Existence of monotone solutions with respect to a preorder and applications

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Plan

Plan

1. Existence of monotone solutions with respect to a preorder and applications
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The problem

Let $K_p \subset \mathbb{R}^m$, $K_q \subset \mathbb{R}^n$ be two convex compact sets. We shall prove the existence of $P(\cdot) \times Q(\cdot)$-monotone solutions of the following mixed system

$$(\mathcal{P}5) \qquad \begin{cases} x'(t) \in \text{proj}_{T_{P(x(t))}(x(t))}(-\partial_x \Gamma(x(t), y(t)), \quad a.e. \ t \in [0, +\infty), \\ y'(t) \in \text{proj}_{T_{Q(y(t))}(y(t))}(\partial_y^+ \Gamma(x(t), y(t)), \quad a.e. \ t \in [0, +\infty), \\ x(0) = x_0 \in K_p, \quad y(0) = y_0 \in K_q. \end{cases}$$

- $\Gamma : K_p \times K_q \to \mathbb{R}_+$ is a convex-concave function,
- $\partial_u \Gamma(u, v)$ is the subdifferential of the convex function $\Gamma(\cdot, v)$ with respect to u,

$$\partial_u \Gamma(u, v) = \{u^* \in \mathbb{R}^n \mid \Gamma(u', v) \geq \Gamma(u, v) + \langle u^*, u' - u \rangle, \quad u' \in \mathbb{R}^n\}.$$

- $\partial_v^+ \Gamma(u, v)$ is the superdifferential of the concave function $\Gamma(u, \cdot)$ with respect to v,

$$\partial_v^+ \Gamma(u, v) = \{v^* \in \mathbb{R}^n \mid \Gamma(u, v') \leq \Gamma(u, v) + \langle v^*, v' - v \rangle, \quad v' \in \mathbb{R}^n\}.$$

The problem

In order to study the planning procedures in mathematical economy, Henry(73) introduced in the 70s the differential inclusion

$$x'(t) \in \operatorname{proj}_{T_C(x(t))} F(x(t)), \qquad x(0) = x_0 \in C.$$

where C is a nonempty closed convex subset of $\mathbb{R}^n$ and $F : C \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous multivalued mapping.

Later on, this inclusion has been associated to the existence of a minimal norm absolutely continuous solution for the following problem

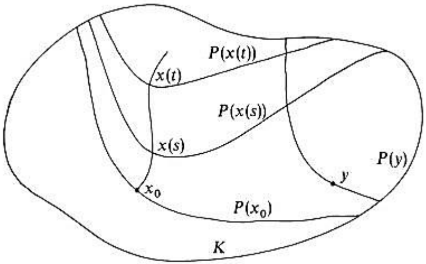
$$x'(t) \in -N_C(x(t)) + F(x(t)), \qquad x(0) = x_0 \in C$$

by Cornet(83).

Monotone trajectories

Let $[0, T]$ be any finite interval ($T > 0$), and K a closed subset of $\mathbb{R}^n$. We say that an absolutely continuous function x from $[0, T]$ into $\mathbb{R}^n$ is a *monotone trajectory* for F starting at $x_0 \in K$ if

$$\begin{cases} (i) & x'(t) \in F(x(t)) \text{ a.e. } t \text{ in } [0, T], \\ (ii) & x(0) = x_0, \\ (iii) & x(t) \in K \text{ for all } t \in [0, T], \\ (iv) & \text{if } t \geq s \text{ then } x(t) \in P(x(s)). \end{cases}$$



Aubin-Cellina-Nohel(77), Aubin(79), Clarke-Aubin(77), Falcone-Siconolfi(83), Haddad(81)

Monotone trajectories

Let $K \subset \mathbb{R}^n$ be a convex set. We recall that a preorder P on K is a multivalued mapping $P : K \rightrightarrows K$ such that

$$\begin{cases} (a) & x \in P(x), \text{ for any } x \in K \quad \text{(reflexivity)}; \\ (b) & z \in P(y), y \in P(x) \Rightarrow z \in P(x) \quad \text{(transitivity)}. \end{cases}$$

The necessary and sufficient condition to have monotone solutions is

$$\forall x \in K, \qquad F(x) \cap T_{P(x)}(x) \neq \emptyset.$$

Definition of $P \times Q$ monotone solutions

We say that trajectories $x : [0, \infty) \to K_p$ and $y : [0, \infty) \to K_q$ of $(\mathcal{P}5)$ are $P \times Q$-monotone, if

(a) $x(\cdot)$ is monotone with respect to $P(\cdot)$,

(b) $y(\cdot)$ is monotone with respect to $Q(\cdot)$.

