nu \in X_\nu$$
$$\min_y \frac{1}{2} y^\top Q_y y - b(x)^\top y$$
$$\text{s.t. } y \geq l(x)$$

$\longleftrightarrow$

Convex NEP

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \sum_{i=1}^{m} a_i \max\{Q_y^{-1} b(x), l(x)\}_i$$
s.t. $x_\nu \in X_\nu$ convex

Assumptions (Follower)

- $Q_y \in \mathbb{R}^{m \times m}$ s.p.d. (and diagonal)
- $b_i, l_i : \mathbb{R}^n \to \mathbb{R}^m$ convex and smooth

$\Rightarrow \exists!$ (Nonsmooth) Best Response

$$y(x) = \max\{Q_y^{-1} b(x), l(x)\}$$

Assumptions (Leader)

- $Q_\nu \in \mathbb{R}^{n_\nu \times n_\nu}$ s.p.d.
- $a \in \mathbb{R}^m_+$
- $X_\nu \neq \emptyset$ convex and closed

Theorem (Existence)

The convex NEP (and therefore the MLFG) has at least one Nash equilibrium for compact X_ν.

Smooth Formulation

MLFG

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + a^\top y$$

$$\text{s.t. } x_\nu \in X_\nu$$

$$\min_{y} \frac{1}{2} y^\top Q_y y - b(x)^\top y$$

$$\text{s.t. } y \geq l(x)$$

For $\nu = 1, \dots, N$

Follower's Best Response

$l(x) = L^\top x, b(x) = B^\top x$

- Nonsmooth

$$y(x) = \max\{(Q_y^{-1} B^\top) x, L^\top x\}$$

- Smooth

$$y_\varepsilon(x) = \frac{1}{2} \left[\left(L^\top + Q_y^{-1} B^\top\right) x + \underbrace{\tilde{\phi}_\varepsilon \left(\left(L^\top - Q_y^{-1} B^\top\right) x\right)}_{\approx \left|\left(L^\top - Q_y^{-1} B^\top\right) x\right|} \right]$$

Smooth NEP(ε)

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \frac{1}{2} \sum_{i=1}^{m} a_i \left[\left(L^\top + Q_y^{-1} B^\top\right) x + \tilde{\phi}_\varepsilon \left(\left(L^\top - Q_y^{-1} B^\top\right) x\right)\right]_i$$

s.t. $x_\nu \in X_\nu$ convex

Smooth Formulation: Existence

Theorem: Existence and Uniqueness

The smoothed NEP with $\nu = 1, \ldots, N$:

$$\min_{x_\nu} \frac{1}{2} x_\nu{}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \frac{1}{2} \sum_{i=1}^{m} a_i \left[\left(L^\top + Q_y^{-1} B^\top \right) x + \tilde{\phi}_\varepsilon \left(\left(L^\top - Q_y^{-1} B^\top \right) x \right) \right]_i$$

s.t. $x_\nu \in X_\nu$ convex and closed

has a unique Nash equilibrium for every smoothing parameter $\varepsilon > 0$.

Proof. We show that $\hat{\theta}' = (\nabla_{x_\nu} \theta_\nu (x_\nu, x_{-\nu}))_{\nu=1}^N$ is uniformly monotone since Q_ν s.p.d. and $\tilde{\phi}_\varepsilon \in C^2$ convex.

Therefore the VI(X, $\hat{\theta}'$) has a unique solution.

Theoretical Results: Summary

For $\nu = 1, \dots, N$

MLFG

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + a^\top y$$
$$\text{s.t. } x_\nu \in X_\nu$$
$$\min_{y} \frac{1}{2} y^\top Q_y y - b(x)^\top y$$
$$\text{s.t. } y \geq l(x)$$

Convex NEP

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \sum_{i=1}^{m} a_i \max\{Q_y^{-1} b(x), l(x)\}_i$$
s.t. $x_\nu \in X_\nu$ convex

Existence for compact X_ν

Existence and Uniqueness

Smooth NEP(ε)

$$\min_{x_\nu} \frac{1}{2} {x_\nu}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + \frac{1}{2} \sum_{i=1}^{m} a_i \left[\left(L^\top + Q_y^{-1} B^\top\right) x + \tilde{\phi}_\varepsilon \left(\left(L^\top - Q_y^{-1} B^\top\right) x \right) \right]_i$$
s.t. $x_\nu \in X_\nu$ convex

