1.

$$2^{x^2+x+\frac{1}{2}} = 4\sqrt{2} = 2^{2+\frac{1}{2}}$$

$\Updownarrow$ 2^x inj.

$$x^2+x+\cancel{\tfrac{1}{2}} = 2+\cancel{\tfrac{1}{2}}$$

$$x^2+x-2=0$$

$x_1 = 1 \qquad x_2 = -2$

$S=\{1,-2\}$

$\boxed{D}$ IGAZ

$\boxed{A}$, $\boxed{B}$, $\boxed{C}$ Hamis

2.

$X \in \mathcal{M}_2(\mathbb{R})$

$$\underbrace{\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}}_{A} X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\boxed{X \neq O_2}$

A invertálható $\Rightarrow$

$$\underbrace{A^{-1}(AX)}_{} = A^{-1} \cdot O_2.$$

$X = O_2$ nem jó

$\det A = 0 \Leftrightarrow 1 - a^2 = 0 \Leftrightarrow a \in \{-1, 1\}.$

$\underline{a=1}$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$a=-1$ $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Tehát $a \in \{-1, 1\} \iff \exists X \neq O_2$ megoldás.

$\boxed{D}$ Igaz

A, B, C hamis

(3.) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2 + x + 1$

A. $\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} \quad f(x) = y$ ✓ Igaz

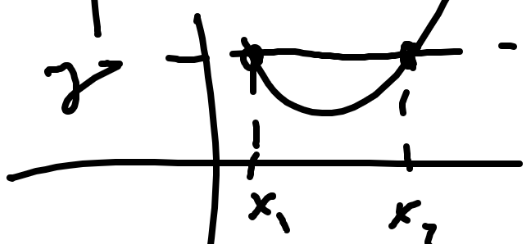
B. $\forall y \in \mathbb{R} \ \exists x \in \mathbb{R} \quad f(x) = y$ hamis

$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$, (f nem szürj).

C. $x_1, x_2 \in \mathbb{R} \Rightarrow f(x_1) \neq f(x_2)$ hamis

$x_1 \neq x_2$

$f(0) = f(-1) = 1$



D. $x_1, x_2 \in \mathbb{R} \quad x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ Igaz

4. $\alpha \in \mathbb{C}$

$\underline{\alpha^3 = 1}$ $\quad A = \begin{pmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{pmatrix}$

$(\alpha-1)(\alpha^2+\alpha+1) = 0$

$\alpha \in \left\{1, \frac{-1 \pm i\sqrt{3}}{2}\right\}$

$\det A = 0 \quad = \begin{pmatrix} 1+\alpha+\alpha^2 & \alpha & \alpha^2 \\ \alpha+\alpha^2+1 & \alpha^2 & 1 \\ \alpha^2+\alpha+1 & 1 & \alpha \end{pmatrix}$

$\begin{matrix} \uparrow \\ 0 \\ 0 \\ 0 \end{matrix}$ $\quad \Rightarrow \underline{\operatorname{rang} A \leq 2}$

$$\begin{vmatrix} \boxed{1} & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix}$$

$$|1| \neq 0$$

$$\underline{\operatorname{rang} A \geq 1}$$

$$\Rightarrow \boxed{\operatorname{rang} A = 1}$$

A – hamis

B – hamis

C igaz

D – hamis

$$\begin{vmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & \alpha \\ \alpha^2 & 1 \end{vmatrix} = 1 - \alpha^3 = 0$$

$$\begin{vmatrix} 1 & \alpha^2 \\ \alpha & 1 \end{vmatrix} = 1 - \alpha^3 = 0$$

$$\begin{vmatrix} 1 & \alpha^2 \\ \alpha^2 & \alpha \end{vmatrix} = \alpha - \alpha^4 = 0$$

5. $(a_n)_{n\geq 1}$

$a_{101} = 695$

$a_{1001} = 6995$

$a_{101} = a_1 + 100r$

$a_{1001} = a_1 + 1000r$

$a_{1001} - a_{101} = 900r$

$63\cancel{00} = 9\cancel{00}r$

$\boxed{r = 7}$

$$\begin{array}{r} 6995 - \\ 695 \\ \hline 6300 \end{array}$$

$a_1 = 695 - 700 = -5$

$\boxed{a_1 = -5}$ A igaz

B - hamis

$a_{2021} = a_1 + r \cdot 2020 = -5 + 14140$
$= 14135 \Rightarrow \boxed{C}$ igaz

D is igaz $\Leftarrow$ $5 \cdot 193 = 965$

$S = 5 \cdot (-10 + 203)$

$\Uparrow$

$= 10(2a_1 + 29r)$

$= 10(a_1 + 10r + a_1 + 19r)$

$2S = 10 \cdot (a_{11} + a_{20})$

$\Uparrow$

$a_{11} + a_{12} + \dots + a_{20} = S$

$a_{20} + \dots + a_{11} = S$

6.