Existence of monotone trajectories

We consider the following differential inclusion

$$(\mathcal{P}) \qquad \begin{cases} x'(t) \in \mathrm{proj}_{T_{R(x(t))}(x(t))}(V(x(t))), & \text{a.e. in } [0,\infty), \\ x(0) = x_0. \end{cases}$$

Let K is a convex compact set of $\mathbb{R}^n$, $V : K \to \mathbb{R}^n$ is an u.s.c. multivalued mapping, and $R(\cdot)$ is a preorder defined on K.

Existence Theorem – Haddad(81)

Let $R(\cdot)$ be a continuous preorder with convex compact values defined on a compact convex subset K in $\mathbb{R}^n$, and let $V : K \to \mathbb{R}^n$ be u.s.c. Then, for any initial point $x_0 \in K$ there exists a R-monotone solution $x(t)$ of $(\mathcal{P})$.

Let us define the multivalued mapping $P : K_p \to K_p$ by:

$$P(x) = \{s \in K_p : \min_{z \in F_p(s)} h_p(z) \leq \min_{z \in F_p(x)} h_p(z)\}, \quad \forall x \in K_p.$$

Hypotheses: $K_p \subseteq \mathbb{R}^m_+$ is a compact convex set.

$F_p : K_p \rightrightarrows \mathbb{R}^n_+$ is a multivalued mapping satisfying;

$$(H_p^1) \quad \begin{cases} (a) & F_p(x) \text{ is a convex compact set, } \forall x \in K_p; \\ (b) & F_p(\cdot) \text{ is concave;} \\ (c) & F_p(\cdot) \text{ is continuous.} \end{cases}$$

$h_p : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a single-valued function satisfying;

$$(H_p^2) \quad \begin{cases} (a) & h_p(\cdot) \text{ is continuous;} \\ (b) & h_p(\cdot) \text{ is strictly convex;} \\ (c) & \text{if } x_1 \geq x_2, \text{ then } h_p(x_1) \leq h_p(x_2), \forall x_1, x_2 \in K_p. \end{cases}$$

Proposition II.1

Assume that Assumptions (H_p^1), and (H_p^2) are satisfied. Then the preorder $P(\cdot)$ defined on K_p is continuous with nonempty compact convex values.

Similarly, let us define the multivalued mapping $Q : K_q \to K_q$ by:

$$Q(y) = \{s \in K_q : \max_{z \in F_q(s)} h_q(z) \geq \max_{z \in F_q(y)} h_q(z)\}, \quad \forall y \in K_q.$$

Hypotheses:

$F_q : K_q \rightrightarrows \mathbb{R}^n_+$ is a multivalued mapping satisfying;

$$(H_q^1) \quad \begin{cases} (a) & F_q(x) \text{ is a convex compact set, } \forall x \in K_q; \\ (b) & F_q(\cdot) \text{ is concave;} \\ (c) & F_q(\cdot) \text{ is continuous.} \end{cases}$$

$h_q : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a single-valued function satisfying;

$$(H_q^2) \quad \begin{cases} (a) & h_q(\cdot) \text{ is continuous;} \\ (b) & h_q(\cdot) \text{ is strictly concave;} \\ (c) & \text{if } y_1 \geq y_2, \text{ then } h_q(y_1) \geq h_q(y_2), \forall y_1, y_2 \in K_q. \end{cases}$$

Proposition II.2

Assume that Assumptions (H_q^1), and (H_q^2) are satisfied. Then the preorder $Q(\cdot)$ defined on K_q is continuous with nonempty compact convex values.

Applications

The problem

To show that (x^*, y^*) is the maximum of the profit of the firm $\Gamma : K_p \times K_q \to \mathbb{R}_+$ given by

$$\Gamma(x,y) = \widetilde{r}(y) - \widetilde{w}(x) \quad \text{for all input-output vector } (x,y).$$

The profit maximization is the process by which the firm determines the price and output level that returns the greatest profit. Therefore, to find (x^*, y^*) such that

$$\Gamma(x^*, y^*) = \widetilde{r}(y^*) - \widetilde{w}(x^*) = \max_{(x,y)\in P(x^*)\times Q(y^*)} \widetilde{r}(y) - \widetilde{w}(x).$$

such that $\widetilde{r}$ is the revenue and $\widetilde{w}$ is the cost function.