Extension to the Model

- Non-diagonal Q_y
  - ➔ auxiliary variable: $z = D^\top y$ where $Q_y = DD^\top$
  - ➔ $y(x) = D^{-\top} \max\{D^{-1}b(x), D^\top l(x)\}$
- Box Constraints $l(x) \leq y \leq u(x)$

$$y_i(x) = \text{median}(l_i(x), \tilde{y}_i(x), u_i(x)) = \begin{cases} \tilde{y}_i(x) & \text{else} \\ l_i(x) & \tilde{y}_i(x) \leq l_i(x) \\ u_i(x) & \tilde{y}_i(x) \geq u_i(x) \end{cases}$$

with $\tilde{y}_i(x) = (Q_y)^{-1}_{ii} b_i(x)$

- Multiple Follower
  - ➔ For $j = 1, \ldots, N_F$ let:

$$\min_{y_j \in \mathbb{R}^{m_j}} \Theta_j(y_j, x) = \frac{1}{2} y_j{}^\top Q_y^j y_j - b_j(x)^\top y_j \quad \text{s.t.} \quad y_j \geq l_j(x)$$

Characterization of NE: KKT(ε)

Joint KKT(ε) System if $X_\nu = \{x_\nu \in \mathbb{R}^{n_\nu} | g_\nu(x_\nu) \leq 0\}$ and assume that LICQ holds.

$$0 = Qx + c_\nu^\top x_\nu + \frac{1}{2}(L^\top + Q_y^{-1}B^\top)^\top a + \frac{1}{2}\sum_{i=1}^{m} a_i (L^\top - Q_y^{-1}B^\top)^\top_{:,i} \, \tilde{\phi}_\varepsilon{}' \left([(L^\top - Q_y^{-1}B^\top)x]_i\right) + \begin{bmatrix} \nabla_{x_{(1)}} g_{(1)}(x_{(1)})\lambda_{(1)} \\ \vdots \\ \nabla_{x_{(N)}} g_{(N)}(x_{(N)})\lambda_{(N)} \end{bmatrix}$$

$$0 = \min \left\{ \begin{bmatrix} \lambda_{(1)} \\ \vdots \\ \lambda_{(N)} \end{bmatrix}, \begin{bmatrix} -g_{(1)}(x_{(1)}) \\ \vdots \\ -g_{(N)}(x_{(N)}) \end{bmatrix} \right\}$$

Lemma

Let $x^*(\varepsilon_k) \in X$ and $\lambda^*(\varepsilon_k) \in \Lambda$ for all $\varepsilon_k > 0$ and for X and Λ compact. Then there exists at least one accumulation point $x^*(0)$ and $\lambda^*(0)$ for a sequence $\varepsilon_k \to 0$.

Characterization of NE: KKT(0)

Fritz-John Conditions of Clarke

Let $\bar{x}$ be a vector of local minimizer $\bar{x}_\nu$ of each leader problem; Slater condition holds for every X_ν, then there exists for $\nu = 1, \ldots, N$ a multiplier $\lambda_\nu \in \mathbb{R}_+^{m_\nu}$:

$$0 \in Q_\nu \bar{x}_\nu + c_\nu^\top \bar{x}_\nu + \frac{1}{2}(L^\top + Q_y^{-1} B^\top)_{\nu,:}^\top a$$

$$+ \frac{1}{2} \sum_{i=1}^{m} a_i (L^\top - Q_y^{-1} B^\top)_{\nu,i}^\top \, \partial^C \tilde{\phi}_0 \left([(L^\top - Q_y^{-1} B^\top)\bar{x}]_i\right) + \nabla_{x_\nu} g_\nu(\bar{x}_\nu)\lambda_\nu$$

$$0 = \min\{\lambda_\nu, -g_\nu(\bar{x}_\nu)\}$$

$$\partial^C \tilde{\phi}_0(z) = \begin{cases} 1, & z > 0 \\ -1, & z < 0 \\ [-1, 1], & z = 0 \end{cases}$$

$$\lim_{\varepsilon \to 0} \tilde{\phi}_\varepsilon{}'(z) = \begin{cases} 1, & z > 0 \\ -1, & z < 0 \\ 0, & z = 0 \end{cases}$$

The limit NE $x^*(0)$ satisfies a necessary optimality condition.

What about MPCC stationarity concepts?

Stationarity Concepts of MPCC

Mathematical Programm with Complementarity Constraints (MPCC)

Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $G_1, G_2 : \mathbb{R}^n \to \mathbb{R}^l$ be smooth functions.