$f_m: \mathbb{R} \to \mathbb{R} \qquad f_m(x) = mx^2 + 2(m+1)x + m - 2$

$\forall x \in \mathbb{R}$

$x_v = -\frac{m+1}{m} = -1 - \frac{1}{m}$

$\Delta = 4\left(\cancel{m^2} + 2m + 1 - \cancel{m^2} + 2m\right)$

$= 4(4m + 1)$

$y_v = \frac{-\Delta}{4a} = -\frac{4m+1}{m} = -4 - \frac{1}{m}$

$\rightarrow \quad x_v = y_v + 3 \qquad \boxed{y_v = x_v - 3}$

C igaz

A, B, D hamis

7. $P = X^4 + aX^3 - 6X^2 + 11X + b$

$Q_1 = X - 1 \qquad Q_2 = X + 3$

- Maradékos osztás tétele

$\forall P, Q \in \mathbb{R}[X] \quad \exists H, R \in \mathbb{R}[X]$

$$P = H \cdot Q + R \qquad \operatorname{gr} R < \operatorname{gr} Q$$

<u>Köv.</u>

$$P(x) = H(x) \cdot (x - a) + r \qquad \operatorname{gr} r < 1$$

$\operatorname{gr} r < 1 \;\Leftrightarrow\; r$ állandó

<u>$P(a) = r$</u>

! $(x-a)$-val való osztási maradék $P(a)$.

$P \,\vdots\, (x-a) \iff P(a)=0$.

$P(1) = P(-3) = 0$

$$\begin{cases} 1+a-6+15+b=0 \\ 81-27a-54-45+b=0 \end{cases}$$

$$\begin{cases} a+b=-10 \Rightarrow \text{C igaz} \\ -27a+b=18 \end{cases}$$

D hamis

$28a=-28$

$a=-1$ $b=-9$

A hamis B igaz

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$$

- $a_0, a_1, \dots, a_n \in \mathbb{Z}$
  $x_0 = \frac{p}{q} \in \mathbb{Q}$ irreducibilis ⟹ $p \mid a_n$, $q \mid a_0$
  $p, q \in \mathbb{Z}$

Ha $a_0 = 1 \Rightarrow x_0 \in \mathbb{Z}$ és $x_0 = p$ $p \mid a_n$

- $a_1, a_2, \dots, a_n \in \mathbb{Q}$
  $x_0 = a + b\sqrt{c}$ gyök ⟹ $x_1 = a - b\sqrt{c}$ is gyök
  $a, b, c \in \mathbb{Z}_{c>0}$ $\sqrt{c} \notin \mathbb{Q}$

$a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$

$z = a + ib$ gyök $\Rightarrow$ $\bar{z} = a - ib$ is gyök.

$a, b \in \mathbb{R}$

8. $f: \mathbb{R} \to (0, +\infty)$

$f(x) = (2a^2+2a+1)^x, \ \forall x \in \mathbb{R}$

$g(y) = (\log_{a^2+4} y), \ \forall y > 0$

$f(x) = y$

$(2a^2+2a+1)^x = y$

$x = \left(\log_{2a^2+2a+1} y\right)$

$f(g(y)) = y, \ \forall y > 0$

$g(f(x)) = x, \ \forall x \in \mathbb{R}$

$\cancel{a^2}+4 = \cancel{2}a^2+2a+1$

$a^2+2a-3=0$

$a_1 = 1 \quad a_2 = -3$

$\underline{a_1=1}$ $\quad f(x)=5^x \quad g(y)=\log_5 y$

$a_2=-3 \quad f(x)=13^x \quad g(y)=\log_{13} y$

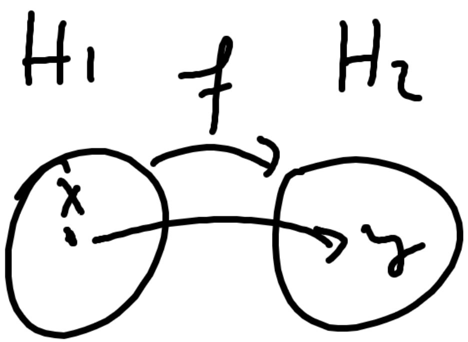
$A=\{1,-3\}$

$-3\notin[-1,4] \Rightarrow$ ~~A~~ hamis

$-3\notin[-2,3] \rightarrow$ B igaz

C hamis

$1,-3\in[-3,2] \Rightarrow$ D igaz



$f(x)=y$

$\Updownarrow$

$x=f^{-1}(y)$

$f^{-1}: H_2 \to H_1$

$$g(f(x)) = x \xrightarrow{x=1} g(2a^2+2a+1) = 1 \Leftrightarrow \log_{a^2+4}(2a^2+2a+1) = 1$$

$$f(g(y)) = y \xrightarrow{y=a^2+4} f(1) = a^2+4 \Longleftrightarrow 2a^2+2a+1 = a^2+4$$

