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A modified Post Widder operators preserving e^{Ax}

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Abstract. In the present paper, we discuss the approximation properties of modified Post-Widder operators, which preserve the test function e^{Ax}. We establish weighted approximation and a direct quantitative estimate for the modified operators.

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1. Post-Widder operators

In the recent years some sequences of linear positive operators and the operators of integral type have been studied in [2], [3] and [4] etc. Also the moments of several operators have been provided in [8]. In the present article, we discuss the vatiant of an integral operators viz. Post-Widder operators. Post-Widder operators are defined for $f \in C[0,\infty)$ as (see [13]):

$$P_n(f,x) \quad := \quad \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty t^n\, e^{-\frac{nt}{x}}\, f(t)\; dt.$$

Following [7], we have

$$P_n(e^{\theta t},x) \quad = \quad \left(1-\frac{x\theta}{n}\right)^{-(n+1)}. \tag{1.1}$$

Very recently Gupta-Agrawal in [6] and Gupta-Tachev in [11] considered different forms of modified Post-Widder operators preserving the test functions $e_r, r \in N$. Gupta-Singh in [9] estimated some quantitative convergence results of Post-Widder operators preserving e^{ax}, e^{bx}.

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Let us consider that the Post-Widder operators preserve the test function e^{Ax}, then we start with the following form

$$\widetilde{P}_n(f,x) := \frac{1}{n!}\left(\frac{n}{a_n(x)}\right)^{n+1}\int_0^\infty t^n\, e^{-\frac{nt}{a_n(x)}}\, f(t)\, dt.$$

Then using (1.1), we have

$$\widetilde{P}_n(e^{At},x) = e^{Ax} = \left(1-\frac{a_n(x)A}{n}\right)^{-(n+1)},$$

implying

$$a_n(x) = \frac{n}{A}(1-e^{-Ax/(n+1)}).$$

Thus our modified operators $\widetilde{P}_n$ take the following form

$$\widetilde{P}_n(f,x) := \frac{1}{n!}\left[\frac{A}{(1-e^{-Ax/(n+1)})}\right]^{(n+1)} \int_0^\infty t^n\, e^{-\frac{At}{(1-e^{-Ax/(n+1)})}}\, f(t)\, dt, \tag{1.2}$$

with $x\in(0,\infty)$ and $\widetilde{P}_n(f,0)=f(0)$, which preserve constant and the test function e^{Ax}.

2. Lemmas

Lemma 2.1. *We have for $\theta>0$ that*

$$\widetilde{P}_n(e^{\theta t},x) = \left(1-\frac{(1-e^{-Ax/(n+1)})\theta}{A}\right)^{-(n+1)}.$$

It may be observed that $\widetilde{P}_n(e^{\theta t},x)$ may be treated as m.g.f. of the operators $\widetilde{P}_n$, which may be utilized to obtain the moments of (1.2). Let $\mu_r^{\widetilde{P}_n}(x)=\widetilde{P}_n(e_r,x)$, where $e_r(t)=t^r$, $r\in N\cup\{0\}$. The moments are given by

$$\begin{aligned}\mu_r^{\widetilde{P}_n}(x) &= \left[\frac{\partial^r}{\partial\theta^r}\widetilde{P}_n(e^{\theta t},x)\right]_{\theta=0}\\ &= \left[\frac{\partial^r}{\partial\theta^r}\left\{\left(1-\frac{(1-e^{-Ax/(n+1)})\theta}{A}\right)^{-(n+1)}\right\}\right]_{\theta=0}.\end{aligned}$$

Few moments are given below:

$$\begin{aligned}\mu_0^{\widetilde{P}_n}(x) &= 1,\\ \mu_1^{\widetilde{P}_n}(x) &= \frac{(n+1)}{A}(1-e^{-Ax/(n+1)}),\\ \mu_2^{\widetilde{P}_n}(x) &= \frac{(n+1)(n+2)}{A^2}(1-e^{-Ax/(n+1)})^2.\end{aligned}$$

Lemma 2.2. *The moments of arbitrary order, satisfy the following*

$$\mu_k^{\widetilde{P}_n}(x) = \frac{(n+1)_k}{A^k}(1-e^{-Ax/(n+1)})^k, k=0,1,....,$$

where the Pochhammer symbol is defined by

$$(c)_0 = 1, \quad (c)_k = c(c+1)\cdots(c+k-1).$$

Further by linearity property and using Lemma 2.2, we have the following lemma:

Lemma 2.3. *The central moments* $U_r^{\widetilde{P}_n}(x) = \widetilde{P}_n((t-x)^r, x)$ *are given below:*

$$U_k^{\widetilde{P}_n}(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x^{k-j}(1-e^{-Ax/(n+1)})^j\frac{(n+1)_j}{A^j}, \quad k=0,1,\dots .$$