$$\begin{aligned} \min_z \ & f(z) \\ \text{s.t. } & g(z) \leq 0 \\ & 0 = \min\{G_1(z), G_2(z)\} \end{aligned} \tag{MPCC}$$

or alternatively: $0 \leq G_1(z) \perp G_2(z) \geq 0$

Stationarity Concepts of MPCC

Stationarity MPCC

$\bar{z}$ is a stationary point if there exist multipliers $(\lambda, \Gamma_1, \Gamma_2) \in \mathbb{R}^{m+l+l}$:

$$0 = \nabla_z f(\bar{z}) + \sum_{i=1}^{m} \lambda_i \nabla_z g_i(\bar{z}) - \sum_{i=1}^{l} \Gamma_{1,i} \nabla_z G_{1,i}(\bar{z}) - \sum_{i=1}^{l} \Gamma_{2,i} \nabla_z G_{2,i}(\bar{z})$$

$$0 \geq g(\bar{z}) \perp \lambda \geq 0$$

$$G_{1,i}(\bar{z})\Gamma_{1,i} = 0, \quad i = 1, \dots, l$$

$$G_{2,i}(\bar{z})\Gamma_{2,i} = 0, \quad i = 1, \dots, l$$

And for $i : G_{1,i}(\bar{z}) = G_{2,i}(\bar{z}) = 0$

S-Stationarity	$\Rightarrow$ M-Stationarity	$\Rightarrow$ C-Stationarity
$\Gamma_{1,i} \geq 0$ and $\Gamma_{2,i} \geq 0$	$\Gamma_{1,i}, \Gamma_{2,i} > 0$ or $\Gamma_{1,i} \cdot \Gamma_{2,i} = 0$	$\Gamma_{1,i} \cdot \Gamma_{2,i} \geq 0$

MPCC Formulation

Leader Mathematical Program with Complementary Constraints

$$\begin{aligned} \min_{x_\nu, y} \ & \frac{1}{2} x_\nu{}^\top Q_\nu x_\nu + c_\nu^\top x_\nu + a^\top y \\ \text{s.t. } & g_\nu(x_\nu) \leq 0 \\ & 0 = \min\{G_1(x_\nu, x_{-\nu}, y), G_2(x_\nu, x_{-\nu}, y)\} \end{aligned}$$

with $G_1(x_\nu, x_{-\nu}, y) = y - (Q_y^{-1} B^\top)^\top x$ and $G_2(x_\nu, x_{-\nu}, y) = y - L^\top x$.

Theorem

The limit $x_\nu^*(0)$ is strongly stationary for the leader ν.

Proof. With the multipliers of the Fritz-John Conditions of Clarke we construct multipliers which fulfill the conditions of strong stationarity.

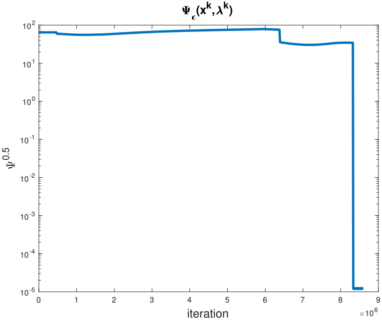
Computation

- Solution of KKT(0)?
- Solve sequence of $\Psi_\varepsilon(x, \lambda) = \frac{1}{2}||\text{KKT}(\varepsilon)||_2^2$ for $\varepsilon_k \to 0$
- Ψ_ε is highly nonlinear and nonsmooth (but locally Lipschitz)
- Standard methods work slowly
- Importance to have good initial values

Computation: NEP(ε)

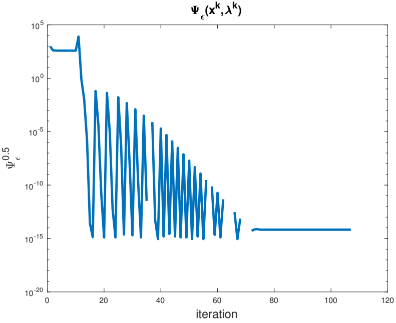
Subgradient

- first order derivatives
- characteristic convergence rate
- convergence guaranty for locally Lipschitz [Bagirov13]



Nonsmooth Newton

- second order derivatives
- very fast
- to be globalized by Subgradient method



Computation: NEP(ε)

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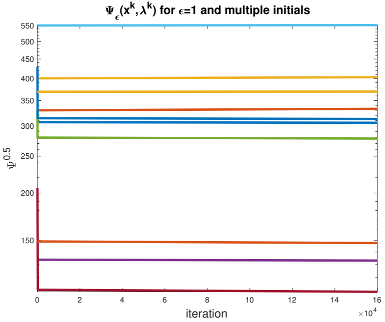