9.

$$x * y = xy - 4x - 4y + 20$$

$$(1*1)*1 = (1-4-4+20)*1 = 13*1 = 13 - 4\cdot 13 \underbrace{-4+20} = -3\cdot 13 + 16 = 16-39+3$$

$x * e = x, \quad \forall x \in \mathbb{R}$

$xe - 4x - 4e + 20 = x$

$x(e-5) = 4(e-5), \ \forall x \in \mathbb{R} \Rightarrow \boxed{e=5}$

A - hamis

B - hamis

$3 * 3 = 9 - 12 - 12 + 20 = 5 \Rightarrow$ 3 inverze $\underline{3}$ (e) igaz

$\underline{D.}$ $(\mathbb{R}, *)$ csoport

$$x * x' = 5$$

$$xx' - 4x - 4x' + 20 = 5$$

$$x'(x-4) = 4x - 15$$

$$x' = \frac{4x-15}{x-4}$$ ha $\underline{x \neq 4}$

$x = 4$-re ellentmondás $x' \cdot 0 = 1$

$\Rightarrow$ $x = 4$-nek $\not\exists$ inverze

$\Rightarrow (\mathbb{R}, *)$ nem csoport

10.

$\overline{x_1 x_2 x_3}$

$x_1 \in \{1,2,3,4,5,6\}$

$x_2 \in \{0,1,2,3,4,5,6\} \setminus \{x_1\}$

$x_3 \in \{0,1,2,3,4,5,6\} \setminus \{x_1, x_2\}$

6

6

5

180

C igaz

A, B hamis

$$C_7^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35$$

minden kombináció pontosan egy olyan számot származtat amelyben a számjegyek erőlkenő sorrendban vannak → D igaz

11.

