Golden Ratio Algorithms for Variational Inequalities

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Games, Dynamics and Optimization
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First-order method for general convex problems

- Lipschitz constants are bad;
- Lipschitz assumptions are worse;
- Linesearch are ugly.

Introduction

Variational inequality problem (VI):

find $x^* \in X = \mathbb{R}^d$ such that

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VI as a monotone operator inclusion:

$$0 \in F(x^*) + \partial g(x^*)$$

Motivation–1

Composite minimization:

$$\min_x f(x) + g(x)$$

- $f \colon X \to \mathbb{R}$ is a convex smooth function with Lipschitz ∇f
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First-order optimality condition:

$$\langle \nabla f(x^*), x - x^* \rangle + g(x) - g(x^*) \geq 0 \quad \forall x \in X.$$

Motivation–2

Saddle point problem:

$$\min_x \max_y \mathcal{L}(x, y) := g_1(x) + K(x, y) - g_2^*(y)$$

- $K\colon X \times Y \to \mathbb{R}$ is smooth convex-concave
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$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(z) = \begin{pmatrix} \nabla_x K(x, y) \\ -\nabla_y K(x, y) \end{pmatrix}, \quad g(z) = g_1(x) + g_2^*(y)$$

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Idea: $\bar{x}^k \in \text{conv}\{x^k, x^{k-1}, \dots, x^0\}$. Let $\varphi = \frac{\sqrt{5}+1}{2} = 1.618\dots$

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Composite minimization:

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$$\sum_{i=1}^{k} \lambda_i (h(x^i) - h_*) \leq C \Longrightarrow h\Big(\frac{\sum_i \lambda_i x^i}{\sum_i \lambda_i}\Big) - h_* \leq \frac{C}{\sum_i \lambda_i}$$

Sparse logistic regression

$$\min_x h(x) := \sum_{i=1}^m \log(1 + e^{-b_i \langle a_i, x\rangle}) + \gamma \|x\|_1,$$

where $x \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$, and $b_i \in \{-1, 1\}$, $\gamma > 0$.

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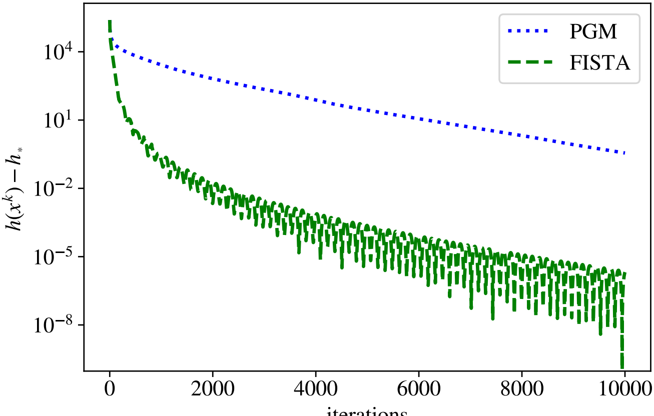
LIBSVM: *kdda-2010*, $n = 2014669$, $m = 510302$



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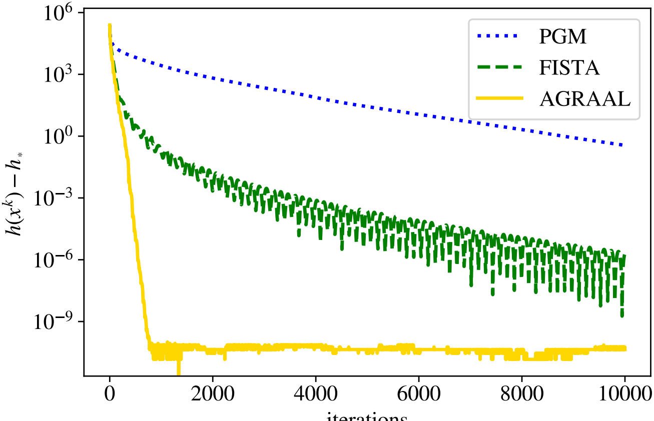

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$$\begin{aligned}\alpha\|\bar{x}^{k+1} - x\|^2 + \beta_k\|x^{k+1} - x^k\|^2 + \gamma_k\|x^k - \bar{x}^k\|^2 \\ \leq \alpha\|\bar{x}^k - x\|^2 + \beta_{k-1}\|x^k - x^{k-1}\|^2 \\ + 2\lambda_k[\langle F(x^k), x^k - x\rangle + g(x^k) - g(x)]\end{aligned}$$

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Beyond monotonicity

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- more general than monotonicity or pseudo-monotonicity

Examples

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$n = 1000, A, B \in \mathbb{R}^{n \times n}$, $A_{ij}, B_{ij} \sim \mathcal{N}(0,1)$

generate 100 random instances $\Rightarrow$ 100% success rate

Much Ado About

- Lipschitz constant
- Global Lipschitz assumption
- Linesearch

Thanks for attention!