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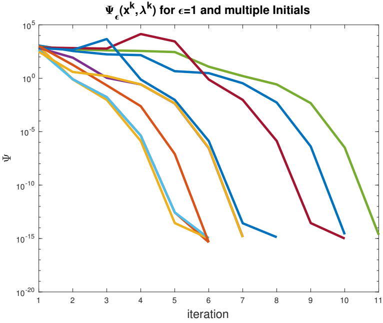
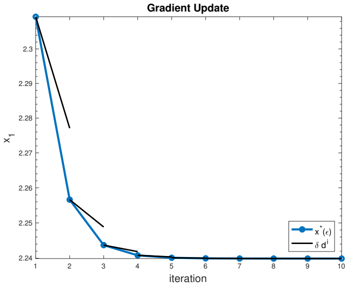
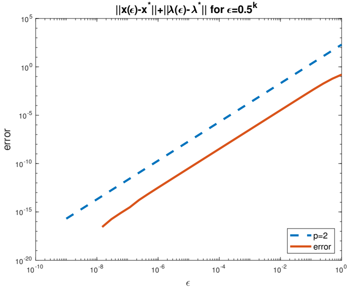


Computation: Good Initials

Algorithm

1. Initialize $k = 0$, $z_0^0 = (x_0^0, \lambda_0^0)$, $\varepsilon_0 > 0$
2. Compute NE (x_k^*, λ_k^*) of NEP(ε_k)
3. Compute $\frac{\partial x}{\partial \varepsilon}$ by solving $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\nabla_{x_\nu} \theta_\nu^\varepsilon(x_\nu(\varepsilon), x_{-\nu}(\varepsilon))\right) = 0$
4. choose $\varepsilon_{k+1} < \varepsilon_k$
5. $x_{k+1}^0 = x_k^* - (\varepsilon_k - \varepsilon_{k+1})\frac{\partial x}{\partial \varepsilon}$
6. $\lambda_{k+1}^0 = \lambda_k^*$
7. $k++$; go to 2





Summary

1. Quadratic Multi-Leader-Follower Game
2. NEP Formulation $\Rightarrow$ Existence of NE for compact strategy sets
3. Smooth NEP(ε) Formulation $\Rightarrow$ Existence&Uniqueness of NE(ε)
4. Limit NEP: $x^*(0)$ is strongly stationary
5. Efficient computation with globalized Nonsmooth Newton with $\frac{\partial x}{\partial \varepsilon}$-based hot start

Outlook

Now: Multi-Level-Games with infinitely many players

For $k = 1, \dots, N_L$

$$\min_{x_k} \int_0^T J_k^L\left(x_k, x_{-k}, m(y_1, \dots, y_{N_F})\right) \, \mathrm{d}t$$

$$\text{s.t. for } i = 1, \dots, N_F :$$

$$\min_{y_i, u_i} \int_0^T \left[\phi(y_i, x) + \frac{\alpha}{2} {u_i}^2\right] \, \mathrm{d}t$$

$$\text{s.t.} \quad \dot{y}_i = u_i, \; y_i(0) = y_{i,0}$$

Outlook: Mean Field Limit

for $i = 1, \ldots, N$:

$$\min_{y_i, u_i} \int_0^T \phi(y_i, x) + \frac{\alpha}{2} {u_i}^2(t)\,\mathrm{d}t$$

s.t. $\dot{y}_i(t) = u_i(t), y_i(0) = y_{i,0}$

$$\xrightarrow{\text{MF}}$$

$$\min_{f,u} \int_0^T \int \left[\phi(y, x) + \frac{\alpha}{2} u^2\right] f \,\mathrm{d}y\,\mathrm{d}t$$

s.t. $\partial_t f + \nabla_y (f u) = 0$

$$f(0, y) = \frac{1}{N} \sum_{i=1}^{N} \delta(y - y_{i,0})$$

Outlook: Mean Field Limit

for $i = 1, \dots, N$:

$$\min_{y_i, u_i} \int_0^T \phi(y_i, x) + \frac{\alpha}{2} {u_i}^2(t)\,\mathrm{d}t$$

$$\text{s.t. } \dot{y}_i(t) = u_i(t),\ y_i(0) = y_{i,0}$$

- empirical measure

$$f(t, y) = \frac{1}{N} \sum_{i=1}^{N} \delta(y - y_i(t))$$

- compactly supported and smooth $\varphi(y)$
- $u(t, y_i(t)) = u_i(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi(y)\ f(t,y)\,\mathrm{d}y = \frac{1}{N} \sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t} \varphi(y) = \frac{1}{N} \sum_{i=1}^{N} \nabla_y \varphi(y_i(t)) = \dot{y}_i(t)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \nabla_y \varphi(y_i(t))\ u(t, y_i(t)) = \int \nabla_y \varphi(y)\ u(t,y)\ f(t,y)\,\mathrm{d}y$$

$$= -\int \nabla_y \left(u(t,y)\ f(t,y)\right) \varphi(y)\,\mathrm{d}y$$

$$\boxed{\partial_t f + \nabla_y (f u) = 0}$$

Thank you for your kind attention!

. . . and see you again at **ICCOPT** in Berlin in the session:
Nash Equilibrium Problems and Extensions

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