Also, for each $n \in N$, *we have*

$$U_1^{\widetilde{P}_n}(x) = \frac{(n+1)}{A}(1-e^{-Ax/(n+1)}-1)-x,$$

$$U_2^{\widetilde{P}_n}(x) = \frac{(n+1)(n+2)}{A^2}(1-e^{-Ax/(n+1)})^2 + x^2 - 2x\frac{(n+1)}{A}(1-e^{-Ax/(n+1)}).$$

Lemma 2.4. *For the central moments* $U_{2k}^{\widetilde{P}_n}(x) = \widetilde{P}_n((t-x)^{2k}, x)$, *we have*

$$U_{2k}^{\widetilde{P}_n}(x) = O(n^{-k}), n \to \infty, k = 1,2,3,\cdots$$

Proof. We observe that

$$\widetilde{P}_n(f,x) = P_n(f,\alpha_n(x)),$$

where

$$a_n(x) = \frac{n}{A}(1-e^{-Ax/(n+1)}).$$

It is easy to verify $y > 1-e^{-y} > y - \frac{y^2}{2}$ for $y \in [0,\infty)$. We set $y = Ax/(n+1)$ and get

$$x\left(\frac{n}{n+1}\right) > \alpha_n(x) > x\left(\frac{n}{n+1}\right) - \left(\frac{Ax}{n+1}\right)^2 \cdot \frac{n}{2A}.$$

Hence

$$\frac{x}{n+1} < x - \alpha_n(x) < \frac{x}{n+1} + \frac{Ax^2 n}{2(n+1)^2} = O(n^{-1}),$$

by fixed $x \in [0,\infty)$. Therefore

$$\begin{aligned}
\widetilde{P}_n((t-x)^{2k},x) &= P_n((t-x)^{2k},\alpha_n(x)) \\
&= P_n((t-\alpha_n(x)+\alpha_n(x)-x)^{2k},\alpha_n(x)) \\
&\le C(k)P_n((t-\alpha_n(x))^{2k},\alpha_n(x)) + P_n((x-\alpha_n(x)^{2k},\alpha_n(x)) \\
&\le C(k).\frac{1}{n^k} + (x-\alpha_n(x))^{2k} = O(n^{-k}).
\end{aligned}$$

This completes the proof of Lemma 2.4. □

3. Weighted approximation

We also analyse the behaviour of the operators on some weighted spaces. Set $\phi(x) = 1 + e^{Ax},\ x \in R^+$ and consider the following weighted spaces:

$$\begin{aligned} B_\phi(R^+) &= \{f : R^+ \to R : |f(x) \leq C_1(1 + e^{Ax})\}, \\ C_\phi(R^+) &= B_\phi(R^+) \cap C(R^+), \\ C_\phi^k(R^+) &= \left\{ f \in C_\phi(R^+) : \lim_{x\to\infty} \frac{f(x)}{1+e^{Ax}} = C_2 < \infty \right\}, \end{aligned}$$

where C_1, C_2 are constants depending on f. The norm is defined as

$$||f||_\phi = \sup_{x\in R^+} \frac{|f(x)|}{1+e^{Ax}}.$$

Theorem 3.1. *For each $f \in C_\phi^k(R^+)$, we have*

$$\lim_{n\to\infty} ||\widetilde{P}_n f - f||_\phi = 0.$$

Proof. Following [1, Th. 1] in order to prove the result we have to prove

$$\lim_{n\to\infty} ||\widetilde{P}_n(e^{iAt/2}) - e^{iAx/2}||_\phi = 0, i = 0, 1, 2.$$

The result is true for $i = 0, i = 2$. It remains to verify it for $i = 1$. By Lemma 2.1 we have

$$\begin{aligned} &||\widetilde{P}_n(e^{At/2}) - e^{Ax/2}||_\phi \\ = &\sup_{x\in R^+} \frac{\left|\left(1 - \frac{(1-e^{-Ax/(n+1)})}{2}\right)^{-(n+1)} - e^{Ax/2}\right|}{1+e^{Ax}} \\ = &\sup_{x\in R^+} \frac{\left|\left(1+e^{-Ax/(n+1)}\right)^{-(n+1)} 2^{n+1} - e^{Ax/2}\right|}{1+e^{Ax}} \\ = &\sup_{x\in R^+} \frac{\left|e^{Ax}\left(1+e^{Ax/(n+1)}\right)^{-(n+1)} 2^{n+1} - e^{Ax/2}\right|}{1+e^{Ax}} \\ = &\sup_{x\in R^+} \left[\frac{e^{Ax}}{1+e^{Ax}}\right] \cdot \left|\left(\frac{2}{1+e^{Ax/(n+1)}}\right)^{n+1} - e^{-Ax/2}\right|. \end{aligned} \quad (3.1)$$