Main existence result

Hypotheses on Γ: Now let us suppose that $\Gamma : K_p \times K_q \to \mathbb{R}_+$ satisfying the following assumptions:

$$\begin{cases} (H^1) \quad \text{For every fixed } y \in K_q, \text{ the function } x \to \Gamma(x,y) \\ \text{is convex and lower semicontinuous.} \\ (H^2) \quad \text{For every fixed } x \in K_p, \text{ the function } y \to \Gamma(x,y) \\ \text{is concave and upper semicontinuous.} \end{cases}$$

Objective: We shall show that the trajectories solutions $x(t)$ and $y(t)$ of the mixed system $(\mathcal{P}5)$ converges to limits points $\tilde{x}$ and $\tilde{y}$ which verifies

$$\Gamma(\tilde{x},\tilde{y}) = \min_{x\in P(\tilde{x})} \max_{y\in Q(\tilde{y})} \Gamma(x,y) = \max_{y\in Q(\tilde{y})} \min_{x\in P(\tilde{x})} \Gamma(x,y).$$

Main existence result

Main Theorem

Let $P(\cdot)$ and $Q(\cdot)$ be two continuous preorders with convex compact values defined on compact convex subsets K_p and K_q respectively, and let $\Gamma : K_p \times K_q \to \mathbb{R}_+$ be a function satisfying (H^1) and (H^2). Then,

(a) there exists a $P(\cdot) \times Q(\cdot)$-monotone solution $(x(\cdot), y(\cdot))$ of $(\mathcal{P}5)$ for any initial points $x_0 \in K_p$, and $y_0 \in K_q$.

(b) Moreover, let $x(\cdot)$ and $y(\cdot)$ be solutions of the mixed problem $(\mathcal{P}5)$, and let

$$\tilde{x} = \lim_{t_n \to +\infty} x(t_n) \quad \text{and} \quad \tilde{y} = \lim_{t_n \to +\infty} y(t_n).$$

Then, we have

$$\Gamma(\tilde{x}, \tilde{y}) = \min_{x \in P(\tilde{x})} \max_{y \in Q(\tilde{y})} \Gamma(x, y) = \max_{y \in Q(\tilde{y})} \min_{x \in P(\tilde{x})} \Gamma(x, y).$$

Proof of the main result

Step 1: Existence of $P(\cdot) \times Q(\cdot)$-monotone solutions

- Assumptions **(H^1)** and **(H^2)**,
- $P(\cdot)$ and $Q(\cdot)$ are two continuous multivalued mappings with convex compact values.

$$\begin{cases} V = (-\partial_x \Gamma, \partial_y^+ \Gamma), & \text{is an u.s.c. multivalued mapping,} \\ R = P \times Q, & \text{is a preorder,} \\ (x', y') \in \text{proj}_{T_R}(V) = (\text{proj}_{T_{P(x)}}(-\partial_x \Gamma(x,y)), \text{proj}_{T_{Q(y)}}(\partial_y^+ \Gamma(x,y))). \end{cases}$$

Step 2: Saddle-point problem

(1) $\psi(x(t), y(t))$ is a measurable selection in $\partial_x \Gamma(x(t), y(t))$, and $\varphi(x(t), y(t))$ is a measurable selection in $\partial_y^+ \Gamma(x(t), y(t))$. We have the following

$$\frac{d}{dt}\Gamma(x(t), y(t)) = \langle \psi(x(t), y(t)), x'(t) \rangle + \langle \varphi(x(t), y(t)), y'(t) \rangle \quad \text{a.e. } t \geq 0.$$

Proof of the main result

(2) points $\tilde{x} = \lim_{t_n \to +\infty} x(t_n)$ and $\tilde{y} = \lim_{t_n \to +\infty} y(t_n)$ verify

$$\min_{x \in P(\tilde{x})} \max_{y \in Q(\tilde{y})} \Gamma(x, y) = \Gamma(\tilde{x}, \tilde{y}) = \max_{y \in Q(\tilde{y})} \min_{x \in P(\tilde{x})} \Gamma(x, y).$$

if and only if

$$\mathrm{proj}_{T_{P(\tilde{x})}}(-\psi(\tilde{x}, \tilde{y})) = 0 \quad \text{and} \quad \mathrm{proj}_{T_{Q(\tilde{y})}}(\varphi(\tilde{x}, \tilde{y})) = 0.$$

Step 3: We use K. Fan(53) minimax theorem for saddle-functions to show that

$$\min_{x \in P(\tilde{x})} \max_{y \in Q(\tilde{y})} \Gamma(x, y) = \max_{y \in Q(\tilde{y})} \min_{x \in P(\tilde{x})} \Gamma(x, y).$$

□

Applications: Game

We consider a two players p and q game with a collective pay-off. The loss function $h_p : \mathbb{R}^n_+ \to \mathbb{R}_+$ represents the negative gain of player p, with the preorder $P : K_p \to K_p$ given by

$$P(x) = \{s \in K_p : \min_{z \in F(s)} h_p(z) \leq \min_{z \in F(x)} h_p(z)\}, \quad \forall x \in K_p.$$

Similarly, the gain function $h_q : \mathbb{R}^n_+ \to \mathbb{R}_+$ represents the positive gain of player q, with the preorder $Q : K_q \to K_q$ given by

$$Q(y) = \{s \in K_q : \max_{z \in F(s)} h_q(z) \geq \max_{z \in F(y)} h_q(z)\}, \quad \forall y \in K_q.$$

Player p seeks to minimize h_p and player q seeks to maximize h_q. The sets K_p and K_q are the sets of strategies of player p and player q respectively. $x(t)$ is a strategy of player p in K_p and $y(t)$ is a strategy of player q in K_q. The pay-off function $\Gamma(x, y)$ represent the collective pay-off of the two players.