$A(2,3)$

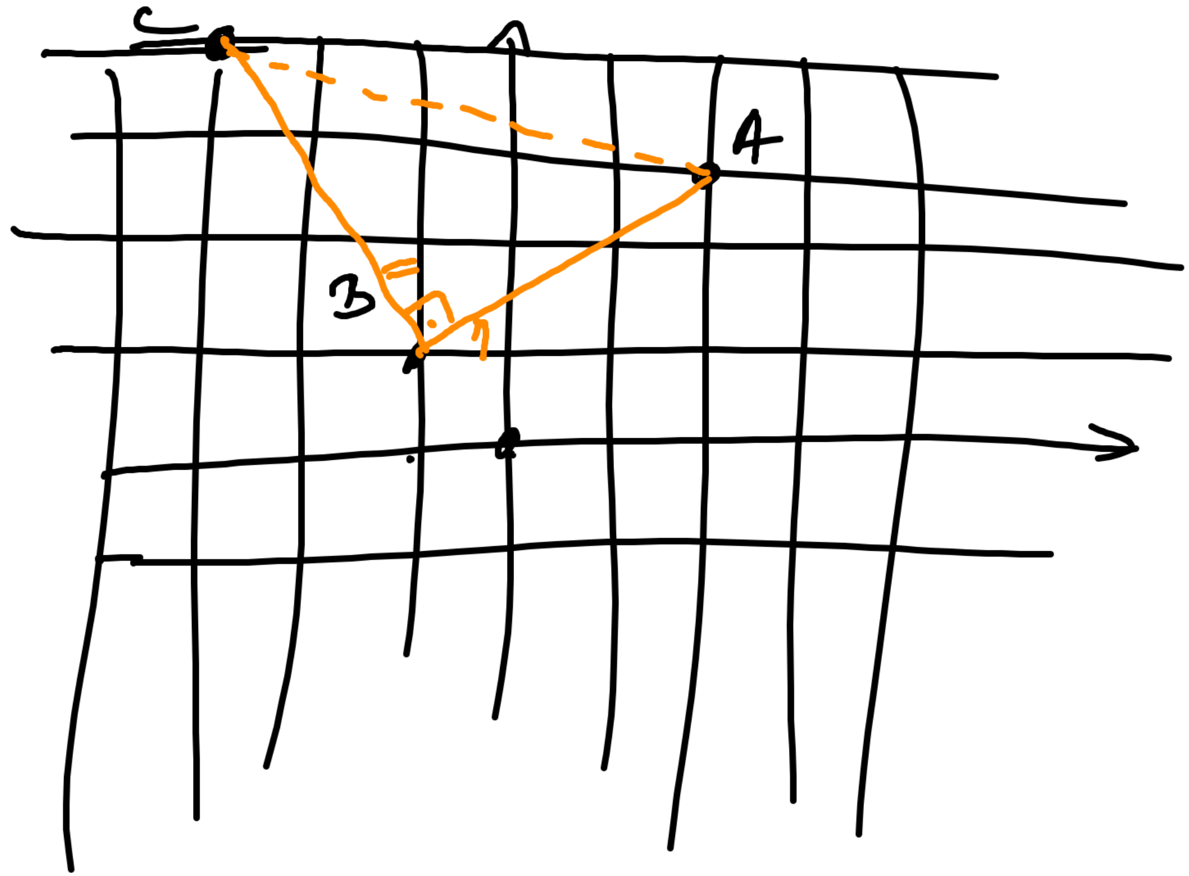
$B(-1,1)$

$C(-3,4)$

$$T=\frac{AB\cdot AB}{2}=\frac{2^2+3^2}{2}=\frac{13}{2}.$$

A, B, C igaz

D hamis



(12)