Obviously $\frac{e^{Ax}}{1+e^{Ax}} \in \left[\frac{1}{2}, 1\right),\ A > 0,\ x > 0$. We set $t = e^{Ax/2}, t \in [1, \infty)$ for $x \in (0, \infty)$. Then (3.1) implies

$$\left|\left(\frac{2}{1+t^{2/(n+1)}}\right)^{n+1} - t^{-1}\right| = t^{-1}\left|\left(\frac{2t^{1/(n+1)}}{1+t^{2/(n+1)}}\right)^{n+1} - 1\right| = g(t). \quad (3.2)$$

In (3.2), we set $t^{1/(n+1)} = y \in [1, \infty)$. Hence

$$\begin{aligned} g(t) = h(y) &= y^{-(n+1)} \left| \left(\frac{2y}{1+y^2} \right)^{n+1} - 1 \right| \\ &= \left| \left(\frac{2}{1+y^2} \right)^{n+1} - y^{-(n+1)} \right| \\ &= y^{-(n+1)} - \left(\frac{2}{1+y^2} \right)^{n+1}. \end{aligned} \tag{3.3}$$

We have $h(1) = 0$, $h(+\infty) = \lim_{y\to\infty} h(y) = 0$. To find the global maxima of $h(y)$ we solve the equation $h'(y) = 0$. Simple calculations imply that $h'(y_0) = 0$ for y_0 satisfying the equation

$$\frac{2}{1+y_0^2} = y_0^{-(n+3)/(n+2)}, y_0 \in (1, \infty). \tag{3.4}$$

The equations (3.3) and (3.4) imply

$$h(y) \le h(y_0) = y_0^{-(n+1)} - y_0^{-(n+3)(n+1)/(n+2)}. \tag{3.5}$$

The proof will be completed if we show

$$h(y_0) < \frac{1}{2(n+3)}, n \to \infty. \tag{3.6}$$

We set in (3.5) $y_0^{n+1} = z_0 \in (1, +\infty)$. Then $h(y_0) = z_0^{-1} - z_0^{-(n+3)/(n+2)} < \max p(z)$ with $p(z) = z^{-1} - z^{-(n+3)/(n+2)}$. We compute that $p'(z_1)$ for $z_1 = \left(\frac{n+3}{n+2}\right)^{n+2}$. Therefore

$$\begin{aligned} p(z_1) &= \left(\frac{n+3}{n+2}\right)^{-(n+2)} - \left(\frac{n+3}{n+2}\right)^{-(n+3)} \\ &= \left(\frac{n+3}{n+2}\right)^{-(n+2)} \left[1 - \left(\frac{n+3}{n+2}\right)^{-1}\right] \\ &= \left(1 + \frac{1}{n+2}\right)^{-(n+2)} \frac{1}{n+3} < \frac{1}{2(n+3)}, \end{aligned}$$

due to $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n+2}\right)^{-(n+2)} = e^{-1} < 1/2$. □

4. A direct quantitative estimate

Our goal in this section is to obtain a quantitative form of the statement in Theorem 3.1. For the sake of simplicity we slightly modify the weight function and instead of $\phi(x) = 1 + e^{Ax}, x \in R^+$ we consider $\phi(x) = e^{Ax}, x \in R^+$, For continuous functions on $[0, \infty)$ with exponential growth i.e.

$$||f||_A := \sup_{x\in[0,\infty)} |f(x) \cdot e^{-Ax}| < \infty, A > 0, \tag{4.1}$$

it is easy to observe that

$$||\widetilde{P}_n f||_A \leq ||f||_A. \tag{4.2}$$

Consequently if the following function series is uniformly convergent on $[0,\infty)$

$$S(x) = \sum_{k=0}^{\infty} u_k(x), x \in [0,\infty),$$

then

$$\widetilde{P}_n(S(t),x) = \sum_{k=0}^{\infty} \widetilde{P}_n(u_k(t),x), x \in [0,\infty), \tag{4.3}$$

where the last series is also uniformly convergent. For our goals in this section we need the first order exponential modulus of continuity, studied by Ditzian in [5] and defined as

$$\omega_1(f,\delta,A) := \sup_{h\leq\delta, 0\leq x<\infty} |f(x) - f(x+h)|e^{-Ax}.$$

We consider the sequence of operators $\widetilde{P}_n : E \to C[0,\infty)$, where the domain of the operator $\widetilde{P}_n$ contains the space of functions f with exponential growth, i.e. $||f||_A < \infty$. Our main result states the following:

Theorem 4.1. *Let $\widetilde{P}_n : E \to C[0,\infty)$ be sequence of linear positive operators of Post-Widder type defined in (1.2). Then*

$$|\widetilde{P}_n(f,x) - f(x)| \leq e^{Ax}[3 + C(n,x)]\omega_1(f, \sqrt{U_2^{\widetilde{P}_n}(x)}, A),$$

where

$$C(n,x) = 2\sum_{k=1}^{\infty} \frac{A^k}{k!}\sqrt{U_{2k}^{\widetilde{P}_n}(x)}, \ n \to \infty \textit{ for fixed } x \in [0,\infty).$$

Proof. We observe that

$$|f(t) - f(x)| \leq \begin{cases} e^{Ax}\omega_1(f,\delta,A), |t-x| \leq \delta \\ e^{Ax}\omega_1(f,k\delta,A), \delta \leq |t-x| \leq k\delta, \end{cases} \tag{4.4}$$

where k is the smallest natural number in the above upper bound. Now [12, Lemma 2.2] (also see [10]) implies

$$\begin{aligned} \omega_1(f,k\delta,A) &\leq ke^{A(k-1)\delta}\omega_1(f,\delta,A) \\ &\leq \omega_1(f,\delta,A)\left[\frac{|t-x|}{\delta} + 1\right]e^{A.|t-x|}. \end{aligned} \tag{4.5}$$

Now (4.4) and (4.5) imply

$$|f(t) - f(x)| \leq \left[1 + \left(\frac{|t-x|}{\delta} + 1\right)e^{A.|t-x|}\right]e^{Ax}\omega_1(f,\delta,A). \tag{4.6}$$

For fixed $x \in [0,\infty)$ the following series is uniformly convergent for $t \in [0,\infty)$

$$\begin{aligned} S_1(t,x) &= e^{A.|t-x|} = \sum_{k=0}^{\infty} \frac{(A|t-x|)^k}{k!} \\ \frac{|t-x|}{\delta} S_1(t,x) &= \frac{|t-x|}{\delta} + \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{A^k |t-x|^{k+1}}{k!}. \end{aligned} \tag{4.7}$$

Obviously for linear positive operators $\widetilde{P}_n$ using (4.4), (4.6) and (4.7), we obtain

$$\begin{aligned} |\widetilde{P}_n(f(t)-f(x)| &\leq \widetilde{P}_n(|f(t)-f(x)|,x) \\ &\leq e^{Ax} \Big\{ 1 + \widetilde{P}_n(S_1(t,x),x) + \frac{1}{\delta} \widetilde{P}_n(|t-x|,x) \\ &\quad + \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{A^k \widetilde{P}_n(|t-x|^{k+1},x)}{k!} \Big\} \omega_1(f,\delta,A). \end{aligned} \tag{4.8}$$

From Cauchy Schwarz inequality, we have

$$\begin{aligned} \widetilde{P}_n(|t-x|^{k+1},x) &\leq \sqrt{\widetilde{P}_n((t-x)^2,x)} \sqrt{\widetilde{P}_n((t-x)^{2k},x)} \\ &= \sqrt{U_2^{\widetilde{P}_n}(x)} \sqrt{U_{2k}^{\widetilde{P}_n}(x)}. \end{aligned} \tag{4.9}$$

Further

$$S_1(t,x) = 1 + A|t-x| + \sum_{k=2}^{\infty} \frac{(A|t-x|)^k}{k!}.$$

Hence

$$\widetilde{P}_n(S_1(t,x),x) \leq 1 + A\sqrt{U_2^{\widetilde{P}_n}(x)} + \sum_{k=2}^{\infty} \frac{A^k \sqrt{U_{2k}^{\widetilde{P}_n}(x)}}{k!}. \tag{4.10}$$

From Lemma 2.4, for fixed $x \in [0,\infty)$, we have

$$U_{2k}^{\widetilde{P}_n}(x) = O(n^{-k}), n \to \infty. \tag{4.11}$$

We set in (4.8) that

$$\delta = \sqrt{U_2^{\widetilde{P}_n}(x)} = O(n^{-1/2}), n \to \infty. \tag{4.12}$$

Therefore estimates (4.8)-(4.12) imply

$$|\widetilde{P}_n(f,x) - f(x)| \leq e^{Ax}[3 + C(n,x)]\omega_1(f, \sqrt{U_2^{\widetilde{P}_n}(x)}, A),$$

where

$$C(n,x) = A\sqrt{U_2^{\widetilde{P}_n}(x)} + \sum_{k=2}^{\infty} \frac{A^k \sqrt{U_{2k}^{\widetilde{P}_n}(x)}}{k!} + \sum_{k=1}^{\infty} \frac{A^k \sqrt{U_{2k}^{\widetilde{P}_n}(x)}}{k!} = O(n^{-1/2}), n \to \infty,$$

by fixed $x \in [0,\infty)$. This completes the proof of theorem. □

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