Applications: Game

The problem

To prove that (x^*, y^*) is the maximum for the collective pay-off function $\Gamma(x, y)$ i. e.

$$\Gamma(x^*, y^*) = \max_{(x,y)\in P(x^*)\times Q(y^*)} \Gamma(x, y).$$

it is sufficient to maximize a collective well-being with $\Gamma(x, y) = \widetilde{r}(y) - \widetilde{w}(x)$, such that $\widetilde{r}$ is the revenue and $\widetilde{w}$ is the cost function.

Plan

The prox-regular case: The problem

This part deals with the existence of P-monotone solutions of the differential inclusion

$$(\mathcal{P}6) \qquad \begin{cases} x'(t) \in \mathrm{proj}_{T_{P(x(t))}(x(t))}(-\partial w(x(t))), & \text{a.e. in } [0,\infty), \\ x(0) = x_0. \end{cases}$$

where,

- $\partial w(\cdot)$ is the proximal subdifferential of the function $w(\cdot)$.

$$\partial w(x) = \{x^* \in \mathbb{R}^n \mid \langle x^*, x' - x\rangle \leq w(x') - w(x) + \frac{c}{2}\|x' - x\|^2, \forall x' \in \mathbb{R}^n\}.$$

$(A0^c)$ $\quad w : K \to \mathbb{R}_+$ is a proper, lower semicontinuous c-prox-regular function.

Notion of monotone solutions

Let K be a convex compact set of $\mathbb{R}^m_+$. We define the following preorder P on K:

$$P(x) = \{y \in K : \min_{z \in F(y)} h(z) \leq \min_{z \in F(x)} h(z)\}, \qquad \forall x \in K.$$

Hypotheses:

$F : K \rightrightarrows \mathbb{R}^n_+$ is a multivalued mapping satisfying;

$$(A1) \qquad \begin{cases} (a) & F(x) \text{ is a convex compact set, } \forall x \in K; \\ (b) & F(\cdot) \text{ is concave;} \\ (c) & F(\cdot) \text{ is continuous.} \end{cases}$$

$h : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a single-valued function satisfying;

$$(A2^c) \qquad \begin{cases} (a) & h(\cdot) \text{ is continuous;} \\ (b) & h(\cdot) \text{ is } c\text{-prox-regular;} \\ (c) & \text{if } x \geq y, \text{ then } h(x) \leq h(y). \end{cases}$$

Proposition II.3

Assume that Assumptions $(A1)$, and $(A2^c)$ are satisfied. Then, the preorder $P(\cdot)$ defined on K is continuous with nonempty compact prox-regular values.

Existence result

Theorem II.2

Let $P(\cdot)$ be a continuous preorder with compact prox-regular values defined on a compact convex subset K, and let $w : K \to \mathbb{R}_+$ be a function satisfying $(A0^c)$. Then,

(a) there exists a $P(\cdot)$-monotone solution $x(\cdot)$ of $(\mathcal{P}6)$ for any initial points $x_0 \in K$.

(b) let $x(\cdot)$ be a solution of the problem $(\mathcal{P}6)$, and let

$$\bar{x} = \lim_{t_n \to +\infty} x(t_n).$$

Then, we have

$$w(\bar{x}) = \min_{x \in P(\bar{x})} w(x).$$

The proof of Theorem II.2 is based on the following:

- Under $(A0^c)$, the subdifferential ∂w is an u.s.c. multivalued map.
- The preorder $P(\cdot)$ is continuous with nonempty compact prox-regular values.
- Let $x \in K$, and let $\widetilde{\psi}(x)$ be a measurable selection in $\partial w(x)$. Then, we have

$$\frac{d}{dt} w(x(t)) = \langle \widetilde{\psi}(x(t)), x'(t) \rangle \quad \text{a.e. } t \geq 0.$$

- A limit point $\bar{x}$ is a minimum of w on $P(\bar{x})$ if and only if

$$\text{proj}_{T_{P(\bar{x})}}(-\widetilde{\psi}(\bar{x})) = 0.$$

$\square$

Aubin-Cellina(84), Falcone-Scinolfi(83)

Plan

Perspectives

- Control theory.
- Second order differential inclusions.
- Applications: economic, game theory.

Thank you for your attention!