$A(-2,3)$

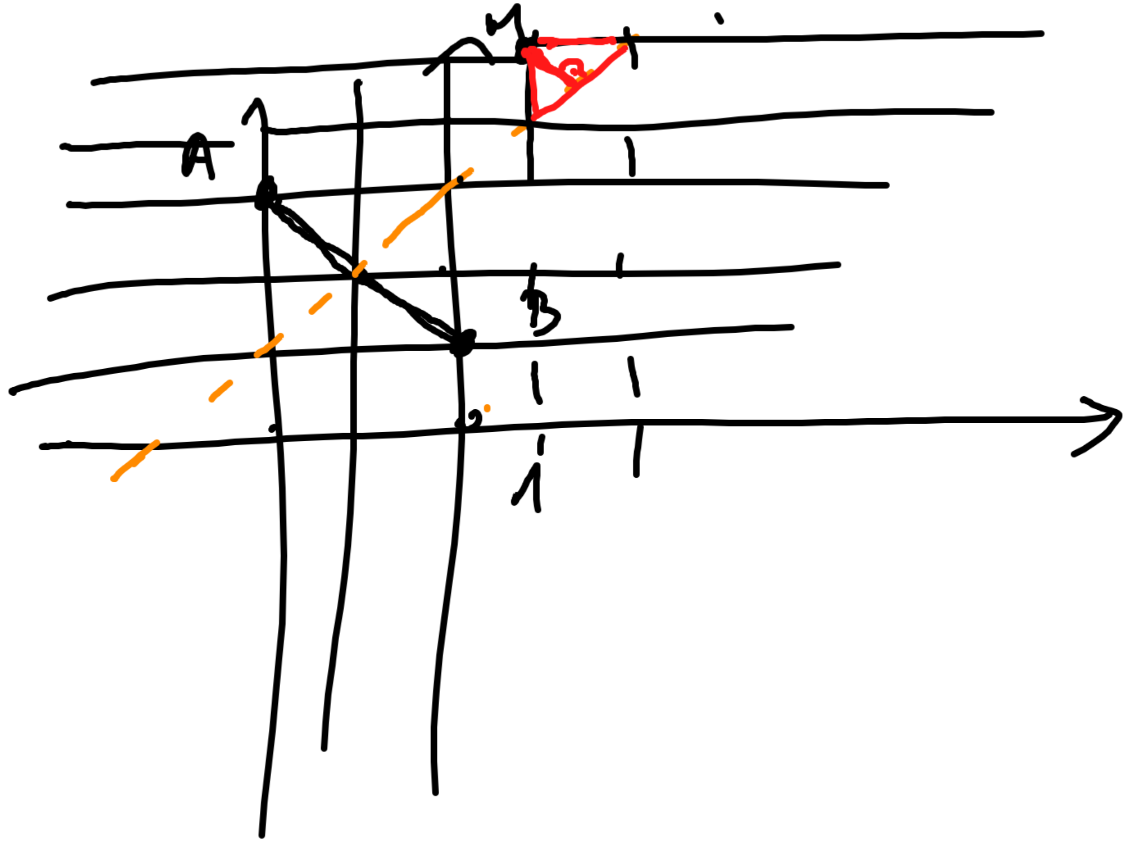
$B(0,1)$

$M(1,5)$

$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

A igaz

B, C, D hamis



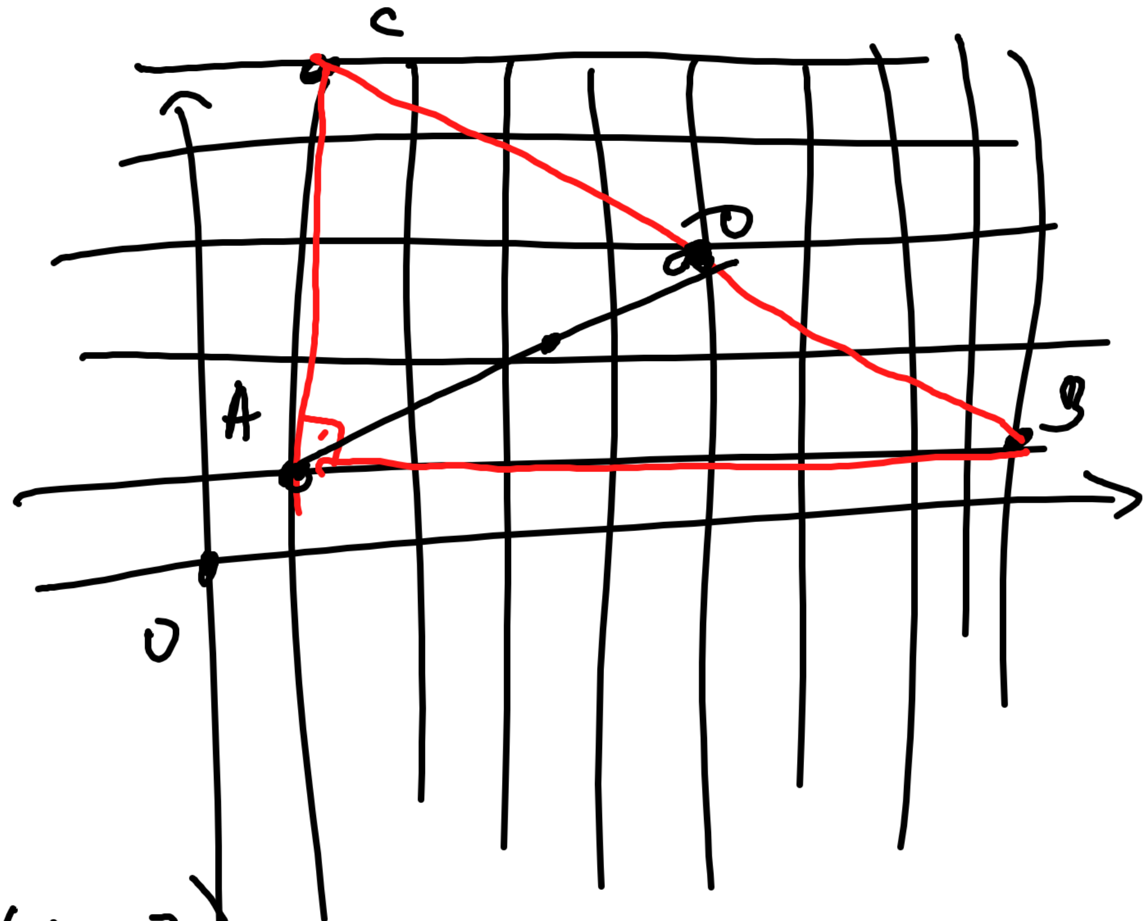
13

A - igaz

B - igaz

C - hamis

D - hamis.



$$G = \left(\frac{1+9+1}{3}, \frac{1+1+5}{3}\right) = \left(\frac{11}{3}, \frac{7}{3}\right)$$

$$O = (5, 3)$$

11.

$$4|\sin x|\cos x = 1 \qquad I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\left.\begin{array}{l} x>0 \\ x \in I \end{array}\right| \Rightarrow \quad 4\sin x \cos x = 1$$

$$2\sin 2x = 1$$

$$\sin 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{6} \qquad x_1 = \frac{\pi}{12} \in I$$

$$2x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$x_2 = \frac{5\pi}{12} \in I$$

$$\left.\begin{array}{l} x<0 \\ x \in I \end{array}\right\} \Rightarrow \quad -4\sin x \cos x = 1$$

$$x_3 = -\frac{\pi}{12}$$

$$x_4 = -\frac{5\pi}{12}$$

(C) IGAZ A, B, D hamis